Homework 4 Solutions

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Total Points: 100

Question 1 (Particle in a Double Well Potential)

Consider a particle with coordinate q and mass m moving in the potential

$$V(q) = \lambda(q^2 - q_0^2)^2. \tag{1.0.1}$$

We are interested in the imaginary time transition amplitude

$$\langle q_0, T/2 | {-q_0}, -T/2 \rangle = \langle q_0 | \, e^{-HT/\hbar} \, | {-q_0} \rangle \,, \tag{1.0.2}$$

in the limit that $T \to \infty$.

1. **[10/35]** The transition amplitude has the path integral form

$$\langle q_0 | \, e^{-HT/\hbar} \, | {-q_0} \rangle = \int\limits_{\substack{q(-T/2)=-q_0 \\ q(T/2)=q_0}} \mathcal{D}q(\tau) \exp\left[-\frac{1}{\hbar} \int_{-T/2}^{T/2} \mathrm{d}\tau \, L_E(q, \dot{q}) \right], \tag{1.1.1a}$$

$$L_E(q, \dot{q}) = \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q) = \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + \lambda(q^2 - q_0^2)^2, \tag{1.1.1b}$$

where the subscripts under the functional integral are the boundary conditions for the sum over configurations $q(-T/2) = -q_0$ and $q(T/2) = +q_0$, and L_E is the Euclidean Lagrangian.

For comparison, the Lagrangian in real time is

$$L = \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - V(q) = \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \lambda(q^2 - q_0^2)^2. \tag{1.1.2}$$

and so the imaginary time Lagrangian looks like a conventional classical Lagrangian with an inverted potential.

2. **[10/35]** The imaginary time equation of motion follows from the Euler-Lagrange equation

$$\frac{\partial L_E}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L_E}{\partial \dot{q}} = 0$$

$$\implies m\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} = \frac{\partial V(q)}{\partial q} = 4\lambda q(q^2 - q_0^2). \tag{1.2.1}$$

This shows that the positions $q = \pm q_0$ are *unstable* equilibria, and that a particle which starts at $q = -q_0$ will "roll" down the inverted potential before rolling up the hill towards $q = +q_0$. In contrast, the real time equation of motion is

$$m\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -4\lambda q(q^2 - q_0^2). \tag{1.2.2}$$

In this case, $q = \pm q_0$ are stable equilibria, and the region in between the double well is classically forbidden for a particle with a small kinetic energy.

The integral of motion which is analogous to the classical energy is the Hamiltonian function corresponding to the Euclidean Lagrangian

$$H_E = \frac{\partial L_E}{\partial \dot{q}}\dot{q} - L_E = \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 - V(q). \tag{1.2.3}$$

Since we are interested in the limit $T \to \infty$, we impose the usual requirement that $\dot{q} = 0$ as $\tau \to \pm\infty$ to ensure a well-behaved/finite action. In this case, since $q(-T/2) = -q_0$, and $V(-q_0) = 0$, the initial value of the "Euclidean energy" is $H_E(-T/2) = 0$, and since this is a constant of motion, we have, for all imaginary time,

$$H_E = 0, \qquad \Longrightarrow \qquad \left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 = \frac{2V(q)}{m}. \tag{1.2.4}$$

We can integrate this to find the solution for the classical trajectory q_c, in the limit $T \to \infty$. Defining τ_0 to be the value of imaginary time when the particle passes through the origin, $q_c(\tau_0) = 0$, we obtain

$$\begin{aligned} \int_0^{q_c(\tau)} \frac{\mathrm{d}q}{q^2 - q_0^2} &= \pm\sqrt{\frac{2\lambda}{m}}\int_{\tau_0}^{\tau} \mathrm{d}\tau', \\ \Longrightarrow \ -\frac{1}{q_0}\,\mathrm{arctanh}\left(\frac{q_c(\tau)}{q_0}\right) &= \pm\sqrt{\frac{2\lambda}{m}}(\tau - \tau_0), \\ \Longrightarrow \ q_c(\tau) = q_0 \tanh&\left[\sqrt{\frac{2\lambda q_0^2}{m}}(\tau - \tau_0)\right], \end{aligned} \tag{1.2.5}$$

where in the last step, we chose the sign so that $q_c(\tau) \to q_0$ as $\tau \to \infty$ and $q_c(\tau) \to -q_0$ as $\tau \to -\infty$, as required.

The solution (1.2.5) is manifestly not unique, since the location τ_0 is arbitrary. Note that there is no contradiction with Picard's theorem on the uniqueness of solutions to ordinary differential equations: Eq. (1.2.4) is a first order nonlinear ODE, and hence, given some initial condition $q(\tau_1) = q_1$, one can only guarantee uniqueness of the corresponding solution on the interval $\tau \in [\tau_1 - \varepsilon, \tau_1 + \varepsilon]$ for some $\varepsilon > 0$. This is precisely the case once we have fixed the crossover point τ_0. Seen another way, it is not sufficient to only specify the behavior at $\tau \to \pm\infty$ in order to have a well-posed initial value problem.

The imaginary time trajectory describes a tunneling event between the minima of the double well potential. This is evident from the real time equations of motion since, as we observed above, the region between $\pm q_0$ is classically forbidden. However, since the potential is inverted in imaginary time, the particle is able to "move" between the two minima. Consequently, $\langle q_0, T/2 | -q_0, -T/2\rangle$ is the probability amplitude for the particle to tunnel quantum mechanically from the well with minimum at $-q_0$ to the well at $+q_0$.

