582 Homework 5

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1 Fermions in one dimension

We will be using the effective continuum Hamiltonian:

$$H = \sum_{\sigma=\pm} \int dx\, \psi_\sigma^\dagger(x)\Big(- i v_f \frac{\partial}{\partial x}\Big)\sigma_3 \psi_\sigma(x) + \int dx \left[\frac{\Pi^2(x)}{8Ma_0^2} + \frac{1}{2}\Delta^2(x)\right] + \sum_{\sigma=\pm}\int dx\, \sqrt{2g}\Delta(x)\bar{\psi}_\sigma(x)\psi_\sigma(x) \tag{1.1}$$

1. **10/20:** To simplify matters even further we consider the case of heavy carbon atoms: $M \to \infty$. When we do this, we can drop the kinetic term, which amounts to disregarding the quantum fluctuations. Therefore, we can work with a constant classical field, so we replace $\Delta(x)$ with Δ_0. Our Hamiltonian reduces to:

$$\begin{aligned} H &= \sum_{\sigma=\pm} \int dx\, \psi_\sigma^\dagger(x)\Big(- i v_f \frac{\partial}{\partial x}\Big)\sigma_3 \psi_\sigma(x) + \int dx \frac{1}{2}\Delta_0^2 + \sum_{\sigma=\pm}\int dx\, \sqrt{2g}\Delta_0\bar{\psi}_\sigma(x)\psi_\sigma(x) \\ &= \sum_{\sigma=\pm} \int dx\, \psi_\sigma^\dagger(x)\Big(- i v_f \frac{\partial}{\partial x}\Big)\sigma_3 \psi_\sigma(x) + \int dx \frac{1}{2}\Delta_0^2 + \sum_{\sigma=\pm}\int dx\, \psi_\sigma^\dagger(x)\Big(\sqrt{2g}\Delta_0\Big)\sigma_2\psi_\sigma(x) \end{aligned} \tag{1.2}$$

Let's group some terms together and write this out in two separate ways:

$$\begin{aligned} H &= \int dx \left(\sum_{\sigma=\pm} i\psi_\sigma^\dagger(x) \begin{bmatrix} -v_f \frac{\partial}{\partial x} & -\sqrt{2g}\Delta_0 \\ \sqrt{2g}\Delta_0 & v_f\frac{\partial}{\partial x}\end{bmatrix} \psi_\sigma(x) + \frac{1}{2}\Delta_0^2 \right) \\ &= \sum_{\sigma=\pm}\int dx \bar{\psi}_\sigma \Big(- i\gamma_1\partial_x + \sqrt{2g}\Delta_0\Big)\psi_\sigma + \int dx \frac{1}{2}\Delta_0^2 \\ &= \sum_{\sigma=\pm}\int dx \bar{\psi}_\sigma \Big(- i\gamma_1\partial_x + \sqrt{2g}\Delta_0\Big)\psi_\sigma + \frac{L}{2}\Delta_0^2 \end{aligned} \tag{1.3}$$

Notice that the second term looks like a Dirac Hamiltonian with a mass term: $m = \sqrt{2g}\Delta_0$. We could solve this as was done in lecture, but we will opt for a straight forward Fourier expansion instead:

$$\psi_\sigma(x) = \int \frac{dp}{2\pi}\frac{m}{\omega(p)}\psi_\sigma(p)e^{ipx} \tag{1.4}$$

We note that this Fourier modes are 2-component *spinors*. $\omega(p)$ will turn out to be the absolute value of our eigen energies. These factors are chosen to make the integration measure Lorentz invariant.

In order for the anticommutation relations to be preserved, we need:

$$\Big\{\psi_{\sigma\alpha}(p), \psi^\dagger_{\sigma'\alpha'}(p')\Big\} = 2\pi\frac{\omega(p)}{m}\delta_{\sigma\sigma'}\delta_{\alpha\alpha'}\delta(p-p') \tag{1.5}$$

With all other anti-commutation relations zero.

Throwing in our Fourier transforms, and performing the necessary manipulations, we arrive at the following Hamiltonian:

$$H = \sum_{\sigma=\pm}\int \frac{dp}{2\pi}\frac{m}{\omega(p)}\, i\psi_\sigma^\dagger(p) \begin{bmatrix} pv_f & -i\sqrt{2g}\Delta_0 \\ i\sqrt{2g}\Delta_0 & -pv_f \end{bmatrix}\psi_\sigma(p) + \frac{L}{2}\Delta_0^2 \tag{1.6}$$

<u>We focus on the matrix equation for now</u>:

$$\text{Det}\begin{bmatrix} pv_f - E & -i\sqrt{2g}\Delta_0 \\ i\sqrt{2g}\Delta_0 & -pv_f - E\end{bmatrix} = -\Big[(pv_f+E)(pv_f-E) + (\sqrt{2g}\Delta_0)^2\Big] = -\Big[(pv_f)^2 - E^2 + (\sqrt{2g}\Delta_0)^2\Big] = 0$$

Which means:

$$E(p) = \pm\sqrt{(pv_f)^2 + 2g\Delta_0^2} \equiv \pm\omega(p)$$

This is the relativistic Pythagorean identity! Let's push this analogy a bit further by defining:

$$\begin{aligned}1 &= \sqrt{(pv_f/\omega(p))^2 + (\sqrt{2g}\Delta_o/\omega(p))^2} \\ &\equiv \sqrt{\sin^2(\theta) + \cos^2(\theta)}\end{aligned}$$

There are many ways to proceed here! Do not worry too much if your method is different. Let's express our field as a linear combination of eigenvectors for positive and negative eigenvalues (energies). We'll call them $u(p)$ and $v(p)$, respectively:

$$\psi_\sigma = \int \frac{dp}{2\pi}\frac{m}{\omega(p)}\left(\hat{a}_\sigma(p)u(p)e^{-ipx} + \hat{b}_\sigma(p)v(p)e^{ipx}\right)$$

I choose to set:

$$\begin{aligned}\{\hat{a}_\sigma(p), \hat{a}^\dagger_{\sigma'}(p')\} &= 2\pi\frac{\omega(p)}{m}\delta_{\sigma\sigma'}\delta(p-p') \\ \{\hat{b}_\sigma(p), \hat{b}^\dagger_{\sigma'}(p')\} &= 2\pi\frac{\omega(p)}{m}\delta_{\sigma\sigma'}\delta(p-p')\end{aligned} \tag{1.7}$$

etc. The factor of ω/m in the mode expansion as well as the above anti-commutation relations will set our normalization of our eigenvectors. Indeed:

$$\begin{aligned}\left\{\psi_\sigma(x), \psi^\dagger_{\sigma'}(x')\right\} &= \int \frac{dpdq}{(2\pi)^2}\frac{m^2}{\omega(p)\omega(q)}\left(\left\{\hat{a}_\sigma(p), \hat{a}^\dagger_{\sigma'}(q)\right\}u(p)u^\dagger(q)e^{-i(px-qx')} + \left\{\hat{b}_\sigma(p), \hat{b}^\dagger_{\sigma'}(q)\right\}v(p)v^\dagger(q)e^{i(px-qx')}\right) \\ &= \delta_{\sigma\sigma'}\int\frac{dp}{2\pi}\frac{m}{\omega(p)}\left(u(p)u^\dagger(p)e^{-ip(x-x')} + v(p)v^\dagger(p)e^{ip(x-x')}\right) \\ &= \delta_{\sigma\sigma'}\delta(x-x') \\ &\Rightarrow u(p)u^\dagger(p) = v(p)v^\dagger(p) = \frac{\omega(p)}{2m}\end{aligned}$$

We will remember this for later.

We solve the following matrix equations:

$$\begin{aligned}\omega(p)u(p) &= \begin{bmatrix} pv_f & -i\sqrt{2g}\Delta_0 \\ i\sqrt{2g}\Delta_0 & -pv_f\end{bmatrix}u(p) \\ -\omega(p)v(p) &= \begin{bmatrix} pv_f & -i\sqrt{2g}\Delta_0 \\ i\sqrt{2g}\Delta_0 & -pv_f\end{bmatrix}v(p)\end{aligned}$$

Re-expressing in terms of sin and cos:

$$\begin{aligned}u(p) &= \begin{bmatrix}\sin(\theta) & -i\cos(\theta) \\ i\cos(\theta) & -\sin(\theta)\end{bmatrix}u(p) \\ -v(p) &= \begin{bmatrix}\sin(\theta) & -i\cos(\theta) \\ i\cos(\theta) & -\sin(\theta)\end{bmatrix}v(p)\end{aligned}$$

These two systems only differ by an overall minus sign on the left. We cast the two systems into the following form and solve[1]:

$$\begin{aligned}\pm\begin{bmatrix}a_\pm \\ b_\pm\end{bmatrix} &= \begin{bmatrix}\sin(\theta) & -i\cos(\theta) \\ i\cos(\theta) & -\sin(\theta)\end{bmatrix}\begin{bmatrix}a_\pm \\ b_\pm\end{bmatrix} \\ \Rightarrow \pm a_\pm &= a_\pm\sin(\theta) - ib_\pm\cos(\theta) \\ \pm b_\pm &= ia_\pm\cos(\theta) - b_\pm\sin(\theta) \\ \Rightarrow b_\pm &= a_\pm\frac{i\cos(\theta)}{\sin(\theta)\pm 1} \\ \therefore \quad \begin{bmatrix}a_\pm \\ b_\pm\end{bmatrix} &= c_\pm\begin{bmatrix}\sin(\theta)\pm 1 \\ i\cos(\theta)\end{bmatrix}\end{aligned}$$

[1]do not confuse $a_\pm$, $b_\pm$ with our creation and annihilation operators!

Here $c_\pm$ is a normalization constant, which can be determined by recalling the normalization we want:

$$\begin{aligned}\frac{\omega(p)}{2m} = \frac{1}{2\cos(\theta)} &= |c_\pm|^2\Big[(\sin(\theta)\pm 1)^2 + \cos^2(\theta)\Big] \\ &= 2|c_\pm|^2(1\pm\sin(\theta)) \\ \Rightarrow |c_\pm| &= \frac{1}{2}\frac{1}{\sqrt{\cos(\theta)\big(1\pm\sin(\theta)\big)}} \\ &= \frac{1}{2}\frac{\omega(p)}{\sqrt{m(\omega(p)\pm pv_f)}}\end{aligned}$$

We arrive at our solutions in terms of our original parameters:

$$u(p) = \frac{1}{2\sqrt{m(\omega(p)+pv_f)}}\begin{bmatrix} pv_f + \omega(p) \\ im \end{bmatrix}$$

and

$$v(p) = \frac{1}{2\sqrt{m(\omega(p)-pv_f)}}\begin{bmatrix} pv_f - \omega(p) \\ im \end{bmatrix}$$

Re-expressing our Hamiltonian in terms of the mode expansion we arrive at:

$$H = \sum_{\sigma=\pm}\int \frac{dp}{2\pi}\frac{m}{\omega(p)}\omega(p)\Big[\hat{a}^\dagger_\sigma\hat{a}_\sigma - \hat{b}^\dagger_\sigma\hat{b}_\sigma\Big] + \frac{L}{2}\Delta_0^2 \tag{1.8}$$

Notice I replaced the integral over x with the length of our polymer chain $L = Na_0$. We can now find the ground state with the usual prescription for a fermionic system: we define hole creation and annihilation operators:

$$\hat{d}_\sigma = \hat{b}^\dagger_\sigma, \;\; \hat{d}^\dagger_\sigma = \hat{b}_\sigma$$

So we have:

$$\hat{b}^\dagger_\sigma\hat{b}_\sigma = \hat{d}_\sigma\hat{d}^\dagger_\sigma$$

using

$$\{\hat{d}_\sigma(p), \hat{d}^\dagger_{\sigma'}(p')\} = 2\pi\frac{\omega(p)}{m}\delta_{\sigma\sigma'}\delta(p-p')$$

We normal order our Hamiltonian:

$$H = \sum_{\sigma=\pm}\int \frac{dp}{2\pi}\frac{m}{\omega(p)}\omega(p)\Big[\hat{a}^\dagger_\sigma(p)\hat{a}_\sigma(p) + \hat{d}^\dagger_\sigma(p)\hat{d}_\sigma(p) - 2\pi\frac{\omega(p)}{m}\delta_{\sigma\sigma}\delta(p-p)\Big] + \frac{1}{2}\int dx\Delta_0^2(x) \tag{1.9}$$

We now define our ground state in the usual manner:

$$\hat{a}_\sigma(p)|\text{gnd}\rangle = \hat{d}_\sigma(p)|\text{gnd}\rangle = 0$$

This can be used to determine an expression for our ground state energy; we have:

$$\begin{aligned}E_{\text{gnd}} &= -\sum_{\sigma=\pm}\int_{-\Lambda}^{\Lambda} dp\delta_{\sigma\sigma}\delta(p-p)\sqrt{2g\Delta_0^2 + (pv_f)^2} + \frac{L}{2}\Delta_0^2 \\ &= -\frac{L}{\pi}\int_{-v_f\Lambda}^{v_f\Lambda}\frac{d\bar{p}}{v_f}\sqrt{2g\Delta_0^2 + \bar{p}^{\,2}} + \frac{L}{2}\Delta_0^2\end{aligned} \tag{1.10}$$

The last line is realized by noting the Dirac delta function in momentum space gives us the factor of $L/2\pi$, and we summed over sigma. Λ is our momentum cutoff.

We can finally find the distortion of the lattice in the ground state by minimizing the above equation in respect to the classical coordinate Δ_0:

$$\begin{aligned}\frac{1}{L}\frac{dE_{\text{gnd}}}{d\Delta_0} &= -\frac{1}{\pi}\int_{-v_f\Lambda}^{v_f\Lambda}\frac{d\bar{p}}{v_f}\frac{2g\Delta_0}{\sqrt{2g\Delta_0^2+\bar{p}^{\,2}}}+\Delta_0 = 0\\ \Rightarrow \frac{\pi v_f}{2g} &= \int_{-v_f\Lambda}^{v_f\Lambda}d\bar{p}\frac{1}{\sqrt{2g\Delta_0^2+\bar{p}^{\,2}}}\end{aligned}$$

Of course $\Delta_0 = 0$ satisfies this equation, but let's find a non-trivial solution. The first integral is easy to recognize if we rewrite it as:

$$\begin{aligned}\int_{-v_f\Lambda}^{v_f\Lambda}d\bar{p}\frac{1}{\sqrt{2g\Delta_0^2+\bar{p}^{\,2}}} &= \int_{-v_f\Lambda}^{v_f\Lambda}\frac{d\bar{p}}{\sqrt{2g}\Delta_0}\frac{1}{\sqrt{1-\big(i\frac{\bar{p}}{\sqrt{2g}\Delta_0}\big)^2}}\\ &= \frac{1}{i}\sin^{-1}\Big(i\frac{\bar{p}}{\sqrt{2g}\Delta_0}\Big)\Big|_{-v_f\Lambda}^{v_f\Lambda}\\ &= \sinh^{-1}\Big(\frac{\bar{p}}{\sqrt{2g}\Delta_0}\Big)\Big|_{-v_f\Lambda}^{v_f\Lambda}\\ &= 2\sinh^{-1}\Big(\frac{v_f\Lambda}{\sqrt{2g}\Delta_0}\Big)\end{aligned}$$

Plugging in we can solve for the Δ_0 which minimizes our ground state energy:

$$\begin{aligned}\frac{\pi v_f}{2g} &= 2\sinh^{-1}\Big(\frac{v_f\Lambda}{\sqrt{2g}\Delta_0}\Big)\\ \Rightarrow \Delta_0 &= \left[\frac{\sqrt{2g}}{\Lambda v_f}\sinh\left(\frac{\pi v_f}{4g}\right)\right]^{-1}\\ &\approx \sqrt{\frac{2}{g}}\Lambda v_f e^{-\frac{\pi v_f}{4g}}\end{aligned}$$

Since we will use this later, we make the simple observation that:

$$\frac{\Delta_0}{\Lambda v_f}\sqrt{g} = \sqrt{2}\exp\Big(-\frac{\pi v_f}{4g}\Big) << 1 \tag{1.11}$$

This is the spontaneous mass generation referred to as dimerization in the original SSH model! You may notice that this gap which opens up is strikingly similar to that found with the BCS model. This is not a coincidence: the phonons play a crucial role in both theories!

We finish up by looking at the ground state energies corresponding to these two solutions. First, for $\Delta_0 = 0$:

$$\begin{aligned}\frac{E_0}{L} &= -\frac{1}{\pi}\int_{-v_f\Lambda}^{v_f\Lambda}\frac{d\bar{p}}{v_f}|\bar{p}|\\ &= -\frac{2}{\pi v_f}\int_0^{v_f\Lambda}d\bar{p}\,\bar{p}\\ &= -\frac{v_f\Lambda^2}{\pi}\end{aligned}$$

We can of course do the more general integral and apply it to the above case. There are some steps involved (trig substitution), so I'll just give the answer for that integral:

$$
\begin{aligned}
\frac{E_{\rm gnd}}{L} &= -\frac{2}{\pi v_f}\int_0^{v_f\Lambda} d\bar{p}\sqrt{2g\Delta_0^2+\bar{p}^{\,2}} + \frac{\Delta_0^2}{2}\\
&= -\frac{1}{\pi v_f}\left[\bar{p}\sqrt{2g\Delta_0^2+\bar{p}^2} + 2g\Delta_0^2 \ln\Big(\sqrt{2g\Delta_0^2+\bar{p}^2}+\bar{p}\Big)\right]_0^{v_f\Lambda} + \frac{\Delta_0^2}{2}\\
&= -\frac{1}{\pi v_f}\left[v_f\Lambda\sqrt{2g\Delta_0^2+v_f^2\Lambda^2} + 2g\Delta_0^2\left(\ln\Big(\sqrt{2g\Delta_0^2+v_f^2\Lambda^2}+v_f\Lambda\Big) - \frac{\ln(2g\Delta_0^2)}{2}\right)\right] + \frac{\Delta_0^2}{2}\\
&= -\frac{\Lambda}{\pi}\sqrt{2g\Delta_0^2+v_f^2\Lambda^2} - \frac{2g\Delta_0^2}{\pi v_f}\left[\ln\Big(\sqrt{2g\Delta_0^2+v_f^2\Lambda^2}+v_f\Lambda\Big) - \ln(v_f\Lambda) - \frac{1}{2}\ln\left(\frac{2g\Delta_0^2}{(v_f\Lambda)^2}\right) - \frac{\pi v_f}{4g}\right]
\end{aligned}
$$

The last line is written in a strange looking form, but it will be helpful below. Let's call:

$$
x = \frac{\pi v_f}{4g}
$$

Using equation (1.11) we can cast the relative difference in the following form:

$$
\begin{aligned}
\frac{E_{\rm gnd}-E_0}{|E_0|} &= -\sqrt{4e^{-2x}+1} - 4e^{-2x}\left[\ln\Big(\sqrt{4e^{-2x}+1}+1\Big) - \frac{1}{2}\ln(4e^{-2x}) - x\right] + 1\\
&= -2e^{-2x} + \dots
\end{aligned}
$$

The final $+1$ in the first line is from the ratio of the E_0's. To get to the last line we just perform a series of Taylor expansions, using e^{-2x} as a small parameter, and group the leading order terms. This is clearly negative! Hence, the energy corresponding to $\Delta_0 \neq 0$ is the lower energy state.

2. **4/20:** We build our single particle states out of the ground state defined above in the usual manner:

$$
\hat{a}_\sigma^\dagger(p)|\text{gnd}\rangle,\ \ \hat{d}_\sigma^\dagger(p)|\text{gnd}\rangle
$$

Recall:

$$
\begin{aligned}
\hat{H} &= \sum_{\sigma'=\pm}\int \frac{dp'}{2\pi}\, E(p')\left[\hat{a}_{\sigma'}^\dagger(p')\hat{a}_{\sigma'}(p') + \hat{d}_{\sigma'}^\dagger(p')\hat{d}_{\sigma'}(p')\right] + E_{\rm gnd}\\
&= \ :\hat{H}: + E_{\rm gnd}
\end{aligned}
$$

Since:

$$
\hat{a}_\sigma^\dagger(p')\hat{a}_{\sigma'}(p')\left[\hat{a}_{\sigma'}^\dagger(p)|\text{gnd}\rangle\right] = \hat{a}_{\sigma'}^\dagger(p')\left[\hat{a}_\sigma^\dagger(p)\hat{a}_{\sigma'}(p') + \delta_{\sigma\sigma'}\delta(p-p')\right]|\text{gnd}\rangle
$$

We get:

$$
\begin{aligned}
:\hat{H}:\hat{a}_\sigma^\dagger(p)|\text{gnd}\rangle &= E(p)\hat{a}_\sigma^\dagger(p)|\text{gnd}\rangle\\
:\hat{H}:\hat{d}_\sigma^\dagger(p)|\text{gnd}\rangle &= E(p)\hat{d}_\sigma^\dagger(p)|\text{gnd}\rangle
\end{aligned}
$$

This tells us that there is a 4-fold degeneracy of the single particle states corresponding to the particles and antiparticles as well as the spin polarization. We can label the particle and antiparticle states explicitly using our conserved charge:

$$
\hat{Q} = -e\sum_{\sigma\pm}\int dx\,\bar{\psi}_\sigma(x)\gamma_0\psi_\sigma(x)
$$

We can express this guy in terms of our above creation and annihilation operators. The normal ordered form is:

$$
:\hat{Q}: = -e\sum_{\sigma=\pm}\int\frac{dp}{2\pi}\frac{m}{\omega(p)}\Big(\hat{a}_\sigma^\dagger(p)\hat{a}_\sigma(p) - \hat{d}_\sigma^\dagger(p)\hat{d}_\sigma(p)\Big)
$$

The result of this operator acting on a single particle excited state is easy to find:

$$
\begin{aligned}
:\hat{Q}: \hat{a}^\dagger_\sigma(p)|\text{gnd}\rangle &= -e\,\hat{a}^\dagger_\sigma(p)|\text{gnd}\rangle \\
:\hat{Q}: \hat{d}^\dagger_\sigma(p)|\text{gnd}\rangle &= +e\,\hat{d}^\dagger_\sigma(p)|\text{gnd}\rangle
\end{aligned}
$$

Thus, we can label our single particle states with the following quantum numbers:

$$
\begin{aligned}
|p, \sigma, -e\rangle &= \hat{a}^\dagger_\sigma(p)\,|\text{gnd}\rangle \\
|p, \sigma, +e\rangle &= \hat{d}^\dagger_\sigma(p)\,|\text{gnd}\rangle
\end{aligned}
$$

3. **4/20:** Let's return to our original continuum Hamiltonian without any approximations. We investigate this model under the combination of the discrete transformations:

$$
\psi(x) \to \gamma_5\psi(x) = \sigma_3\psi(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} R \\ L \end{bmatrix} = \begin{bmatrix} R \\ -L \end{bmatrix} \tag{1.12}
$$

$$
\Delta(x) \to -\Delta(x) \tag{1.13}
$$

This obviously leaves $\Pi^2(x)$, and $\Delta^2(x)$ invariant. Recall:

$$
H = \int \frac{dp}{2\pi} \sum_{\sigma=\pm} \begin{bmatrix} R^\dagger & L^\dagger \end{bmatrix} \begin{bmatrix} pv_f & -i\sqrt{2g}\Delta(x) \\ i\sqrt{2g}\Delta(x) & -pv_f \end{bmatrix} \begin{bmatrix} R \\ L \end{bmatrix} + \int dx \left[\frac{\Pi^2(x)}{8Ma_0^2} + \frac{1}{2}\Delta^2(x)\right]
$$

Thus, our Hamiltonian transforms as:

$$
\begin{aligned}
H \to & \int \frac{dp}{2\pi} \sum_{\sigma=\pm} \begin{bmatrix} R^\dagger & -L^\dagger \end{bmatrix} \begin{bmatrix} pv_f & i\sqrt{2g}\Delta(x) \\ -i\sqrt{2g}\Delta(x) & -pv_f \end{bmatrix} \begin{bmatrix} R \\ -L \end{bmatrix} + \int dx \left[\frac{\Pi^2(x)}{8Ma_0^2} + \frac{1}{2}\Delta^2(x)\right] \\
&= \int \frac{dp}{2\pi} \sum_{\sigma=\pm} \begin{bmatrix} R^\dagger & -L^\dagger \end{bmatrix} \sigma_3 \begin{bmatrix} pv_f & -i\sqrt{2g}\Delta(x) \\ i\sqrt{2g}\Delta(x) & -pv_f \end{bmatrix} \sigma_3 \begin{bmatrix} R \\ -L \end{bmatrix} + \int dx \left[\frac{\Pi^2(x)}{8Ma_0^2} + \frac{1}{2}\Delta^2(x)\right] \\
&= H
\end{aligned}
$$

Our Hamiltonian is invariant under the combination of these transformations! Notice the following term has an *odd parity* under this combined transformation:

$$
\bar{\psi}\psi(x) = \psi^\dagger(x)\sigma_2\psi(x) \to \psi^\dagger(x)\sigma_3\sigma_2\sigma_3\psi(x) = -\psi^\dagger(x)\sigma_2\psi(x) = -\bar{\psi}(x)\psi(x)
$$

We could easily deduce this from our Hamiltonian being invariant since $\sqrt{2}\Delta\bar{\psi}\psi(x)$ is present there. Thus, the operator $\bar{\psi}\psi(x)$ is odd under this discrete symmetry, and so it is a good candidate for an *order parameter*.