3. **[5/35]** Using the identity (1.2.4), we can easily evaluate the classical action:

$$
\begin{aligned}
S[q_c(\tau)] &= \int_{-\infty}^{\infty} \mathrm{d}\tau\, L_E \\
&= \int_{-\infty}^{\infty} \mathrm{d}\tau\, m \left(\frac{\mathrm{d}q_c}{\mathrm{d}\tau}\right)^2 \\
&= \int_{-q_0}^{q_0} \mathrm{d}q_c\, m \frac{\mathrm{d}q_c}{\mathrm{d}\tau} \\
&= \int_{-q_0}^{q_0} \mathrm{d}q\, \sqrt{2mV(q)} \\
&= \frac{4}{3} q_0^3 \sqrt{2m\lambda},
\end{aligned}
\tag{1.3.1}
$$

where we have chosen the same sign for the square root as above, which ensures the imaginary time action is strictly non-negative, as it must be.

4. **[10/35]** We now expand the path integral around the classical trajectory by defining

$$
q(\tau) = q_c(\tau) + \xi(\tau), \tag{1.4.1}
$$

where $\xi(-\infty) = \xi(\infty) = 0$, since the boundary conditions are carried by the classical solution. We can start by expanding the kinetic term:

$$
\begin{aligned}
\int \mathrm{d}\tau\, \frac{m}{2} \left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 &= \int \mathrm{d}\tau\, \frac{m}{2} \left[\left(\frac{\mathrm{d}q_c}{\mathrm{d}\tau}\right)^2 + 2\frac{\mathrm{d}q_c}{\mathrm{d}\tau}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + \left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2\right] \\
&= \int \mathrm{d}\tau\, \frac{m}{2} \left[\left(\frac{\mathrm{d}q_c}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2\right] \\
&\qquad + \int \mathrm{d}\tau\, \frac{\mathrm{d}}{\mathrm{d}\tau}\left(m\xi \frac{\mathrm{d}q_c}{\mathrm{d}\tau}\right) - \int \mathrm{d}\tau\, m\xi \frac{\mathrm{d}^2 q_c}{\mathrm{d}\tau^2} \\
&= \int \mathrm{d}\tau\, \frac{m}{2} \left[\left(\frac{\mathrm{d}q_c}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2\right] - \int \mathrm{d}\tau\, m\xi \frac{\mathrm{d}^2 q_c}{\mathrm{d}\tau^2},
\end{aligned}
\tag{1.4.2}
$$

where the boundary term evaluates to zero because $\xi(-\infty) = \xi(\infty) = 0$. Then, we expand the potential term:

$$
\begin{aligned}
\int \mathrm{d}\tau\, V(q) = \int \mathrm{d}\tau\, V(q_c) &+ \int \mathrm{d}\tau\, \left.\frac{\partial V(q)}{\partial q}\right|_{q_c} \xi(\tau) \\
&+ \frac{1}{2} \int \mathrm{d}\tau \int \mathrm{d}\tau'\, \left.\frac{\partial^2 V(q)}{\partial q(\tau)\partial q(\tau')}\right|_{q_c} \xi(\tau)\xi(\tau') + \ldots
\end{aligned}
\tag{1.4.3}
$$

When added to the kinetic term above, the second term in this expansion gives the contribution

$$
-\int \mathrm{d}\tau\, \xi(\tau) \left[m\frac{\mathrm{d}^2 q_c}{\mathrm{d}\tau^2} - \left.\frac{\partial V(q)}{\partial q}\right|_{q_c}\right] = 0, \tag{1.4.4}
$$

which vanishes since the term in the brackets is simply the Euler-Lagrange equation for the classical trajectory. Then, since the potential $V(q)$ is a simple polynomial in q, the third term in the expansion of the potential yields

$$\int \mathrm{d}\tau \int \mathrm{d}\tau' \left.\frac{\partial^2 V(q)}{\partial q(\tau)\partial q(\tau')}\right|_{q_c} \xi(\tau)\xi(\tau') = \int \mathrm{d}\tau \int \mathrm{d}\tau' \left.\frac{\partial^2 V(q)}{\partial q(\tau)^2}\right|_{q_c} \delta(\tau-\tau')\xi(\tau)\xi(\tau')$$
$$= \int \mathrm{d}\tau\, V''(q_c)\xi^2. \tag{1.4.5}$$

Therefore, putting everything together, the semi-classical expansion of the action is

$$S[q(\tau)] \simeq \int \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\mathrm{d}q_c}{\mathrm{d}\tau}\right)^2 + V(q_c)\right] + \int \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}V''(q_c)\xi^2\right]$$
$$= \frac{4}{3}q_0^3\sqrt{2m\lambda} + \int \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 + 2\lambda(3q_c^2(\tau) - q_0^2)\xi^2\right]. \tag{1.4.6}$$