How can we interpret these transformations when viewing the original lattice model? The best way to find out is to just do it. First, recall $p_F = \frac{\pi}{2a_0}$, and we set $a_0 = 1$. So, starting with the fermion operator we see:

$$
\begin{aligned}
c_\sigma(n) &= e^{ip_F n}R_\sigma(n) + e^{-ip_F n}L_\sigma(n) \\
&\to e^{ip_F n}R_\sigma(n) - e^{-ip_F n}L_\sigma(n) \\
&= e^{i\pi/2}e^{-i\pi/2}e^{ip_F n}R_\sigma(n) + e^{-i\pi}e^{-ip_F n}L_\sigma(n) \\
&= -i\big(e^{ip_F(n+1)}R_\sigma(n) + e^{-ip_F(n+1)}L_\sigma(n)\big)
\end{aligned}
$$

The factor of $-i$ is no problem since these operators respect a global $U(1)$ symmetry. The phase factors of the n^{th} site becomes that of the $(n+1)^{st}$ site. We interpret this as a shift in the dimerized structure.

A similar result holds for the phonons:

$$
\begin{aligned}
x(n) &= \delta(n) + e^{2ip_F n}\Delta_+(n) + e^{-2ip_F n}\Delta_-(n) \\
&\to \delta(n) - e^{2ip_F n}\Delta_+(n) - e^{-2ip_F n}\Delta_-(n) \\
&= \delta(n) + e^{i\pi}e^{2ip_F n}\Delta_+(n) + e^{-i\pi}e^{-2ip_F n}\Delta_-(n) \\
&= \delta(n) + e^{2ip_F (n+1)}\Delta_+(n) + e^{-2ip_F (n+1)}\Delta_-(n)
\end{aligned}
$$

We effectively shift even sites to odd sites, and odd sites to even sites under the combination of this global discrete symmetry. Our original lattice model had a asymmetrical hopping parameter corresponding to short and long bond lengths, which results in an accumulation of electrons on even or odd sites.[2]

4. **2/20:** Let's now find the expectation value of the order parameter of question 1.3 in the ground state. We simply substitute in our mode expansion:

$$
\begin{aligned}
\sum_\sigma \langle \text{gnd}|\bar{\psi}_\sigma(x)\psi_\sigma(x)|\text{gnd}\rangle &= \sum_\sigma \int \frac{dpdq}{(2\pi)^2}\frac{m}{E(q)}\frac{m}{E(p)}\bar{v}(q)v(p)\langle \text{gnd}|\hat{d}_\sigma(q)\hat{d}^\dagger_\sigma(p)|\text{gnd}\rangle e^{i(q-p)\cdot x} \\
&= \sum_\sigma \int \frac{dp}{2\pi}\frac{m}{E(p)}\bar{v}(p)v(p) \\
&= -\int \frac{dp}{\pi}\frac{\sqrt{2g}\Delta_0}{\sqrt{(v_F p)^2 + 2g\Delta_0^2}} \\
&= -\int \frac{dp}{\pi}\frac{1}{\sqrt{(\frac{v_F p}{\sqrt{2g}\Delta_0})^2 + 1}} \\
&= -2\frac{\sqrt{2g}\Delta_0}{v_F}\sinh^{-1}\left(\frac{v_F\Lambda}{\sqrt{2g}\Delta_0}\right)
\end{aligned}
$$

We see the expectation value is only nonzero if Δ_0 is non-zero! This order parameter is thus related to a spontaneous generation of mass $m = \sqrt{2g}\Delta_0$, or *dimerization*. This is observed as a distortion of our lattice: the so called *Peierls instability*.

2 Grassmann Stuff

1. **2/20:** Let a and a^* be a pair of Grassmann variables and consider: $g(a^*) = g_0 + g_1 a^*$, $f(a) = f_0 + f_1 a$. Furthermore, define:

$$
\langle f|g\rangle = \int da^* da\, e^{-a^* a} f(a^*)^* g(a^*)
$$

Recall: $a^2 = (a^*)^2 = 0$; thus, $e^{-a^*a} = 1 - a^* a$. Now let's expand the product of the two functions:

$$
\begin{aligned}
f(a^*)^* g(a^*) &= \left[f_0 + f_1 a^*\right]^* \left[g_0 + g_1 a^*\right] \\
&= \left[\bar{f}_0 + \bar{f}_1 a\right]\left[g_0 + g_1 a^*\right] \\
&= \bar{f}_0 g_0 + \bar{f}_1 g_0 a + \bar{f}_0 g_1 a^* + \bar{f}_1 g_1 a a^*
\end{aligned}
$$

From here we can see (I set $a^2 = (a^*)^2 = 0$ without writing our explicitly) :

$$
\begin{aligned}
e^{-a^* a} f(a^*)^* g(a^*) &= \left[1 - a^* a\right]\left[\bar{f}_0 g_0 + \bar{f}_1 g_0 a + \bar{f}_0 g_1 a^* + \bar{f}_1 g_1 a a^*\right] \\
&= \left[1 - a^* a\right]\bar{f}_0 g_0 + \left[\bar{f}_1 g_0 a + \bar{f}_0 g_1 a^* + \bar{f}_1 g_1 a a^*\right]
\end{aligned}
$$

[2] Since the above transformation is a symmetry of our Hamiltonian, we see there is an ambiguity of where our electrons reside due to the fact this operation shifts our discrete lattice sites! With open boundary conditions the two different states have a relative polarization however (classified by their Berry phase), and this ambiguity gives rise to a *charge pump*. Polyacetylene can be described as a conductive polymer.

Now we recall (analogous rules exist for a^*):

$$\int da\ \kappa a = \kappa$$

and,

$$\int da\ \kappa \partial_a a = \int da\ \kappa = 0$$

It follows, after integration, that only terms which survive are those with products aa^*.

Finally, we note that we need to reorder the term $-a^*a = aa^*$, which uses the anticommutation relations of Grassmann variables, since the a integral needs to be done first. The resulting expression is:

$$\langle f|g\rangle = \int da^* da \left[\bar{f}_0 g_0 + \bar{f}_1 g_1\right] aa^* = \int da^* \left[\bar{f}_0 g_0 + \bar{f}_1 g_1\right] a^* = \bar{f}_0 g_0 + \bar{f}_1 g_1$$

2. **5/20:** Now let's consider the matrix equation:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix}$$

This vector space is spanned by:

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

There is a clear isomorphism staring us in the face: $\vec{b}_1 \to 1$, $\vec{b}_2 \to a^*$.

Then:

$$\vec{f} = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} \to f(a^*) = f_0 + f_1 a^*$$

We know that the most general function of two Grassmann variables has the form:

$$A(a^*, \alpha) = A_{00} + A_{01}\alpha + A_{10}a^* + A_{11}a^*\alpha$$

Using properties of the Grassmann variables we can actually compute this matrix product in an alternative way; indeed:

$$(Af)(a^*) = \int d\alpha^* d\alpha A(a^*, \alpha) f(\alpha^*) e^{-\alpha^*\alpha}$$

The integrand can be expanded to (I drop terms which do not have the product $\alpha^*\alpha$):

$$\begin{aligned} \left[A_{00} + A_{01}\alpha + A_{10}a^* + A_{11}a^*\alpha\right]\left[f_0 + f_1\alpha^*\right]\left[1 - \alpha^*\alpha\right] &\to \left[A_{00} + A_{10}a^*\right] f_0 \Big(- \alpha^*\alpha\Big) + \left[A_{01} + A_{11}a^*\right] f_1 \alpha\alpha^* \\ &= \Big\{\left[A_{00}f_o + A_{01}f_1\right] + \left[A_{10}f_0 + A_{11}f_1\right]a^*\Big\}\alpha\alpha^* \end{aligned}$$

And, since: $\int d\alpha^* d\alpha \alpha\alpha^* = 1$, we see:

$$(Af)(a^*) = \int d\alpha^* d\alpha A(a^*, \alpha) f(\alpha^*) e^{-\alpha^*\alpha} = \left[A_{00}f_0 + A_{01}f_1\right] + \left[A_{10}f_0 + A_{11}f_1\right]a^* = g(a^*)$$

We can also construct a rule for matrix multiplication:

$$(A\ B)(a^*, a) = \int d\alpha^* d\alpha e^{-\alpha^*\alpha} A(a^*, \alpha) B(\alpha^*, a)$$

Where: $B(\alpha^*, a) = B_{00} + B_{01}a + B_{10}\alpha^* + B_{11}\alpha^* a$. Let's follow the same protocol and expand the integrand (again dropping terms which clearly integrate to zero):

$$e^{-\alpha^*\alpha} A(a^*, \alpha) B(\alpha^*, a) = \Big[1 - \alpha^*\alpha\Big]\Big[A_{00} + A_{01}\alpha + A_{10}a^* + A_{11}a^*\alpha\Big]\Big[B_{00} + B_{01}a + B_{10}\alpha^* + B_{11}\alpha^* a\Big]$$

$$\begin{aligned}
&\rightarrow \Big[A_{01}\alpha + A_{11}a^*\alpha\Big]\Big[B_{10}\alpha^* + B_{11}\alpha^* a\Big] - \alpha^*\alpha\Big[A_{00} + A_{10}a^*\Big]\Big[B_{00} + B_{01}a\Big] \\
&= \Big[\Big(A_{00}B_{00} + A_{01}B_{10}\Big) + \Big(A_{01}B_{11} + A_{00}B_{01}\Big)a + \Big(A_{11}B_{10} + A_{10}B_{00}\Big)a^* + \Big(A_{11}B_{11} + A_{10}B_{01}\Big)a^* a\Big]\alpha\alpha^* \\
&= C(a^*, a)\alpha\alpha^*
\end{aligned}$$

It follows:

$$(A\ B)(a^*, a) = \int d\alpha^* d\alpha e^{-\alpha^*\alpha} A(a^*, \alpha) B(\alpha^*, a) = C(a^*, a)$$

3. **3/20:** Let's now define two operators $\hat{a}^*$ and $\hat{a}$ which satisfy:

$$\hat{a}^* f(a^*) = a^* f(a^*), \quad \hat{a} f(a^*) = \frac{d}{da^*} f(a^*)$$

As a result of these definitions, we see:

$$\hat{a}^*\hat{a}^* f(a^*) = a^*\hat{a} f(a^*) = (a^*)^2 f(a^*) = 0$$

Thus, $(\hat{a}^*)^2 = 0$. Similarly:

$$(\hat{a})^2 f(a^*) = \frac{d}{da^*}\Big[\frac{d}{da^*} f(a^*)\Big] = 0$$

This last equality can be viewed as a consequence of differentials of Grassmann variables acting like Grassmann variables, or the second derivative of any function of Grassmann variables is zero since the most general function of a Grassmann variable is linear in Grassmann variables (that is a lot of Grassmann variables). In any case: $(\hat{a})^2 = 0$.

Now let's go ahead and see what happens if we combine these operators, we will need $f(a^*) = f_0 + a^* f_1$. So we have two combinations to calculate:

(a) $\hat{a}\hat{a}^* f(a^*) = \hat{a} a^* f(a^*) = \frac{d}{da^*}\big[f_0 a^* + f_1 (a^*)^2\big] = \frac{d}{da^*}\big[f_0 a^*\big] = f_0$

(b) $\hat{a}^*\hat{a} f(a^*) = \hat{a}^* \frac{d}{da^*} f(a^*) = \hat{a}^* \frac{d}{da^*}\big[f_0 + f_1(a^*)\big] = \hat{a}^*\big[f_1\big] = a^* f_1$

Putting these together we see:

$$\{\hat{a}, \hat{a}^*\} f(a^*) = f_0 + a^* f_1 = f(a^*) \rightarrow \{\hat{a}, \hat{a}^*\} = 1$$

These operators have the same properties as creation and annihilation operators for fermions!

4. **5/20:** Now let's consider the set of 2N Grassmann variables: $\big\{\{\xi_i\}, \{\bar{\xi}_i\}\big\}$. Let's compute the following integral:

$$\mathcal{Z} = \int \prod_{j=1}^{N} \Big[d\bar{\xi}_j d\xi_j\Big] e^{-\sum_{k,l} \bar{\xi}_k M_{kl} \xi_l}$$

The integrand can be expanded in a Taylor series, which results in terms of the following form:

$$e^{-\sum_{k,l} \bar{\xi}_k M_{kl} \xi_l} = \sum_{n \geq 0} \frac{1}{n!}\Big(-\sum_{k,l} \bar{\xi}_k M_{kl} \xi_l\Big)^n$$

Using properties of Grassmann variables we can greatly simplify this expression. For one, certain products will contain terms with squares of identical Grassmann variables, so these will automatically go to zero. Furthermore, we recall:

$$\int d\eta \, 1 = 0$$

Thus, the only nonzero combinations, after integration, are those terms which have exactly one of each Grassmann variable.

It is not hard to see we are forced to look at the term:

$$\begin{aligned}
\frac{1}{N!}\Big(-\sum_{k,l}\bar{\xi}_k M_{kl}\xi_l\Big)^N &= \frac{(-1)^N}{N!}\Big(\sum_{l_1}\bar{\xi}_1 M_{1l_1}\xi_{l_1} + \sum_{l_2}\bar{\xi}_2 M_{2l_2}\xi_{l_2} + \cdots + \sum_{l_N}\bar{\xi}_N M_{Nl_N}\xi_{l_N}\Big)^N \\
&= (-1)^N \sum_{l_1 \dots l_N} M_{1l_1}M_{2l_2}\dots M_{Nl_N}\bar{\xi}_1\xi_{l_1}\bar{\xi}_2\xi_{l_2}\dots\bar{\xi}_N\xi_{l_N}
\end{aligned}$$

This last line comes about from noting that we have N products of terms of the form in the parentheses and if we multiply any identical sum, then these integrate to zero. Thus, we drop those terms. Furthermore, there are $N!$ ways to achieve these products. We permute Grassmann variables in pairs to get everything in the above form, so we do not need to worry about keeping track of minus signs.

The determinant is now staring us right in the face! We can recast the above sum into the following form:

$$\begin{aligned}
\frac{1}{N!}\Big(-\sum_{k,l}\bar{\xi}_k M_{kl}\xi_l\Big)^N &= (-1)^N \sum_{l_1 \dots l_N} M_{1l_1}M_{2l_2}\dots M_{Nl_N}\bar{\xi}_1\xi_{l_1}\bar{\xi}_2\xi_{l_2}\dots\bar{\xi}_N\xi_{l_N} \\
&= (-1)^N \Bigg[\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)M_{1\sigma(1)}M_{2\sigma(2)}\dots M_{N\sigma(N)}\Bigg]\bar{\xi}_1\xi_1\bar{\xi}_2\xi_2\dots\bar{\xi}_N\xi_N
\end{aligned}$$

Here σ is a element of the symmetry group of N elements. In contrast to what we saw above we are not permuting in pairs, so the permutation will matter! The $\mathrm{sgn}(\sigma)$ term comes from reordering the Grassmann variables in the order indicated outside the brackets, so the accumulation in minus signs will have to be the same as the parity of the symmetry element.

We are now ready to insert this into the partition function. The integral is trivial after we permute the Grassmann variables to match the order of the differentials for the integration. We just end up with an additional factor of $(-1)^N$; thus, an overall factor of 1! Our desired result is almost immediate:

$$\begin{aligned}
\mathcal{Z} &= \int \prod_{j=1}^{N}\Big[d\bar{\xi}_j d\xi_j\Big] e^{-\sum_{k,l}\bar{\xi}_k M_{kl}\xi_l} \\
&= \int \prod_{j=1}^{N}\Big[d\bar{\xi}_j d\xi_j\Big](-1)^N\Bigg[\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)M_{1\sigma(1)}M_{2\sigma(2)}\dots M_{N\sigma(N)}\Bigg]\bar{\xi}_N\xi_N\dots\bar{\xi}_2\xi_2\bar{\xi}_1\xi_1 \\
&= \int \prod_{j=1}^{N}\Big[d\bar{\xi}_j d\xi_j\Big]\Bigg[\sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)M_{1\sigma(1)}M_{2\sigma(2)}\dots M_{N\sigma(N)}\Bigg]\xi_N\bar{\xi}_N\dots\xi_2\bar{\xi}_2\xi_1\bar{\xi}_1 \\
&= \sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)M_{1\sigma(1)}M_{2\sigma(2)}\dots M_{N\sigma(N)} \\
&= \det(M)
\end{aligned} \tag{2.1}$$

3 Dirac Fermions

Consider the following Lagrangian density:

$$\mathcal{L} = \bar{\psi}\Big(i\partial\!\!\!/ - m\Big)\psi$$

1. **5/20:** As before, let's consider the amplitude ($\hbar = 1$):

$$\langle \Psi_f, t_f | \Psi_i, t_i \rangle = \langle \Psi_f | e^{-iH(t_f - t_i)} | \Psi_i \rangle \equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS(\bar{\psi},\psi)} \times \text{projection operators}$$

Where:

$$S(\bar{\psi}, \psi) = \int_{t_i}^{t_f} dt \big[i\bar{\psi}\partial_t \psi - H(\bar{\psi}, \psi) \big]$$

From our Lagrangian we see that the momentum canonically conjugate to ψ is $i\bar{\psi}$; thus:

$$S(\bar{\psi}, \psi) = \int d^4x \mathcal{L}$$

Now let's couple our above Lagrangian to some external sources η and $\bar{\eta}$:

$$\mathcal{L} = \bar{\psi}\Big(i\partial\!\!\!/ - m\Big)\psi + \bar{\psi}\eta + \bar{\eta}\psi$$

As was the case with a scalar field theory, we can now find the Generating functional:

$$\mathcal{Z}[\bar{\eta}, \eta] = \frac{1}{\langle 0|0\rangle} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS}$$

For simplicity, let's define $\hat{A} = i\partial\!\!\!/ - m$. There are several ways to go about solving this problem; for instance, we could complete the square. I will opt to shift the fields as we did in the previous homework, but these two methods are essentially the same. Let's define:

$$\psi(x) = \psi_0(x) + \xi(x) \Rightarrow \ \bar{\psi}(x) = \bar{\psi}_0(x) + \bar{\xi}(x)$$

This means:

$$\begin{aligned}\mathcal{L} =& \Big[\bar{\psi}_0 \hat{A}\psi_0 + \psi_0(x)\bar{\eta}(x) + \bar{\psi}_0(x)\eta(x)\Big] + \bar{\xi}(x)\hat{A}\xi(x) \\ &+ \Big[\bar{\xi}(x)\hat{A}\psi_0(x) + \bar{\xi}(x)\eta(x)\Big] + \Big[\bar{\psi}_0(x)\hat{A}\xi(x) + \xi(x)\bar{\eta}(x)\Big]\end{aligned}$$

Using $\hat{A}G(x-y) = \delta(x-y)$ we note that:

$$\psi_0(x) = -\int d^4y G(x-y)\eta(y) \Rightarrow \bar{\xi}(x)\hat{A}\psi_0(x) + \bar{\xi}(x)\eta(x) = 0 \tag{3.1}$$

Since the Greens function is symmetric in x and y and we can integrate by parts, it follows for:

$$\bar{\psi}_0(x) = -\int d^4y G(x-y)\bar{\eta}(y)$$

that,

$$\bar{\psi}_0(x)\hat{A}\xi(x) + \xi(x)\bar{\eta}(x) = 0$$

Substituting these solutions back into our Lagrangian yields the uncoupled (or completed square) version of our original Lagrangian (these steps are trivial since $\hat{A}\psi_0 = -\eta(x)$):

$$\mathcal{L} = -\int d^4y \bar{\eta}(x)G(x-y)\eta(y) + \bar{\xi}(x)\hat{A}\xi(x)$$

Finally, we find:

$$\mathcal{Z}[\bar{\eta}, \eta] = \left[\int \mathcal{D}\bar{\xi}\mathcal{D}\xi e^{i\int d^4x \bar{\xi}(x)\hat{A}\xi(x)}\right] e^{-i\iint d^4x d^4y \bar{\eta}(x)G(x-y)\eta(y)}$$

We now note this looks a lot like the functional integral we met in the last problem except the summation is now an integration; this here is just the continuum case of what we already know. Thus, we say the functional integral evaluates to the determinate of our operator:

$$\mathcal{Z}[\bar{\eta},\eta] = \text{Det}\Big(i\not{\partial} - m\Big)e^{-i\iint d^4x d^4y \bar{\eta}(x)G(x-y)\eta(y)} \tag{3.2}$$

$$= \text{Det}\Big(i\not{\partial} - m\Big)e^{-i\iint d^4x d^4y \bar{\eta}_\alpha(x)G_{\alpha\beta}(x-y)\eta_\beta(y)} \tag{3.3}$$

2. **10/20:** Let's go ahead and find the Feynman propagator defined by:

$$iS_F^{\alpha\beta}(x-y) = \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle$$

We'll derive this using the generating functional for our Dirac theory; for convenience I'll copy its original form below:

$$\mathcal{Z}[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\bigg\{i\int d^4z\Big(\bar{\psi}(z)\Big(i\not{\partial} - m\Big)\psi(z) + \bar{\psi}(z)\eta(z) + \bar{\eta}(z)\psi(z)\Big)\bigg\}$$

First, let's note that when taking derivatives of products of distinct Grassmann variables, we must keep in mind for $F[\eta,\bar{\eta}] = \int d^4z\bar{\eta}(z)\eta(z)$ we have:

$$\frac{\delta F[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)} = \int d^4z\frac{\delta\bar{\eta}(z)}{\delta\bar{\eta}(x)}\eta(z) = \eta(x)$$

but,

$$\begin{aligned}\frac{\delta F[\eta,\bar{\eta}]}{\delta\eta(x)} &= \frac{\delta}{\delta\eta(x)}\int d^4z\bar{\eta}(z)\eta(z)\\ &= -\frac{\delta}{\delta\eta(x)}\int d^4z\eta(z)\bar{\eta}(z)\\ &= -\int d^4z\frac{\delta\eta(z)}{\delta\eta(x)}\bar{\eta}(z)\\ &= -\bar{\eta}(x)\end{aligned}$$

That is, we must permute the Grassmann variable that we are differentiating to match the order in which we are differentiating said variables. This will result in an accumulation of various minus signs. Recall that each of these guys have four components, so we say:

$$\eta(z)\bar{\psi}(z) = \eta_\gamma(z)\bar{\psi}_\gamma(z)$$

Subscripts will be used from here on out. **We can make our lives easier if we perform the following trick**:

$$\frac{\delta^2\mathcal{Z}[\eta,\bar{\eta}]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)} = \frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}i\frac{\delta}{\delta\eta_\beta(y)}\mathcal{Z}[\eta,\bar{\eta}]$$

That is, we accompany derivatives in respect to our Grassmann variables with an imaginary number in a specified way (indicated above). Why? Let's see it in action:

$$\begin{aligned}i\frac{\delta\mathcal{Z}[\bar{\eta},\eta]}{\delta\eta_\beta(y)} &= i\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\int d^4z\, i\frac{\delta\big(\bar{\psi}_\gamma(z)\eta_\gamma(z)\big)}{\delta\eta_\beta(y)}e^{iS}\\ &= -i^2\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\int d^4z\;\delta_{\beta\gamma}\delta(y-z)\bar{\psi}_\gamma(z)e^{iS}\\ &= \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\;\bar{\psi}_\beta(y)e^{iS}\end{aligned}$$

There is no accumulation in minus sign, or an i, as a result of pairing an i with derivatives in respect to η! The $\bar{\eta}$ definition kills imaginary numbers too:

$$\begin{aligned}\frac{\delta^2\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)} &= \frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\;\bar{\psi}_\beta(y)e^{iS}\\ &= -\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\;\bar{\psi}_\beta(y)\psi_\alpha(x)e^{iS}\\ &= \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\;\psi_\alpha(x)\bar{\psi}_\beta(y)e^{iS}\end{aligned}$$

Finally, we set $\eta = \bar{\eta} = 0$; thus:

$$\frac{1}{\mathcal{Z}[0,0]}\frac{\delta^2\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}\Bigg|_{\eta=\bar{\eta}=0} = \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle$$

Now that we have this expression it is time to use our result from the previous part of this question; this will give an expression for the Feynman propagator in terms of our Green function for the Dirac operator. Let's get going:

$$\begin{aligned}i\frac{\delta\mathcal{Z}[\bar{\eta},\eta]}{\delta\eta_\beta(y)} &= \Bigg[\det(i\partial\!\!\!/ - m)\Bigg]i\frac{\delta}{\delta\eta_\beta(y)}e^{-i\iint d^4z d^4z'\bar{\eta}(z)G(z-z')\eta(z)}\\ &= \Bigg[\iint d^4z d^4z'\frac{\delta}{\delta\eta_\beta(y)}\Big[\bar{\eta}(z)G(z-z')\eta(z')\Big]\Bigg]\mathcal{Z}[\bar{\eta},\eta]\end{aligned}$$

Let's take a closer look at our functional derivative in the integral. First, let's note that this function must be a scalar. We perform the following contraction:

$$\bar{\eta}(z)G(z-z')\eta(z') = \sum_{\kappa\gamma}\Big[[\bar{\eta}(z)]_\kappa[G(z-z')]_{\kappa\gamma}[\eta(z')]_\gamma\Big] \equiv \bar{\eta}_\kappa(z)G_{\kappa\gamma}(z-z')\eta_\gamma(z')$$