Thus, the leading order tunneling amplitude is

$$\lim_{T\to\infty} \langle q_0, T/2|{-q_0}, -T/2\rangle = \int\limits_{\substack{q(-\infty)=-q_0\\ q(\infty)=q_0}} \mathcal{D}q(\tau)\, e^{-S[q(\tau)]/\hbar}$$
$$\simeq e^{-\frac{4q_0^3\sqrt{2m\lambda}}{3\hbar}} \int\limits_{\substack{\tilde\xi(-\infty)=0\\ \tilde\xi(\infty)=0}} \mathcal{D}\tilde\xi(\tau) \exp\left(-\int_{-\infty}^{\infty} \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\mathrm{d}\tilde\xi}{\mathrm{d}\tau}\right)^2 + 2\lambda(3q_c^2(\tau) - q_0^2)\tilde\xi^2\right]\right), \tag{1.4.7}$$

where we have re-scaled $\tilde\xi = \xi/\sqrt{\hbar}$. Integrating the kinetic term in the action by parts gives

$$-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \tilde\xi(\tau) \left[-m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + 4\lambda(3q_c^2(\tau) - q_0^2)\right]\tilde\xi(\tau) \doteq -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \tilde\xi(\tau)\, \hat{A}_E\, \tilde\xi(\tau), \tag{1.4.8}$$

which lets us identify that the functional integral over $\tilde\xi(\tau)$ yields

$$\lim_{T\to\infty} \langle q_0, T/2|{-q_0}, -T/2\rangle = e^{-\frac{4q_0^3\sqrt{2m\lambda}}{3\hbar}} \left(\mathrm{Det}\,\hat{A}_E\right)^{-1/2}, \tag{1.4.9}$$

where the fluctuation determinant is

$$\mathrm{Det}\,\hat{A}_E = \mathrm{Det}\left[-m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + 4\lambda(3q_c^2(\tau) - q_0^2)\right]$$
$$= \mathrm{Det}\left[-m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + 4\lambda q_0^2\left(3\tanh^2\left[\sqrt{\frac{2\lambda q_0^2}{m}}(\tau-\tau_0)\right] - 1\right)\right], \tag{1.4.10}$$

over the space of functions obeying the boundary conditions $\tilde\xi(-\infty) = \tilde\xi(\infty) = 0$. It turns out that the eigenfunctions of this operator can be solved for exactly (see this Wikipedia page).

Question 2 (Charged Particle in a Magnetic Field)

Consider a particle with mass m and charge $-e$ confined to a plane, such that it has position $\boldsymbol{r} = (x_1, x_2)$ and momentum $\boldsymbol{p} = (p_1, p_2)$. An external magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ is applied, so that the Hamiltonian is

$$H(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2m}\Big(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r})\Big)^2, \tag{2.0.1a}$$

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{B}{2}\left(x_2\hat{\boldsymbol{x}} - x_1\hat{\boldsymbol{y}}\right). \tag{2.0.1b}$$

1. **[15/30]** Consider the return amplitude

$$\langle \boldsymbol{r}_0, t_f | \boldsymbol{r}_0, t_i\rangle = \langle \boldsymbol{r}_0 | \, e^{-iH(t_f - t_i)/\hbar} \, | \boldsymbol{r}_0\rangle. \tag{2.1.1}$$

In constructing the (one-dimensional) path integral, one considers intermediate matrix elements of the form

$$\langle q_j | \, H \, | q_{j-1}\rangle = \int \frac{\mathrm{d}p_j}{2\pi\hbar} \langle q_j | \, H \, | p_j\rangle \langle p_j | q_{j-1}\rangle, \tag{2.1.2}$$

and $\langle q_j | \, H \, | p_j\rangle$ is simple to evaluate for a Hamiltonian of the form $H = p^2/2m + V(q)$, where p and q appear separately. The Hamiltonian for our present problem does not have this form. However, by expanding the Hamiltonian, we find

$$\begin{aligned} H &= \frac{1}{2m}\left(p_1 - \frac{eB}{2c}x_2\right)^2 + \frac{1}{2m}\left(p_2 + \frac{eB}{2c}x_1\right)^2 \\ &= \frac{1}{2m}(p_1^2 + p_2^2) + \frac{eB}{2mc}(x_1 p_2 - x_2 p_1) + \frac{1}{2m}\left(\frac{eB}{2c}\right)^2 (x_1^2 + x_2^2), \end{aligned} \tag{2.1.3}$$

since $[x_1, p_2] = [x_2, p_1] = 0$. Therefore, we can make all position coordinates appear to the left of the momenta, and hence, the usual path integral derivation is still valid. From this observation, we can immediately write down

$$\langle \boldsymbol{r}_0, t_f | \boldsymbol{r}_0, t_i\rangle = \int\limits_{\substack{\boldsymbol{r}(t_i) = \boldsymbol{r}_0 \\ \boldsymbol{r}(t_f) = \boldsymbol{r}_0}} \mathcal{D}\boldsymbol{r}(t)\mathcal{D}\boldsymbol{p}(t) \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t \Big(\boldsymbol{p}\cdot\dot{\boldsymbol{r}} - H(\boldsymbol{r}, \boldsymbol{p})\Big)\right]. \tag{2.1.4}$$

Since the exponent is quadratic in the momenta, we can perform the functional integral over $\boldsymbol{p}$. We can re-write the integrand in the exponent as

$$\begin{aligned} \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - H(\boldsymbol{r}, \boldsymbol{p}) &= \left(\dot{x}_1 + \frac{eB}{2mc}x_2\right)p_1 - \frac{p_1^2}{2m} \\ &\quad + \left(\dot{x}_2 - \frac{eB}{2mc}x_1\right)p_2 - \frac{p_2^2}{2m} \\ &\quad - \frac{1}{2m}\left(\frac{eB}{2c}\right)^2 (x_1^2 + x_2^2), \end{aligned} \tag{2.1.5}$$

from which we can infer the following useful change of coordinates:

$$P_1 = \frac{1}{\sqrt{2m}}\left[p_1 - m\left(\dot{x}_1 + \frac{eB}{2mc}x_2\right)\right], \tag{2.1.6a}$$