That is, repeated indices contract.

$$\begin{aligned}\frac{\delta}{\delta\eta_\beta(y)}\Big[\bar{\eta}(z)G(z-z')\eta(z')\Big] &= \frac{\delta}{\delta\eta_\beta(y)}\Big[\bar{\eta}_\kappa(z)G_{\kappa\gamma}(z-z')\eta_\gamma(z')\Big]\\ &= -\bar{\eta}_\kappa(z)G_{\kappa\gamma}(z-z')\delta(y-z')\delta_{\beta\gamma}\end{aligned}$$

The overall minus sign comes from my spiel on taking derivatives of a product of Grassmann variables. Thus, we see:

$$i\frac{\delta\mathcal{Z}[\bar{\eta},\eta]}{\delta\eta_\beta(y)} = -\Bigg[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\beta}(z-y)\Bigg]\mathcal{Z}[\bar{\eta},\eta]$$

Differentiating this expression again:

$$\begin{aligned}\frac{\delta^2\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)} &= -\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\Bigg[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\beta}(z-y)\Bigg]\mathcal{Z}[\bar{\eta},\eta]\\ &= \Bigg[i\int d^4z\frac{\delta\bar{\eta}_\kappa(z)}{\delta\bar{\eta}_\alpha(x)}G_{\kappa\beta}(z-y)\Bigg]\mathcal{Z}[\bar{\eta},\eta] + \Bigg[i\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\beta}(z-y)\Bigg]\frac{\delta\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x)}\\ &\xrightarrow{\eta,\bar{\eta}\to 0} iG_{\alpha\beta}(x-y)\mathcal{Z}[0,0]\end{aligned}$$

Thus, the Feynman propagator of the Dirac theory is given by:

$$\begin{aligned}S_F^{\alpha\beta}(x-y) &= -\frac{i}{\mathcal{Z}[0,0]}\frac{\delta^2\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}\Bigg|_{\eta=\bar{\eta}=0}\\ &= G_{\alpha\beta}(x-y)\\ &= \langle x,\alpha|\frac{1}{i\partial\!\!\!/ - m}|y,\beta\rangle\end{aligned} \tag{3.4}$$

3. **5/20:** Now we want to find:

$$S^4_{F,\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) = \langle 0|T\psi_\alpha(x_1)\psi_\beta(x_2)\bar{\psi}_\gamma(x_3)\bar{\psi}_\delta(x_4)|0\rangle$$
$$= \frac{1}{\mathcal{Z}[0,0]}\frac{\delta^4\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x_1)\delta\bar{\eta}_\beta(x_2)\eta_\gamma(x_3)\delta\eta_\delta(x_4)}\Bigg|_{\eta=\bar{\eta}=0}$$

Because of the accumulation of minus signs, we will have to be extra careful here! Proceeding as before:

$$i\frac{\delta\mathcal{Z}[\bar{\eta},\eta]}{\delta\eta_\delta(x_4)} = -\left[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\delta}(z-x_4)\right]\mathcal{Z}[\bar{\eta},\eta]$$

$$i^2\frac{\delta^2\mathcal{Z}[\bar{\eta},\eta]}{\delta\eta_\gamma(x_3)\delta\eta_\delta(x_4)} = i\left[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\delta}(z-x_4)\right]\frac{\delta\mathcal{Z}[\bar{\eta},\eta]}{\delta\eta_\gamma(x_3)}$$
$$= -\left[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\delta}(z-x_4)\right]\left[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\gamma}(z-x_3)\right]\mathcal{Z}[\bar{\eta},\eta]$$

For now on any derivatives of $\mathcal{Z}[\bar{\eta},\eta]$ will be neglected since this will give a factor of η, and we only differentiate in respect to $\bar{\eta}$ from here on out[3]:

$$i\frac{\delta^3\mathcal{Z}[\bar{\eta},\eta]}{\bar{\eta}_\beta(x_2)\delta\eta_\gamma(x_3)\delta\eta_\delta(x_4)} = -\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\beta(x_2)}\left[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\delta}(z-x_4)\right]\left[\int d^4z\bar{\eta}_\kappa(z)G_{\kappa\gamma}(z-x_3)\right]\mathcal{Z}[\bar{\eta},\eta]$$
$$\to\frac{1}{i}\int d^4z\left[G_{\beta\gamma}(x_2-x_3)\bar{\eta}_\kappa(z)G_{\kappa\delta}(z-x_4) - G_{\beta\delta}(x_2-x_4)\bar{\eta}_\kappa(z)G_{\kappa\gamma}(z-x_3)\right]\mathcal{Z}[0,0]$$

Finally, we get:

$$\begin{aligned}S^4_{F,\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) &= \langle 0|T\psi_\alpha(x_1)\psi_\beta(x_2)\bar{\psi}_\gamma(x_3)\bar{\psi}_\delta(x_4)|0\rangle\\ &= \frac{1}{\mathcal{Z}[0,0]}\frac{\delta^4\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x_1)\delta\bar{\eta}_\beta(x_2)\eta_\gamma(x_3)\delta\eta_\delta(x_4)}\Bigg|_{\eta=\bar{\eta}=0}\\ &= G_{\beta\delta}(x_2-x_4)G_{\alpha\gamma}(x_1-x_3) - G_{\beta\gamma}(x_2-x_3)G_{\alpha\delta}(x_1-x_4)\end{aligned} \tag{3.5}$$

4 Functional Determinants and the Casimir Effect

Consider the free scalar field $\phi(x,t)\equiv\phi(x)$ in 1+1 space-time dimensions; we know:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi(x)^2$$

Furthermore, suppose for all time t:

$$\phi(x,t) = \phi(x+L,t)$$

1. **1/20:** Let's begin by looking for the classical ground state energy of the system; thus, let's find the Hamiltonian density:

$$\mathcal{H} = \frac{\delta\mathcal{L}}{\delta\partial_o\phi(x)}\partial_o\phi(x) - \mathcal{L} = \Pi(x)^2 - \mathcal{L} = \frac{1}{2}\Big[\Pi(x)^2 + \big(\nabla\phi(x)\big)^2 + m^2\phi(x)^2\Big]$$

[3]The factors of i here have the potential of being confusing. There's a single factor on the left due to the pairing of specific differentials with factors of i and the fact there's three derivatives total. The right side is only differentiating in respect to a single Grassmann variable since the action of the first two differentials have been accounted for.

In general, to minimize a Hamiltonian of the form:

$$\mathcal{H} = \frac{1}{2}\Big[\Pi(x)^2 + \big(\nabla\phi(x)\big)^2 + V(\phi(x))\Big]$$

we see that the first two terms should be zero since they are positive definite; thus, $\phi(x)$ =constant. *We then choose* $\phi(x)$ *in such a way that* $V(\phi(x)) = V_{min}$. In our case this is very simple: $\phi(x) = 0$. Thus, our ground state energy is zero; this is believable since our Hamiltonian is positive definite.

2. **7/20:** Here we are asked to use the path integral methods to derive an energy density for our real scalar field, but we did this in lecture. Furthermore, this procedure is very similar to that of the complex scalar field we did in the previous homework. Because of this, I will go through the derivation in a telegraphic way:

   (a) We start with the usual prescription (equation 5.109 in Fradkin's tesxt):

   $${}_J\langle\{\phi(\vec{x}, x_o)\}|\{\phi'(\vec{y}, y_o)\}\rangle_J = \mathcal{N}\int_{b.c.} \mathcal{D}\phi e^{iS(\phi,\partial_\mu\phi,J)}$$

   Where: $S(\phi, \partial_\mu\phi, J) = \int d^4x\Big[\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + J\phi\Big]$, and $\mathcal{N}$ is a normalization constant.

   (b) We then relate this to the Vacuum Persistent Amplitude by projecting out the excitations with the Gell-Mann-Low Theorem. We thus work with our best friend $Z[J]$, the vacuum persistent amplitude, being the generating functional of our theory:

   $$Z[J] = {}_J\langle 0|0\rangle_J = \mathcal{N}\int \mathcal{D}\phi e^{iS(\phi,\partial_\mu\phi,J)}$$

   (c) We now perform a wick rotation:

   $$\mathcal{L} \xrightarrow{t\to -i\tau} \mathcal{L}_E = \frac{1}{2}(\nabla_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 - J\phi$$

   So then we work with:

   $$Z_E[J] = {}_J\langle 0|0\rangle_J = \mathcal{N}\int \mathcal{D}\phi e^{-S_E(\phi,\partial_\mu\phi,J)}$$

   Where: $S_E(\phi, \partial_\mu\phi, J) = \int d^4x\mathcal{L}_E$.

   (d) Next we integrate by parts to put the Euclidean Lagrangian density in a bilinear form:

   $$\mathcal{L}_E \to \frac{1}{2}\phi[-\nabla^2 + m^2]\phi - J\phi$$

   (e) We now shift the field and use the Green's function, $G_E(x - x')$ of our operator $\frac{1}{2}[-\nabla^2 + m^2]$ to express our shifted field in terms of our source, which decouples the source from the field. The result is:

   $$Z_E[J] = Z_E[0]e^{\frac{1}{2}\int d^Dxd^Dx'J(x)G_E(x-x')J(x')}$$

   where:

   $$Z_E[0] = \int \mathcal{D}\delta e^{-\frac{1}{2}\int d^Dx\delta(x)[-\nabla^2+m^2]\delta(x)}$$

   (f) Finally, we expand $\delta(x)$ in terms of the eigen functions of our operator $-\nabla^2 + m^2$. The result of this is just a product of a bunch of Gaussian integrals with exponents of the form $\frac{1}{2}A_n c_n^2$ where A_n is the eigenvalue of the n^{th} eigenfunciton and c_n is the n^{th} expansion coefficient of $\delta(x)$. The resulting integration is simply given by:

   $$Z_E[0] = \prod_n A_n^{-1/2} \equiv \det\big[-\nabla^2 + m^2\big]^{-1/2}$$

   (g) In conclusion:

   $$Z_E[J] = \det\big[-\nabla^2 + m^2\big]^{-1/2} e^{\frac{1}{2}\int d^Dxd^Dx'J(x)G_E(x-x')J(x')}$$

Now let's go ahead and use our partition function to calculate the ground state energy. This can be found with the following equation:

$$E_G = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln\,\mathrm{tr}e^{-\beta H} = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln Z_E[J]$$

We know from lecture, and the last homework, that we can expand our Lagrangian around the classical path and calculate the quantum corrections. For the case of a field $\phi_c(x)+\phi(x)$, where $\phi_c(x)$ is the classical path and $\phi(x)$ is the quantum correction, a Lagrangian of the form $\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)$ takes the following form (to leading order):

$$\mathcal{L} = \mathcal{L}[\phi_c] + \frac{1}{2}\phi(x)\big[-\partial^2 - V''[\phi_c]\big]\phi(x) + \dots$$

The term with the quantum corrections is bi-linear in this field; thus, from our previous experience with real scalar fields we see:

$$\mathcal{Z} = \int \mathcal{D}\phi e^{iS[\phi]} = \mathcal{N}\Big[\det\big(-\partial^2 - V''[\phi_c]\big)\Big]^{-1/2} e^{iS[\phi_c]}[1+\dots]$$

In our case we have $V(\phi)=\frac{1}{2}m^2\phi^2(x)$, so then: $V''[\phi_c]=m^2$

We however prefer to work in imaginary time, so our corresponding Euclidean action can be achieved with the following substitutions:

(a) $iS[\phi_c]\to -S[\phi_c]$

(b) $\frac{1}{2}\phi(x)\big[-\partial^2 - V''[\phi_c]\big]\phi(x) \to -\frac{1}{2}\phi(x)\big[-\nabla^2+m^2\big]\phi(x)$

Here: $\nabla^2 = \partial_\tau^2 + \partial_x^2$. From these substitutions we see:

$$\mathcal{Z} = \mathcal{N}\Big[\det\big(-\nabla^2 + V''[\phi_c]\big)\Big]^{-1/2} e^{-S[\phi_c]}[1+\dots] = \mathcal{N}e^{-\beta E_0}\Big[\det\big(-\nabla^2+m^2\big)\Big]^{-1/2}[1+\dots]$$

Above I made the following identification between the classical action and the ground state energy for a free field:

$$S[\phi_c] = -\beta^{-1}E_0$$

Our equation for our ground state energy above becomes:

$$E_G = E_0 + \frac{1}{2\beta}\ln\det(-\nabla^2+m^2) + \dots$$

Thus, the term which arises from our quantum corrections amounts to evaluating a determinant for the operator: $(-\nabla^2+m^2)\equiv \hat{A}$.