$$P_2 = \frac{1}{\sqrt{2m}}\left[p_2 - m\left(\dot{x}_2 - \frac{eB}{2mc}x_1\right)\right], \tag{2.1.6b}$$

such that the integrand becomes

$$-P_1^2-P_2^2+\frac{1}{2}m\left(\dot{x}_1+\frac{eB}{2mc}x_2\right)^2+\frac{1}{2}m\left(\dot{x}_2-\frac{eB}{2mc}x_1\right)^2-\frac{1}{2m}\left(\frac{eB}{2c}\right)^2(x_1^2+x_2^2), \quad (2.1.7)$$

in which case we have an elementary Gaussian integral over the two now-factorized shifted momenta P_1 and P_2. This leaves only the terms which are independent of P_1 and P_2, which, when expanded out, give

$$\frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2)-\frac{eB}{2c}(x_1\dot{x}_2-x_2\dot{x}_1). \quad (2.1.8)$$

Therefore, the path integral for the return amplitude is

$$\langle \boldsymbol{r}_0, t_f|\boldsymbol{r}_0, t_i\rangle = \int\limits_{\substack{\boldsymbol{r}(t_i)=\boldsymbol{r}_0\\ \boldsymbol{r}(t_f)=\boldsymbol{r}_0}} \mathcal{D}\boldsymbol{r}(t)\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}\mathrm{d}t\left(\frac{1}{2}m\dot{\boldsymbol{r}}^2-\frac{e}{c}\dot{\boldsymbol{r}}\cdot\boldsymbol{A}\right)\right]. \quad (2.1.9)$$

2. **[10/30]** In the limit $m\to 0$, the action [the integrand of (2.1.9)] becomes

$$S=-\frac{e}{c}\int_{t_i}^{t_f}\mathrm{d}t\,\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\boldsymbol{A}, \quad (2.2.1)$$

where $\boldsymbol{r}(t)$ is any trajectory which satisfies $\boldsymbol{r}(t_i)=\boldsymbol{r}(t_f)=\boldsymbol{r}_0$. This integral can be simplified using Stokes' theorem. Denoting the region with boundary defined by the trajectory $\boldsymbol{r}(t)$ by Ω,

$$\begin{aligned} S &= -\frac{e}{c}\oint_{\partial\Omega}\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{A} \\ &= -\frac{e}{c}\int_{\Omega}\mathrm{d}\boldsymbol{S}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) \\ &= \mp\frac{eB}{c}\mathrm{Area}(\Omega), \\ &= \mp 2\pi\hbar\frac{\Phi}{\Phi_0} \end{aligned} \quad (2.2.2)$$

where the $\mp$ sign corresponds to the orientation of the enclosed area—the direction of the path induces an orientation for the unit normal $\mathrm{d}\boldsymbol{S}$ in either the $\hat{\boldsymbol{z}}$ (counter-clockwise when viewed from above) or $-\hat{\boldsymbol{z}}$ (clockwise from above) direction—$\Phi = B\,\mathrm{Area}(\Omega)$ is the flux enclosed by the trajectory, and $\Phi_0 = hc/e$ is the flux quantum; the superconducting flux quantum differs by a factor of 2 since Cooper pairs have charge $2e$. Therefore, the action measures the (signed) area enclosed by the trajectory of the particle.

3. **[5/30]** Consider a trajectory which is negatively oriented with respect to the $\hat{\boldsymbol{z}}$ unit normal. On the one hand, as noted in the previous part, the amplitude associated with this trajectory must be

$$e^{iS_{\mathrm{in}}/\hbar}=\exp\left(2\pi i\frac{\Phi}{\Phi_0}\right)=\exp\left(2\pi i\frac{B\,\mathrm{Area}(\Omega)}{\Phi_0}\right), \quad (2.3.1)$$

where Area(Ω) is area enclosed on the *inside* of the trajectory.

On the other hand, if we suppose that the particle is actually on the surface of a sphere with large radius R—the radius must be large for the Hamiltonian (2.0.1), which is written in terms of Cartesian coordinates, to be a good approximation—then the trajectory also forms the boundary of another closed region which is positively oriented with respect to $\hat{\boldsymbol{z}}$, with area $[4\pi R^2 - \text{Area}(\Omega)]$; this is the region left "outside" of the trajectory. Therefore, we could equally associate to this same trajectory the amplitude

$$e^{iS_{\text{out}}/\hbar} = \exp\left(-2\pi i \frac{B[4\pi R^2 - \text{Area}(\Omega)]}{\Phi_0}\right). \tag{2.3.2}$$

To prevent any ambiguities, we should require that these two amplitudes be equal. That is, the phase associated with the "inside" and "outside" regions must be equal, modulo 2π,

$$(S_{\text{in}} - S_{\text{out}})/\hbar = 2\pi n, \qquad \text{for } n \in \mathbb{Z}, \tag{2.3.3}$$

which implies that the field strength must be quantized according to

$$B = n\frac{\Phi_0}{4\pi R^2}, \qquad \text{for } n \in \mathbb{Z}. \tag{2.3.4}$$

In other words, the flux through the surface of the sphere is quantized in units of Φ_0. This is the same phenomenon as the quantization of magnetic flux through superconductors with a non-trivial topology (e.g., SQUIDs).