3. **10/20:** We can find an explicit expression for the energy density via a number of methods. Here I will use the method covered in class which uses the generalized ζ-function associated with $\hat{A}$:

$$\zeta_A(s) = \sum_n \frac{1}{a_n^s}$$

These a_n's are eigenvalues for $\hat{A}$; that is:

$$\hat{A}f_n(x) = a_n f_n(x)$$

As a quick reminder, I write out the use of the ζ-function:

$$\lim_{s\to 0^+}\frac{d\zeta_A}{ds} = -\lim_{s\to 0^+}\sum_n \frac{\ln a_n}{a_n^s} = -\ln\prod_n a_n \equiv -\ln\det\hat{A}$$

We thus need to find our generalized ζ-function *to compute the functional determinant.* As we did in class, we can accomplish this by *using the generalize heat kernel*:

$$G_A(x,y;\tau) = \sum_n e^{-a_n\tau} f_n(x) f_n^*(y) \equiv \langle x|e^{-\tau\hat{A}}|y\rangle$$

Here $\tau > 0$. The heat kernel has the following initial condition:

$$\lim_{\tau \to 0^+} G_A(x, y; \tau) = \sum_n f_n(x) f_n^*(y) = \delta(x - y)$$

We now relate our heat kernel to our generalized ζ-function in the following way:

$$\int d^D x \lim_{y \to x} G_A(x, y; \tau) = \sum_n e^{-a_n \tau} \int d^D x\, f_n(x) f_n^*(x) = \sum_n e^{-a_n \tau} \equiv \text{tr}\, e^{-\tau \hat{A}}$$

We know that:

$$\frac{\Gamma(s)}{a_n^s} = \int_0^\infty d\tau\, \tau^{s-1} e^{-a_n \tau}$$

So then we see by the above procedure:

$$\sum_n \frac{\Gamma(s)}{a_n^s} = \int_0^\infty d\tau\, \tau^{s-1} \sum_n e^{-a_n \tau} = \int_0^\infty d\tau\, \tau^{s-1} \int d^D x \lim_{y \to x} G_A(x, y; \tau)$$

We finally have:

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \int d^D x \lim_{y \to x} G_A(x, y; \tau)$$

There are two ways we can proceed:

(a) We find the eigenfunctions for $\hat{A}$ to build the generalized heat kernel.

(b) We use the eigenvalues of $\hat{A}$ to build the generalized ζ-function.

I will find the eigenfunctions, but we will construct the generalized ζ-function. Also, recall we are in $1 + 1$-D, so we only have two components.

To begin, we note that our eigenfunctions need to satisfy periodic boundary conditions in our spatial coordinate; i.e. (two labels for the two coordinates) $f_{ln}(x + L, x_0) = f_{ln}(x, x_0)$. **Here x_0 is the label I choose to use for imaginary time since τ is already taken**. This function also needs to be periodic in imaginary time with period β, which we will take to infinity in due time.

Our eigenfunctions are very simple here! They are just complex exponentials. We could separate variables and show the imaginary time and the spatial components are both complex exponentials. If we did that, we'd get for $f_{ln}(\vec{x}) \equiv f_{ln}(x, x_0) = h_l(x_0) g_n(x)$:

$$\hat{A} f_{ln}(x, x_0) = a_n f_{ln}(x, x_0) \Rightarrow \frac{1}{h_l(x_0)} \partial_0^2 h_l(x_0) + \frac{1}{g_n(x)} \partial_x^2 g_n(x) = a_{ln} - m^2 \equiv k^2$$

We then define $k_x^2 + \omega^2 = k^2$, and do all the things we learned way back in intro to ODE's. I'm just going to cut to the chase and say our normalized eigenfunctions are:

$$f_{ln}(\vec{x}) = \frac{e^{i\boldsymbol{x} \cdot \boldsymbol{k}}}{\sqrt{L\beta}}$$

with eigenvalues: $a_{ln} = k^2 + m^2$. Note, the eigenvalues are: $\boldsymbol{k} = (k_n, \omega_l) = (\frac{2\pi}{L} n, \frac{2\pi}{\beta} l)$. This gives us our periodic boundary conditions.

Since we know our eigenvalues, we can now write out our expression for the *generalized ζ-function*:

$$\begin{aligned}
\zeta_{\hat{A}}(s) &= \sum_{l,n} \frac{1}{a_{ln}^s} \\
&= \frac{1}{\Gamma(s)} \sum_{l,n} \int_0^\infty d\tau \tau^{s-1} e^{-a_{ln} \tau} \\
&= \frac{\beta}{\Gamma(s)} \sum_n \int_0^\infty \int_{-\infty}^\infty \frac{d\tau d\omega}{2\pi} \tau^{s-1} e^{-(\omega^2 + k_n^2 + m^2)\tau}
\end{aligned}$$

Notice that we replaced the sum with an integral in the last step by using the density of states; hence, $\omega_l \to \omega$. We can easily perform the integral over ω since it is just Gaussian:

$$\begin{aligned}\zeta_{\hat{A}}(s) &= \frac{\beta}{\Gamma(s)} \sum_n \int_0^\infty \frac{d\tau d\omega}{2\pi} \tau^{s-1} \sqrt{\frac{\pi}{\tau}} e^{-(k_n^2+m^2)\tau} \\ &= \frac{\beta}{\Gamma(s)} \sum_n \int_0^\infty \frac{d\tau}{2\sqrt{\pi}} \tau^{s-3/2} e^{-(k_n^2+m^2)\tau}\end{aligned}$$

We are now in a position to use Poisson's Summation formula:

$$\sum_n f(n) = \sum_{\bar{m}} \int_{-\infty}^{\infty} dy e^{2\pi i \bar{m} y} f(y)$$

We identify: $f(n) = \exp(-k_n^2 \tau)$. Thus, substituting y in for n in k_n explicitly we find:

$$\begin{aligned}\zeta_{\hat{A}}(s) &= \frac{\beta}{\Gamma(s)} \sum_{\bar{m}} \int_{-\infty}^{\infty} dy \int_0^\infty \frac{d\tau}{2\sqrt{\pi}} e^{2\pi i \bar{m} y} \tau^{s-3/2} e^{-\left((\frac{2\pi}{L})^2 y^2 + m^2\right)\tau} \\ &= \frac{\beta L}{4\pi\Gamma(s)} \sum_n \int_0^\infty d\tau \tau^{s-3/2} \int_{-\infty}^{\infty} dk \left[\frac{1}{\sqrt{\pi}} e^{inkL} e^{-\left(k^2+m^2\right)\tau}\right] \\ &= \frac{\beta L}{4\pi\Gamma(s)} \sum_n \int_0^\infty d\tau \tau^{s-2} e^{-m^2\tau - \frac{(nL)^2}{4\tau}}\end{aligned}$$

The penultimate line was achieved simply from a change of variables, $k = (2\pi/L)y$, and **I sent $\bar{m} \to n$ to avoid confusion with our mass term** (It is not an eigenvalue either!). The final step just relies on completing the square to find the Gaussian integral.

We need to consider the massless limit, but there will be issues at $n = 0$; essentially, we have no regulator to tame the term τ^{s-2}. Thus, **we imagine we have a very, very small mass which allows us to neglect it for all nonzero n. Specifically, we are taking the limit where we fix L, but send m to zero**. Applying this line of reasoning, and recalling the Euler Γ-function:

$$\Gamma(s) = \int_0^\infty dz z^{s-1} e^{-z}$$

allows us to finish the calculation. The $n = 0$ term is (almost) automatic, but the $n \neq 0$ terms only require a substitution of variables: $z = \frac{(nL)^2}{4\tau}$. Furthermore, this is even in n, so we just sum over positive values. Thus:

$$\zeta_{\hat{A}}(s) = \frac{\beta L}{4\pi\Gamma(s)} \left[\int_0^\infty d\tau \tau^{s-2} e^{-m^2\tau} + 2\sum_{n=1} \int_0^\infty d\tau \tau^{s-2} e^{-\frac{(nL)^2}{4\tau}}\right] \tag{4.1}$$

$$= \frac{\beta L}{4\pi\Gamma(s)} \left[\frac{\Gamma(s-1)}{(m^2)^{s-1}} + 2\sum_{n=1} \left(\frac{2}{nL}\right)^{2(1-s)} \int_0^\infty d\tau z^{-s} e^{-z}\right] \tag{4.2}$$

$$= \frac{\beta L}{4\pi\Gamma(s)} \left[\frac{\Gamma(s-1)}{(m^2)^{s-1}} + 2\left(\frac{2}{L}\right)^{2(1-s)} \zeta(2-2s)\Gamma(1-s)\right] \tag{4.3}$$

We can rewrite this expression in several ways; I chose the following. First, we'll use (Euler's reflection formula):

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$$

We can also use the relation which is used to analytically continue the Γ-function to the negative real numbers:

$$\Gamma(s+1) = s\Gamma(s)$$

which implies:

$$\frac{\Gamma(s-1)}{\Gamma(s)} = \frac{1}{s-1}$$

$$\zeta_{\hat{A}}(s) = \frac{\beta L}{4\pi}\left[\frac{1}{(m^2)^{s-1}(s-1)} + 2\Big(\frac{2}{L}\Big)^{2(1-s)}\zeta(2-2s)\frac{\Gamma^2(1-s)\sin(\pi s)}{\pi}\right] \tag{4.4}$$

We need to take a derivative of this expression, then take the limit as $s \to 0$. Mathematica is a good choice here. However, we can do this by hand if we keep linear order terms in a Taylor expansion. We start off by defining a mass scale μ (used momentarily), so that we can expand the first term as follows:

$$\begin{aligned}\frac{(m^2)^{-s}}{1-s} =&(m^2)^{-s}(1+s+\dots)\\ =&(e^{-s\cdot\log(m^2/\mu^2)})(1+s+\dots)\\ =&(1-s\cdot\log(m^2/\mu^2)+\dots)(1+s)+\dots\\ =&1+s(1-\log(m^2/\mu^2))+\dots\end{aligned}$$

That was fairly easy, but the next term looks like a nightmare! We're going to cheat a little by permuting the limit and the derivative (or use Mathematica). This happens to work out fine here, but these do not commute in general! I repeat: these are dangerous actions I am taking here! The terms containing s are:

$$\Big(\frac{2}{L}\Big)^{2(1-s)}\zeta(2-2s)\frac{\Gamma^2(1-s)\sin(\pi s)}{\pi} \xrightarrow{s\to 0} \Big(\frac{2}{L}\Big)^{2}\zeta(2)\Gamma^2(1)s$$

Using the famous result:

$$\zeta(2) = \frac{\pi^2}{6}$$

We find our final expression:

$$\begin{aligned}\ln\det\hat{A} = -\lim_{s\to 0^+}\frac{d\zeta_A}{ds} &= -\frac{\beta L}{4\pi}\left[-m^2\Big(1-\log\big(m^2/\mu^2\big)\Big)+\frac{4\pi^2}{3L^2}\right]\\ \Rightarrow E_{\text{fluc}} = E_G - E_0 &= \frac{1}{2\beta}\ln\det(\hat{A})+\dots\\ &= \frac{m^2 L}{8\pi}\Big(1-\log\big(m^2/\mu^2\big)\Big)-\frac{\pi}{6L}+\dots\end{aligned} \tag{4.5}$$

We thus have a extensive piece and a term proportional to L^{-1} which goes to zero as $m \to 0$. Note that E_0 is singular, but I basically just redefined my reference of energy.

4. **2/20:** Ignoring the fact we used periodic boundary conditions, opposed to vanishing boundary conditions, we analyze the pressure exerted by the *zero point fluctuations* on the walls enclosing the system.

   We expect the "outside" contributions to the pressure to overtake the "inside" contributions because there are more modes allowed in the infinite region "outside" the system: the momenta does not need to be multiples of $2\pi/L$. The "outside" contribution should be divergent; hence, we'll associate the divergent part of E_{fluc} with the "outside" region. Similarly, the contribution from the "inside" region is due to the second term.

   Finally, in $1+1D$ pressure and force have the same units, so we have:

$$F_{\text{Casmir}} = \frac{E_{\text{in}} - E_{\text{out}}}{L} = -\frac{\pi}{6L^2} \tag{4.6}$$

   The minus sign tells us there is an attractive force between the walls, due to the above considerations. This is the famous *Casmir effect*!

5 Weakly interacting Bose Gas

Consider the Hamiltonian:

$$H = \int d^3\boldsymbol{x}\Big(\hat{\phi}^\dagger(\hbar^2\boldsymbol{x})\Big[\frac{\boldsymbol{p}^2}{2m} - \mu\Big]\hat{\phi}(\boldsymbol{x}) + \frac{1}{2}\int d^3\boldsymbol{x}'\hat{n}(\boldsymbol{x})V(\boldsymbol{x}-\boldsymbol{x}')\hat{n}(\boldsymbol{x}')\Big)$$

We'll take: $V(\boldsymbol{x}-\boldsymbol{x}') = \lambda\delta^3(\boldsymbol{x}-\boldsymbol{x}')$.