Question 3 (Path Integrals for a Scalar Field Theory)

Consider the action for a free complex scalar field in $D = 4$ spacetime dimensions

$$S = \int \mathrm{d}^4x \Big[|\partial_\mu \phi(x)|^2 - m^2 |\phi(x)|^2 - J(x)^* \phi(x) - J(x)\phi(x)^* \Big]. \tag{3.0.1}$$

Note: The sign of the source coupling terms is the opposite to the conventional definition in Minkowski space. In this question, we work in units of $\hbar = 1$.

1. [4/35] Writing the field and sources as

$$\phi(x) = \frac{1}{\sqrt{2}}\Big[\phi_1(x) + i\phi_2(x)\Big], \qquad \phi(x)^* = \frac{1}{\sqrt{2}}\Big[\phi_1(x) - i\phi_2(x)\Big], \tag{3.1.1a}$$

$$J(x) = \frac{1}{\sqrt{2}}\Big[J_1(x) + iJ_2(x)\Big], \qquad J(x)^* = \frac{1}{\sqrt{2}}\Big[J_1(x) - iJ_2(x)\Big], \tag{3.1.1b}$$

where $\phi_{1,2}$ and $J_{1,2}$ are real, the action decouples into the sum of the actions of two independent real scalar fields:

$$S[\phi, \phi^*, J, J^*] = S[\phi_1, J_1] + S[\phi_2, J_2], \tag{3.1.2a}$$

$$S[\phi_\alpha, J_\alpha] = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu \phi_\alpha(x))^2 - \frac{1}{2} m^2 \phi_\alpha(x)^2 - J_\alpha(x)\phi_\alpha(x) \right]. \tag{3.1.2b}$$

Therefore, the vacuum persistence function in Minkowski space will take the simple form

$$\begin{aligned} {}_J\langle 0|0\rangle_J \doteq \mathcal{Z}_M[J, J^*] &= \left(\int \mathcal{D}\phi_1(x)\, e^{iS[\phi_1, J_1]} \right) \left(\int \mathcal{D}\phi_2(x)\, e^{iS[\phi_2, J_2]} \right) \\ &= \int \mathcal{D}\phi_1(x)\mathcal{D}\phi_2(x)\, e^{i(S[\phi_1, J_1] + S[\phi_2, J_2])} \\ &= \int \mathcal{D}\phi(x)\mathcal{D}\phi(x)^*\, e^{iS[\phi, \phi^*, J, J^*]}, \end{aligned} \tag{3.1.3}$$

where $S[\phi, \phi^*, J, J^*]$ is the action given at the beginning of this question. In the last step, we have changed the functional integral to be over ϕ and ϕ^* instead of $\phi_1 = \mathrm{Re}(\phi)$ and $\phi_2 = \mathrm{Im}(\phi)$, but this is a trivial change of basis with Jacobian equal to 1.

The vacuum persistence function in Euclidean space follows by analytic continuation of the action from real to imaginary time $t = -i\tau$:

$$\begin{aligned} \mathcal{Z}_E[J, J^*] = \int \mathcal{D}\phi(x)\mathcal{D}\phi(x)^* \exp\bigg(- \int \mathrm{d}^4x \Big[&|\partial_\tau \phi(x)|^2 + |\boldsymbol{\nabla}\phi(x)|^2 \\ &+ m^2|\phi(x)|^2 + J(x)^*\phi(x) + J(x)\phi^*(x) \Big] \bigg). \end{aligned} \tag{3.1.4}$$

2. [10/35] To evaluate these path integrals, define

$$\phi(x) = \bar{\phi}(x) + \xi(x), \qquad \phi(x)^* = \bar{\phi}(x)^* + \xi^*(x). \tag{3.2.1}$$

Then the Lagrangian (in Minkowski spacetime) becomes

$$\begin{aligned} \mathcal{L} = |\partial_\mu \bar{\phi}|^2 + \partial_\mu \bar{\phi} \partial^\mu \xi^* + \partial_\mu \bar{\phi}^* \partial^\mu \xi + |\partial_\mu \xi|^2 - m^2\big(|\bar{\phi}|^2 + \bar{\phi}\xi^* + \bar{\phi}^*\xi + |\xi|^2\big) \\ - J^*\big(\bar{\phi} + \xi\big) - J\big(\bar{\phi}^* + \xi^*\big). \end{aligned} \tag{3.2.2}$$

We can integrate this by parts to obtain

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\bar{\phi}^*\big(\partial^2+m^2\big)\bar{\phi} - \frac{1}{2}\bar{\phi}\big(\partial^2+m^2\big)\bar{\phi}^* - \xi\big(\partial^2+m^2\big)\bar{\phi}^* - \xi^*\big(\partial^2+m^2\big)\bar{\phi} \\ &- \frac{1}{2}\xi^*\big(\partial^2+m^2\big)\xi - \frac{1}{2}\xi\big(\partial^2+m^2\big)\xi^* - J^*\big(\bar{\phi}+\xi\big) - J\big(\bar{\phi}^*+\xi^*\big),\end{aligned} \tag{3.2.3}$$

where $\partial^2 = \partial_\mu\partial^\mu$, and we split up the $|\partial_\mu\bar{\phi}|^2$ and $|\partial_\mu\xi|^2$ terms to make the final expression more symmetric between the fields and their complex conjugates. Now, we impose that $\bar{\phi}$ and $\bar{\phi}^*$ obey the classical equations of motion generated by the Lagrangian:

$$\big(\partial^2+m^2\big)\bar{\phi} = -J, \qquad \big(\partial^2+m^2\big)\bar{\phi}^* = -J^*, \tag{3.2.4}$$

so that the Lagrangian greatly simplifies to

$$\mathcal{L} = -\frac{1}{2}\xi^*\big(\partial^2+m^2\big)\xi - \frac{1}{2}\xi\big(\partial^2+m^2\big)\xi^* - \frac{1}{2}J^*\bar{\phi} - \frac{1}{2}J\bar{\phi}^*. \tag{3.2.5}$$