1. **5/20:** In this problem we will be using the Bose coherent states:

$$|\{\phi(\boldsymbol{x})\}\rangle = e^{\int d^3\boldsymbol{x}\phi(\boldsymbol{x})\hat{\phi}(\boldsymbol{x})}|0\rangle$$

As was the case with creation and annihilation operators, we replace the field operators with complex scalar fields (after doing the necessary work of course). Slicing up the time intervals in the usual way, one can show[4]:

$$\langle f|e^{-\frac{i}{\hbar}\hat{H}\Delta t}|i\rangle = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\exp\Bigg\{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\Bigg(\int d^3\boldsymbol{x}\frac{\hbar}{i}\Big[\phi(\boldsymbol{x},t)\partial_t\bar{\phi}(\boldsymbol{x},t) - \bar{\phi}(\boldsymbol{x},t)\partial_t\phi(\boldsymbol{x},t)\Big] - H[\phi,\bar{\phi}]\Bigg)\Bigg\}$$
$$\times\, \bar{\psi}_f(\phi(\boldsymbol{x},t_f))\psi_i(\bar{\phi}(\boldsymbol{x},t_i))e^{\frac{1}{2}\int d^3x\Big(|\phi(\boldsymbol{x},t_f)|^2+|\phi(\boldsymbol{x},t_i)|^2\Big)}$$

I decided to carry around the $\hbar$ for this problem (mainly because it was there when I took the class). We can write this in a simpler form, as was done in lecture:

$$S = \int d^4x\bar{\phi}(x)\Big(i\hbar\partial_t + \frac{\hbar^2}{2m}\vec{\nabla}^2 + \mu\Big)\phi(x) - \frac{1}{2}\int d^4x\int d^4x'|\phi(x)|^2|\phi(x')|^2V(x-x')$$

Where, $V(x-x') = \lambda\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t')$, and $d^4x = dtd^3\boldsymbol{x}$.

We can use the coherent-state path integrals to represent the partition function at finite temperature T:

$$Z = \text{tr}e^{-\beta\hat{H}}$$

We'd also like to work with the Euclidean version of the partition function. What we need to do is:

(a) Set: $|i\rangle = |f\rangle$

(b) Sum over all possible intermediate states.

(c) Perform a Wick rotation $t \to -i\tau$. Since we are eventually going to derive a partition function, we are going to need a trace of our path integral. This is the same thing as saying our particle starts and ends in the same state; i.e. periodic boundary conditions in time. As we know from chapter 5, equation (5.130), **Bosonic theories which have been Wick rotated obey periodic boundary conditions in imaginary time**. That is:

$$\phi(\boldsymbol{x},\tau) = \phi(\boldsymbol{x},\tau+\beta)$$

We compactified our imaginary time coordinate.

The Wick rotated action is achieved in the usual way:

$$\begin{aligned} &t \to -i\tau \\ \Rightarrow &\partial_t \to i\partial_\tau \\ \Rightarrow &dtd^3\boldsymbol{x} \to -id\tau d^3\boldsymbol{x} \end{aligned}$$

Implementing this is easy enough. To indicate these substitutions I use small square braces around terms which change (in first line):

[4] The set up is in an Appendix.

$$iS \to i\int_0^\beta [-id\tau]\int d^3\boldsymbol{x}\bar{\phi}(x)\Big(i\hbar[i\partial_\tau] + \frac{\hbar^2}{2m}\vec{\nabla}^2 + \mu\Big)\phi(x) - i\frac{1}{2}\int_0^\beta [-id\tau]\int d^3\boldsymbol{x}\int d^3\boldsymbol{y}|\phi(x)|^2|\phi(y)|^2 V(\boldsymbol{x}-\boldsymbol{y})$$
$$= -\Bigg[\int_0^\beta d\tau\int d^3\boldsymbol{x}\bar{\phi}(\boldsymbol{x})\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \mu\Big)\phi(\boldsymbol{x}) + \frac{1}{2}\int_0^\beta d\tau d^3\boldsymbol{x}\int d^3\boldsymbol{y}|\phi(x)|^2|\phi(y)|^2 V(\boldsymbol{x}-\boldsymbol{y})\Bigg]$$

Note the use of the interaction potential (a delta function) to collapse the y-integral below. Our Euclidean action is given by:

$$\begin{aligned}
S_E &= \int_0^\beta d\tau\int d^3\boldsymbol{x}\bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \mu\Big)\phi(x) + \frac{1}{2}\int_0^\beta d\tau d^3\boldsymbol{x}\int d^3\boldsymbol{y}|\phi(x)|^2|\phi(y)|^2 V(x-y) \\
&= \int_0^\beta d\tau\int d^3\boldsymbol{x}\bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \mu\Big)\phi(x) + \frac{\lambda}{2}\int_0^\beta d\tau\int d^3\boldsymbol{x}|\phi(x)|^4 \\
&= \int_0^\beta d\tau\int d^3\boldsymbol{x}\bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \mu\Big)\phi(x) + \frac{\lambda}{2}\int_0^\beta d\tau\int d^3\boldsymbol{x}|\phi(x)|^4 \\
&= \int_0^\beta d\tau\int d^3\boldsymbol{x}\Big(\hbar\bar{\phi}(x)\partial_\tau\phi(x) + H(\phi,\bar{\phi})\Big)
\end{aligned}$$

The trailing product in our above expression, which depends on the initial and final states, drops out because it is just a constant due to our boundary conditions. We arrive at our integral formula for our partition function:

$$Z = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}e^{-S_E(\phi,\bar{\phi})}$$

2. **2/20:** Now that we have a form for our generating functional let's go ahead and find our saddle points of our Euclidean action above. We can hand pick the Lagrangian density from the above expression for S_E:

$$\begin{aligned}
\mathcal{L}_E &= \bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2\Big)\phi(x) + \left(\frac{\lambda}{2}|\phi(x)|^2 - \mu\right)|\phi(x)|^2 \\
&= \bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2\Big)\phi(x) + V_{eff}\big(\phi(x)\big)
\end{aligned}$$

This guy is clearly *invariant under the transformation*:

$$\phi'(x) = \phi(x)e^{i\theta}$$

So we have a global $U(1)$ symmetry, and so our ground state is not unique (just change the phase). Furthermore, we can view the Lagrangian as the energy density for the Euclidean theory. The first term is a kinetic term, so it is positive definite. For the classical path we have $\phi_c(x)$ is a constant (which clearly satisfies our boundary conditions).

To figure out what constant we minimize V_{eff}. This implies[5]:

$$\begin{aligned}
\phi_c(x) &= \sqrt{\rho_0}e^{i\theta_0/2} \\
&= \sqrt{\frac{\mu}{\lambda}}e^{i\theta_0/2}
\end{aligned}$$

We also write the field in terms of the square-root of its density (at the classical level) so that we can say:

$$\mu = \rho_0\lambda e^{-i\theta_0}$$

This will be used below. Finally, we also note θ_0 is an arbitrary constant phase, and we will eventually take $\theta_0 = 0$. For now we leave it undetermined.

In the limit as $T \to 0$, these configurations correspond to the degenerate ground states of the system since they minimize the energy.

[5]The phase angle is divided by 2 for later convenience. It's purpose is shown immediately below.

3. **5/20:** Let's analyze the Green function in the semi-classical limit:

$$\phi(x) = \phi_c(x) + \delta\phi(x)$$
$$\bar{\phi}(x) = \bar{\phi}_c(x) + \delta\bar{\phi}(x)$$

Here $\delta\phi(x)$ is a small, but arbitrary, fluctuation about the classical path. Furthermore, since the fluctuations are small (semi-classical), we will Taylor expand the action to quadratic order, and then find the corresponding (real time) Green function. First, recall:

$$S = \int_0^\beta d\tau \int d^3\boldsymbol{x}\bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2 - \mu\Big)\phi(x) + \frac{\lambda}{2}\int_0^\beta d\tau \int d^3\boldsymbol{x}|\phi(x)|^4$$

For simplicity, we write $dx_E^4 = d\tau d^3\boldsymbol{x}$, but leave the function dependence as $\phi(x)$. To expand about the classical path we use the following Taylor expansion:

$$\begin{aligned}
S =& S(\phi_c) + \int d^4x_E\left[\frac{\delta S}{\delta\phi(x)}\bigg|_{\phi_c}\delta\phi(x) + \delta\bar{\phi}(x)\frac{\delta S}{\delta\bar{\phi}(x)}\bigg|_{\phi_c}\right] + 2\int\int d^4x_E d^4y_E\ \delta\bar{\phi}(x)\frac{\delta^2 S}{\delta\bar{\phi}(x)\delta\phi(y)}\bigg|_{\phi_c}\delta\phi(y)\\
&+\int\int d^4x_E d^4y_E\ \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi_c}\delta\phi(x)\delta\phi(y) + \int\int dx_E^4 d^4y_E\ \delta\bar{\phi}(x)\delta\bar{\phi}(y)\frac{\delta^2 S}{\delta\bar{\phi}(x)\delta\bar{\phi}(y)}\bigg|_{\phi_c} + \dots\\
=& S(\phi_c) + \int d^4x_E\left[2\delta\bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu + 2\lambda|\phi_c|^2\Big)\delta\phi(x) + \lambda\Big(\delta\phi(x)\delta\phi(x)\bar{\phi}_c^2 + \delta\bar{\phi}(x)\delta\bar{\phi}(x)\phi_c^2\Big)\right] + \dots\\
=& S(\phi_c) + \int d^3x_E\left[\delta\bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + \rho_0\lambda\Big)\delta\phi(x) + \delta\phi(x)\Big(-\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + \rho_0\lambda\Big)\delta\bar{\phi}(x)\right.\\
&\left.\qquad + \rho_0\lambda\Big(e^{-i\theta_0}\,\delta\phi(x)\delta\phi(x) + e^{i\theta_0}\,\delta\bar{\phi}(x)\delta\bar{\phi}(x)\Big)\right] + \dots\\
\equiv& S(\phi_c) + \delta S(\delta\phi, \delta\bar{\phi})
\end{aligned}$$

Notice I integrated by parts in the last line; the purpose is to write the fluctuating part of the action as a matrix:

$$\delta S(\delta\phi,\delta\bar{\phi}) = \int d^4x_E\left[\delta\bar{\phi}(x),\quad \delta\phi(x)\right]\begin{bmatrix}\rho_0\lambda e^{i\theta_0} & \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + \rho_0\lambda\\ -\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + \rho_0\lambda & \rho_0\lambda e^{-i\theta_0}\end{bmatrix}\begin{bmatrix}\delta\bar{\phi}(x)\\ \delta\phi(x)\end{bmatrix} \tag{5.1}$$

We now use this result to find our Green function[6]:

$$\begin{aligned}
G(x-y) &= -i\langle T\big(\phi(x)\phi^\dagger(y)\big)\rangle\\
&= -i\langle T\big(\phi_c(x)\bar{\phi}_c(y)\big)\rangle - i\langle T\big(\delta\phi(x)\bar{\delta}\phi(y)\big)\rangle
\end{aligned}$$

The first term is easy to find:

$$\langle T\big(\phi_c(x)\bar{\phi}_c(y)\big)\rangle = \left|\frac{\mu}{\lambda}\right| = \rho_0$$

The other propagator is going to take some work! The focus will be on the fluctuating part of the action. We begin by introducing some sources:

$$\begin{aligned}
\delta S(\delta\phi,\delta\bar{\phi}) &= \int d^4x_E\left[\delta\bar{\phi}(x),\quad \delta\phi(x)\right]\begin{bmatrix}\rho_0\lambda e^{i\theta_0} & \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + \rho_0\lambda\\ -\hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 + \rho_0\lambda & \rho_0\lambda e^{-i\theta_0}\end{bmatrix}\begin{bmatrix}\delta\bar{\phi}(x)\\ \delta\phi(x)\end{bmatrix} - \left[\delta\bar{\phi}(x),\quad \delta\phi(x)\right]\begin{bmatrix}J\\ \bar{J}\end{bmatrix}\Bigg)\\
&\equiv \int d^4x_E\Big(\boldsymbol{\delta\phi}^T\hat{\boldsymbol{H}}\boldsymbol{\delta\phi} - \boldsymbol{\delta\phi}^T\boldsymbol{J}\Big)
\end{aligned}$$

The last line is short hand for our full expression. The first thing we note is simply:

$$\langle T\big(\delta\phi(x)\bar{\delta}\phi(y)\big)\rangle = \frac{1}{Z[0,0]}\frac{\delta^2 Z[J,\bar{J}]}{\delta\bar{J}(x)\delta J(y)}\bigg|_{\boldsymbol{J}=\boldsymbol{0}}$$

[6]You should, in principle, also look at the mixed propagators. These terms will correspond to expectation values of our fluctuating field, but our generating functional will be shown to be quadratic. It follows these terms are zero.