To go further, first note that, defining $\hat{A} = \partial^2 + m^2$, and $\xi = \xi_1 + i\xi_2$,

$$\frac{1}{2}\big(\xi^*\hat{A}\xi + \xi\hat{A}\xi^*\big) = \frac{1}{2}\big(\xi_1\hat{A}\xi_1 + \xi_2\hat{A}\xi_2\big). \tag{3.2.6}$$

Then, observe that the solutions to the equations of motion above are

$$\begin{Bmatrix}\bar{\phi}(x) \\ \bar{\phi}(x)^*\end{Bmatrix} = -\int \mathrm{d}^4y\, G_0^M(x-y)\begin{Bmatrix}J(y) \\ J(y)^*\end{Bmatrix}, \tag{3.2.7}$$

where

$$G_0^M(x-y) = \langle x|\,\frac{1}{\partial^2+m^2}\,|y\rangle\,. \tag{3.2.8}$$

Therefore, we can write the path integral as

$$\begin{aligned}\mathcal{Z}_M[J,J^*] &= \left[\int \mathcal{D}\xi\,\exp\left(-\frac{i}{2}\int \mathrm{d}^4x\,\xi(x)\big[\partial^2+m^2\big]\xi(x)\right)\right]^2 \\ &\qquad\qquad \times \exp\left(i\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)^*G_0^M(x-y)J(y)\right) \\ &= \big[\mathrm{Det}\big(\partial^2+m^2\big)\big]^{-1}\exp\left(i\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)^*G_0^M(x-y)J(y)\right),\end{aligned} \tag{3.2.9}$$

up to a formally divergent normalization constant. The corresponding expression in Euclidean spacetime is then obtained via analytic continuation:

$$\begin{aligned}\mathcal{Z}_E[J,J^*] &= \left[\int \mathcal{D}\xi\,\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\xi(x)\big[-\partial^2+m^2\big]\xi(x)\right)\right]^2 \\ &\qquad\qquad \times \exp\left(\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)^*G_0^E(x-y)J(y)\right) \\ &= \big[\mathrm{Det}\big(-\partial^2+m^2\big)\big]^{-1}\exp\left(\int \mathrm{d}^4x\mathrm{d}^4y\,J(x)^*G_0^E(x-y)J(y)\right),\end{aligned} \tag{3.2.10}$$

where $\partial^2 = \partial_\tau^2 + \nabla^2$ and the Euclidean Green's function is

$$G_0^E(x-y) = \langle x|\,\frac{1}{-\partial^2+m^2}\,|y\rangle\,. \tag{3.2.11}$$

3. **[5/35]** In Minkowski spacetime, the two-point functions are

$$\begin{aligned}
G_2(x-x') &= \langle 0|\, T\phi(x)\phi(x')^*\, |0\rangle \\
&= \frac{1}{(i)^2}\frac{1}{\mathcal{Z}[0]}\left.\frac{\delta^2\mathcal{Z}[J,J^*]}{\delta J(x)^*\delta J(x')}\right|_{J=0} \\
&= \frac{1}{i}\frac{\delta}{\delta J(x)^*}\left(\int \mathrm{d}^4y J(y)^* G_0^M(y-x')\right) \\
&\qquad\qquad \times \exp\left.\left(i\int \mathrm{d}^4y\mathrm{d}^4y'\; J(y)^* G_0^M(y-y')J(y')\right)\right|_{J=0} \\
&= -iG_0^M(x-x'), && (3.3.1\text{a}) \\
G_2^*(x-x') &= \langle 0|\, T\phi(x)^*\phi(x')\, |0\rangle \\
&= \frac{1}{(i)^2}\frac{1}{\mathcal{Z}[0]}\left.\frac{\delta^2\mathcal{Z}[J,J^*]}{\delta J(x)\delta J(x')^*}\right|_{J=0} \\
&= \frac{1}{i}\frac{\delta}{\delta J(x)}\left(\int \mathrm{d}^4y' G_0^M(x'-y')J(y')\right) \\
&\qquad\qquad \times \exp\left.\left(i\int \mathrm{d}^4y\mathrm{d}^4y'\; J(y)^* G_0^M(y-y')J(y')\right)\right|_{J=0} \\
&= -iG_0^M(x'-x), && (3.3.1\text{b}) \\
G_2'(x-x') &= \langle 0|\, T\phi(x)\phi(x')\, |0\rangle \\
&= \frac{1}{(i)^2}\frac{1}{\mathcal{Z}[0]}\left.\frac{\delta^2\mathcal{Z}[J,J^*]}{\delta J(x)^*\delta J(x')^*}\right|_{J=0} \\
&= \frac{1}{i}\frac{\delta}{\delta J(x)^*}\left(\int \mathrm{d}^4y' G_0^M(x'-y')J(y')\right) \\
&\qquad\qquad \times \exp\left.\left(i\int \mathrm{d}^4y\mathrm{d}^4y'\; J(y)^* G_0^M(y-y')J(y')\right)\right|_{J=0} \\
&= \left(\int \mathrm{d}^4y' G_0^M(x'-y')J(y')\right)\left(\int \mathrm{d}^4y' G_0^M(x-y')J(y')\right) \\
&\qquad\qquad \times \exp\left.\left(i\int \mathrm{d}^4y\mathrm{d}^4y'\; J(y)^* G_0^M(y-y')J(y')\right)\right|_{J=0} \\
&= 0, && (3.3.1\text{c}) \\
G_2'^*(x-x') &= \langle 0|\, T\phi(x)^*\phi(x')^*\, |0\rangle \\
&= \frac{1}{(i)^2}\frac{1}{\mathcal{Z}[0]}\left.\frac{\delta^2\mathcal{Z}[J,J^*]}{\delta J(x)\delta J(x')}\right|_{J=0} \\
&= 0, && (3.3.1\text{d})
\end{aligned}$$

with the Green's function $G_0^M(x-x')$ defined in the previous part.

The Euclidean two-point functions follow by the same logic, without the factors of i:

$$G_2^E(x-x') = G_0^E(x-x'), \tag{3.3.2a}$$

$$G_2^{E\,*}(x-x') = G_0^E(x'-x), \tag{3.3.2b}$$

$$G_2'^{E}(x-x') = G_2'^{E\,*}(x-x') = 0, \tag{3.3.2c}$$

with $G_0^E(x-x')$ defined in the previous part.

4. [12/35] The equations satisfied by the Green's functions follow from the matrix elements of the inverse differential operators shown above:

$$G_0^M(x-y) = \langle x| \frac{1}{\partial^2 + m^2} |y\rangle\,,$$
$$\implies \quad \big(\partial_x^2 + m^2\big)G_0^M(x-y) = \delta^{(4)}(x-y), \tag{3.4.1a}$$
$$G_0^E(x-y) = \langle x| \frac{1}{-\partial^2 + m^2} |y\rangle\,,$$
$$\implies \quad \big(-\partial_x^2 + m^2\big)G_0^E(x-y) = \delta^{(4)}(x-y), \tag{3.4.1b}$$

where in the Minkowski case $\partial^2 = \partial_0^2 - \nabla^2$, and in the Euclidean case $\partial^2 = \partial_\tau^2 + \nabla^2$.

To solve for the Euclidean Green's function, we expand both sides of the differential equation using a Fourier transform:

$$\begin{aligned}
\big(-\partial_x^2 + m^2\big)G_0^E(x-y) &= \delta^{(4)}(x-y),\\
\implies \big(-\partial_x^2 + m^2\big)\int \frac{\mathrm{d}^4p}{(2\pi)^4} G_0^E(p) e^{ip\cdot(x-y)} &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)},\\
\implies \int \frac{\mathrm{d}^4p}{(2\pi)^4}\big(p^2 + m^2\big) G_0^E(p) e^{ip\cdot(x-y)} &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)},\\
\implies G_0^E(p) &= \frac{1}{p^2+m^2},\\
\implies G_0^E(x-y) &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2+m^2}.
\end{aligned} \tag{3.4.2}$$

We can then re-write this integral using a Feynman-Schwinger parameter to bring the denominator into the exponential:

$$\begin{aligned}
G_0^E(x-y) &= \frac{1}{2}\int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} \exp\left[-\frac{\alpha}{2}(p^2+m^2) + ip\cdot(x-y)\right]\\
&= \frac{1}{2}\int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^4p}{(2\pi)^4} \exp\left[-\frac{1}{2}\left(\sqrt{\alpha}p - i\frac{x-y}{\sqrt{\alpha}}\right)^2 - \frac{|x-y|^2}{2\alpha} - \frac{1}{2}m^2\alpha\right]\\
&= \frac{1}{2}\int_0^\infty \mathrm{d}\alpha \frac{1}{(2\pi\alpha)^2} \exp\left[-\frac{|x-y|^2}{2\alpha} - \frac{1}{2}m^2\alpha\right].
\end{aligned} \tag{3.4.3}$$

Then, changing variables by letting

$$\alpha = \frac{|x-y|}{m}\frac{1}{t}, \tag{3.4.4}$$

the integral can then be evaluated

$$\begin{aligned}
G_0^E(x-y) &= \frac{1}{(2\pi)^2}\left(\frac{m}{|x-y|}\right)\frac{1}{2}\int_0^\infty \mathrm{d}t\, t \exp\left[-\frac{m|x-y|}{2}\left(t+\frac{1}{t}\right)\right]\\
&= \frac{1}{4\pi^2}\left(\frac{m}{|x-y|}\right) K_1\big(m|x-y|\big),
\end{aligned} \tag{3.4.5}$$

where

$$K_\nu(z) = \frac{1}{2}\int_0^\infty \mathrm{d}t\, t^{\nu-1} \exp\left[-\frac{z}{2}\left(t+\frac{1}{t}\right)\right], \tag{3.4.6}$$

is the modified Bessel function of the second kind.

At large separations $m|x-y| \gg 1$, we can use the asymptotic expansion

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z}, \tag{3.4.7}$$

which yields

$$G_0^E(x-y) \simeq \frac{\sqrt{\pi/2}}{4\pi^2} \frac{m^2}{(m|x-y|)^{3/2}} e^{-m|x-y|}, \tag{3.4.8}$$

which is the usual exponential decay at large (Euclidean) distances of a theory with a mass gap, with correlation length $\xi = 1/m$. At short distances $m|x-y| \ll 1$, we can use

$$K_\nu(z) \simeq \frac{\Gamma(\nu)}{2} \left(\frac{2}{z}\right)^\nu, \tag{3.4.9}$$

which yields

$$G_0^E(x-y) \simeq \frac{1}{4\pi^2 |x-y|^2}. \tag{3.4.10}$$

On these very short scales, the theory is insensitive to the mass gap, which corresponds to a much longer length scale than $|x-y|$. Therefore, the correlations have the same power law decay as a massless theory.

We can then analytically continue back to Minkowski spacetime to find the Green's function $G_0^M(x-y)$. We write

$$|x-y| = \sqrt{|x-y|^2} = \sqrt{-s^2}, \tag{3.4.11}$$

where $s^2 = (x_0 - y_0)^2 - |\boldsymbol{x} - \boldsymbol{y}|^2$ is the relativistic interval. From the previous part, we also note that the Green's function in Minkowski space also picks up an additional factor of i relative to the time ordered vacuum expectation value. Therefore, we find that

$$G_0^M(x-y) = \frac{i}{4\pi^2} \frac{m}{\sqrt{-s^2}} K_1(m\sqrt{-s^2}). \tag{3.4.12}$$

For space-like separations, $s^2 < 0$, so the Minkowski propagator has the exact same asymptotic behavior as the Euclidean propagator (up to a factor of i):

$$G_0^M(x-y) \simeq i \frac{\sqrt{\pi/2}}{4\pi^2} \frac{m^2}{(m\sqrt{-s^2})^{3/2}} e^{-m\sqrt{-s^2}}, \qquad \text{for } m\sqrt{-s^2} \gg 1, \tag{3.4.13a}$$

$$G_0^M(x-y) \simeq \frac{i}{4\pi^2(-s^2)}, \qquad \text{for } m\sqrt{-s^2} \ll 1. \tag{3.4.13b}$$

Therefore, the two-point functions share the same asymptotic form in both Euclidean and Minkowski space in this regime.

For time-like separations, $s^2 > 0$, we can simplify the expression above to give

$$G_0^M(x-y) = \frac{1}{4\pi^2} \frac{m}{\sqrt{s^2}} K_1(im\sqrt{s^2}) = -\frac{1}{8\pi} \frac{m}{\sqrt{s^2}} H_1^{(2)}(m\sqrt{s^2}), \tag{3.4.14}$$

where $H_1^{(2)}(z) = J_1(z) - iY_1(z)$ is a Hankel function. For large separations $m\sqrt{s^2} \gg 1$, we can use the asymptotic expansion (see, for example, https://dlmf.nist.gov/10.7)

$$H_\nu^{(2)}(z) \simeq \sqrt{\frac{2}{\pi z}} \exp\left[-i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right], \tag{3.4.15}$$

so that,

$$G_0^M(x-y) \simeq \frac{e^{-i\pi/4}\sqrt{\pi/2}}{4\pi^2} \frac{m^2}{(m\sqrt{s^2})^{3/2}} e^{-im\sqrt{s^2}}. \tag{3.4.16}$$

Therefore, we see that the two-point functions in Minkowski space are oscillatory, with a length scale of oscillations once again set by the mass: $\lambda \sim 1/m$. For small separations, we use

$$H_\nu^{(1)}(z) \simeq \frac{\Gamma(\nu)}{i\pi}\left(\frac{2}{z}\right)^\nu, \tag{3.4.17}$$

so that

$$G_0^M(x-y) \simeq \frac{i}{4\pi^2 s^2} \tag{3.4.18}$$

Therefore, the short distance behavior of the two-point function is the same for both space-like and time-like separations.

5. **[4/35]** To find the four-point functions, it suffices to apply Wick's theorem, since we are working with a free field theory. First, observe that the Green's function (in Minkowski or Euclidean spacetime) is even in its argument: $G_0(x-y) = G_0(y-x)$. Therefore, the two non-trivial two-point functions are equal: $G_2(x-x') = G_2^*(x-x')$. And, recalling that the two other two-point functions vanish identically, we find that

$$\begin{aligned} G_4^a(x_1,x_2,x_3,x_4) &= \langle 0|\, T\phi(x_1)^*\phi(x_2)^*\phi(x_3)\phi(x_4)\,|0\rangle \\ &= G_2(x_1-x_3)G_2(x_2-x_4) + G_2(x_1-x_4)G_2(x_2-x_3), \qquad (3.5.1a) \\ G_4^b(x_1,x_2,x_3,x_4) &= \langle 0|\, T\phi(x_1)^*\phi(x_2)\phi(x_3)^*\phi(x_4)\,|0\rangle \\ &= G_2(x_1-x_2)G_2(x_3-x_4) + G_2(x_1-x_4)G_2(x_2-x_3) \qquad (3.5.1b) \\ G_4^c(x_1,x_2,x_3,x_4) &= \langle 0|\, T\phi(x_1)^*\phi(x_2)\phi(x_3)\phi(x_4)^*\,|0\rangle \\ &= G_2(x_1-x_2)G_2(x_3-x_4) + G_2(x_1-x_3)G_2(x_2-x_4) \qquad (3.5.1c) \\ G_4'(x_1,x_2,x_3,x_4) &= G_4'^*(x_1,x_2,x_3,x_4) = 0. \qquad (3.5.1d) \end{aligned}$$

In this form, it is simple to deduce the relations

$$G_4^a(x_1,x_2,x_3,x_4) = G_4^b(x_1,x_3,x_2,x_4) = G_4^c(x_1,x_4,x_3,x_2). \tag{3.5.2}$$

That is, the three non-trivial four-point functions are equivalent up to a permutation of their arguments.