We should be very familiar with the procedure at this point: we want the action to be in quadratic form, so we shift the fields:

$$\delta\phi(x) = \delta\phi_0(x) + \xi(x)$$
$$\delta\bar{\phi}(x) = \delta\bar{\phi}_0(x) + \bar{\xi}(x)$$

We can define column vectors in analogy to before

$$\boldsymbol{\delta\phi_0} = \begin{bmatrix} \delta\bar{\phi}_0(x) \\ \delta\phi_0(x) \end{bmatrix}$$
$$\boldsymbol{\xi} = \begin{bmatrix} \delta\bar{\xi}(x) \\ \delta\xi(x) \end{bmatrix}$$

and we make the usual requirement that our shift satisfies (look at which currents are correlated with which fields by referencing our above definition):

$$\boldsymbol{\hat{H}\delta\phi_0} = \boldsymbol{J}$$

Making the necessary substitutions we find:

$$\delta S = \int d^3\boldsymbol{x}d\tau\Big(\boldsymbol{\xi}^T\boldsymbol{\hat{H}\xi} - \boldsymbol{\delta\phi_0}^T\boldsymbol{J}\Big)$$

In order to find the final term we need to explicitly solve for $\boldsymbol{\delta\phi_0}$ in $\boldsymbol{\hat{H}\delta\phi_0} = \boldsymbol{J}$. This is a coupled set of equations which can be solved via the introduction of the following Fourier transforms[7]:

$$\delta\phi_0(x) = \int \frac{d^4p_E}{(2\pi)^4}\delta\phi_0(p)e^{ip\cdot x}, \ \ J(x) = \int \frac{d^4p_E}{(2\pi)^4}J(p)e^{ip\cdot x}$$

$$\delta\bar{\phi}_0(x) = \int \frac{d^4p_E}{(2\pi)^4}\delta\bar{\phi}_0(-p)e^{ip\cdot x}, \ \ \bar{J}(x) = \int \frac{d^4p_E}{(2\pi)^4}\bar{J}(-p)e^{ip\cdot x}$$

Here: $p = (\omega, \boldsymbol{p})$. Substituting in we find the following matrix equation[8]:

$$\begin{bmatrix} \rho_0\lambda e^{i\theta_0} & i\hbar\omega + \frac{\hbar^2}{2m}\boldsymbol{p}^2 + \rho_0\lambda \\ -i\hbar\omega + \frac{\hbar^2}{2m}\boldsymbol{p}^2 + \rho_0\lambda & \rho_0\lambda e^{-i\theta_0} \end{bmatrix} \begin{bmatrix} \delta\bar{\phi}_0(-p) \\ \delta\phi_0(p) \end{bmatrix} = \begin{bmatrix} J(p) \\ \bar{J}(-p) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \delta\bar{\phi}_0(-p) \\ \delta\phi_0(p) \end{bmatrix} = \boldsymbol{\tilde{G}}(p)\begin{bmatrix} J(p) \\ \bar{J}(-p) \end{bmatrix}$$

Where[9]:

$$\boldsymbol{\tilde{G}}(p) = \frac{1}{(\rho_0\lambda)^2 - \left[\Big(\hbar\omega\Big)^2 + \Big(\frac{\hbar^2\boldsymbol{p}^2}{2m} + \rho_0\lambda\Big)^2\right]}\begin{bmatrix} \rho_0\lambda e^{-i\theta_0} & -i\hbar\omega - \frac{\hbar^2}{2m}\boldsymbol{p}^2 - \rho_0\lambda \\ i\hbar\omega - \frac{\hbar^2}{2m}\boldsymbol{p}^2 - \rho_0\lambda & \rho_0\lambda e^{i\theta_0} \end{bmatrix}$$

We can relate this to our fluctuations:

$$\begin{aligned} \boldsymbol{\delta\phi_0}(x) = \int \frac{d^4p_E}{(2\pi)^4}\begin{bmatrix} \delta\bar{\phi}_0(-p) \\ \delta\phi_0(p) \end{bmatrix} e^{ipx} &= \int \frac{d^4p_E}{(2\pi)^4}\boldsymbol{\tilde{G}}(p)\begin{bmatrix} J(p) \\ \bar{J}(-p) \end{bmatrix} e^{ipx} \\ &= \int \frac{d^4p_E}{(2\pi)^4}\boldsymbol{\tilde{G}}(p)\left[\int d^4x' \begin{bmatrix} J(x') \\ \bar{J}(x') \end{bmatrix} e^{-ipx'}\right]e^{ipx} \\ &= \int d^4x'_E\,\boldsymbol{G}(x-x')\begin{bmatrix} J(x') \\ \bar{J}(x') \end{bmatrix} \end{aligned}$$

[7]Note the dot products in the exponents are in Euclidean 4-space! Also, our fields/currents are dressed up enough so we just denote the Fourier transforms by their arguments, and likewise for the currents sources.

[8]Understanding orthogonality of distinct Fourier components are at play here.

[9]Take the determinant to come up with the prefactor, then you can eye ball it. Of course, Gauss-Jordan elimination works too.

Where:

$$\boldsymbol{G}(x-x') = \int \frac{d^4p_E}{(2\pi)^4}\tilde{\boldsymbol{G}}(p)e^{ip(x-x')}$$

Making the above substitutions we can see:

$$Z[J,\bar{J}] = Z[0,0]e^{-\int\int d^4x_E\, d^4x'_E \boldsymbol{J}(x)\boldsymbol{G}(x-x')\boldsymbol{J}(x')}$$

<u>We are finally in a position to calculate our Green function</u>. Going back to a previous part:

$$\begin{aligned}
\langle T\big(\delta\phi(x)\bar{\delta}\phi(y)\big)\rangle = \frac{1}{Z[0,0]}\frac{\delta^2 Z[J,\bar{J}]}{\delta\bar{J}(x)\delta J(y)}\Bigg|_{\boldsymbol{J}=\boldsymbol{0}} &= \frac{\delta^2}{\delta\bar{J}(x)\delta J(y)}\Bigg[e^{-\int\int d^4z_E\, d^4z'_E \boldsymbol{J}(z)\boldsymbol{G}(z-z')\boldsymbol{J}(z')}\Bigg]_{\boldsymbol{J}=\boldsymbol{0}} \\
&= \frac{\delta^2}{\delta\bar{J}(x)\delta J(y)}\Bigg[-\int\int d^4z_E\, d^4z'_E \boldsymbol{J}(z)\boldsymbol{G}(z-z')\boldsymbol{J}(z')\Bigg]_{\boldsymbol{J}=\boldsymbol{0}} \\
&= \Big[\boldsymbol{G}(x-y)\Big]_{21} + \Big[\boldsymbol{G}(y-x)\Big]_{12} \\
&= 2\Big[\boldsymbol{G}(x-y)\Big]_{21}
\end{aligned}$$

The last line can be seen by expanding the components of the Green functions in terms of their Fourier transforms and performing a change of variables. Looking at our above Green function we see we still need to evaluate an integral. We will do this by residues, but first let's explicitly state the integral:

$$\begin{aligned}
\Big[\boldsymbol{G}(x-y)\Big]_{21} &= \int \frac{d^3\boldsymbol{p}d\omega}{(2\pi)^4}\frac{i\hbar\omega - \frac{\hbar^2}{2m}\boldsymbol{p}^2 - \rho_0\lambda}{(\rho_0\lambda)^2 - \Bigg[\Big(\hbar\omega\Big)^2 + \Big(\frac{\hbar^2\boldsymbol{p}^2}{2m}+\rho_0\lambda\Big)^2\Bigg]}e^{ip(x-y)} \\
&= \int \frac{d^3\boldsymbol{p}d\omega}{(2\pi)^4}\frac{-i\hbar\omega + \frac{\hbar^2}{2m}\boldsymbol{p}^2 + \rho_0\lambda}{\big(\hbar\omega\big)^2 + \big(\hbar\omega_p\big)^2}e^{ip(x-y)}
\end{aligned}$$

where:

$$\hbar\omega_p = \sqrt{\frac{\hbar^2\boldsymbol{p}^2}{2m}\Big(\frac{\hbar^2\boldsymbol{p}^2}{2m}+2\rho_0\lambda\Big)}$$

We can evaluate the ω integral with contours using the residue theorem. The poles are at:

$$\omega = \pm i\omega_p$$

We next need to determine our contour. The goal is to use a semicircular contour and Jordan's lemma. The choice of contour boils down to what the sign of the relevant exponent is. This will depend on the difference:

$$x - y \equiv \boldsymbol{x} - \boldsymbol{y} + \Delta\tau$$

and in particular on $\Delta\tau$. If $\Delta\tau > 0$, Jordan's lemma calls for a semi-circle in the positive half plane giving a pole at just $\omega = i\omega_p$. Similarly, a $\Delta\tau < 0$ calls for a semi-circle in the negative half plane, and a pole at $\omega = -i\omega_p$. Using the residue theorem for both of these cases gives:

$$\Big[\boldsymbol{G}(x-y)\Big]_{21} = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\text{sgn}(\Delta\tau)\hbar\omega_p + \frac{\hbar^2}{2m}\boldsymbol{p}^2 + \rho_0\lambda}{2\omega_p}e^{-\omega_p|\Delta\tau|}e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}$$

We are not out of the woods yet. We next convert to spherical polar coordinates[10]:

$$\begin{aligned}
\Big[\boldsymbol{G}(x-y)\Big]_{21} &= e^{-\omega_p|\Delta\tau|}\int_0^\infty \frac{p^2\,dp}{(2\pi)^2}\frac{\text{sgn}(\Delta\tau)\hbar\omega_p + \frac{\hbar^2}{2m}p^2 + \rho_0\lambda}{2\omega_p}\int_{-1}^1 d(\cos(\theta))e^{ip\cos(\theta)|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \frac{e^{-\omega_p|\Delta\tau|}}{4\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\int_0^\infty p\,dp\frac{\text{sgn}(\Delta\tau)\hbar\omega_p + \frac{\hbar^2}{2m}p^2 + \rho_0\lambda}{\omega_p}\sin(p|\boldsymbol{x}-\boldsymbol{y}|)
\end{aligned}$$

[10] Do not confuse p here with the 4-momenta! It is the magnitude of the 3-momenta.

The integral, as it stands, is still a mess; however, we are interested in equal times! We then say:

$$\Big[\boldsymbol{G}(x-y)\Big]_{21} = \frac{1}{4\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\int_0^\infty p\,dp\,\frac{\frac{\hbar^2}{2m}p^2+\rho_0\lambda}{\omega_p}\,\sin(p|\boldsymbol{x}-\boldsymbol{y}|)$$

For some characteristic length scale ξ of our system we consider the limit: $\xi << |\boldsymbol{x}-\boldsymbol{y}|$. The rapid oscillation which arise cause the integral to be dominated by small values of p. This tells us that:

$$\hbar\omega_p = \hbar p\sqrt{\frac{\rho_0\lambda}{m}}\sqrt{\frac{\hbar^2p^2}{4\rho_0\lambda m}+1} \sim \hbar p\sqrt{\frac{\rho_0\lambda}{m}}$$

So then:

$$p\frac{\frac{\hbar^2}{2m}p^2+\rho_0\lambda}{\omega_p} \sim \sqrt{m\rho_0\lambda}$$

Our above integral then has the following behavior[11]:

$$\begin{aligned}
\Big[\boldsymbol{G}(x-y)\Big]_{21} &\sim \frac{\sqrt{m\rho_0\lambda}}{4\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\int_0^\infty dp\,\sin(p|\boldsymbol{x}-\boldsymbol{y}|)\\
&= \frac{\sqrt{m\rho_0\lambda}}{4\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{Im}\left(\int_0^\infty dp\, e^{(i|\boldsymbol{x}-\boldsymbol{y}|-\epsilon)p}\right)\\
&= -\frac{\sqrt{m\rho_0\lambda}}{4\pi^2|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{Im}\left(\frac{1}{i|\boldsymbol{x}-\boldsymbol{y}|-\epsilon}\right)\\
&\sim \frac{\sqrt{m\rho_0\lambda}}{4\pi^2|\boldsymbol{x}-\boldsymbol{y}|^2}
\end{aligned}$$

After all that work, we can easily see that at equal time and large separations our Green function becomes[12]:

$$G(\boldsymbol{x}-\boldsymbol{y}) \to -i\rho_0$$

This is known as *off diagonal long range order* and is a signature of a superfluid![13]

4. **5/20:** We expand the field as:

$$\phi(x) = \sqrt{\rho_0+\delta\rho(x)}e^{i\theta(x)}$$

so then,

$$\bar{\phi}(x) = \sqrt{\rho_0+\delta\rho(x)}e^{-i\theta(x)}$$

Here $\rho_0 = \frac{\mu}{\lambda}$, and $\delta\rho(x)$ and $\theta(x)$ are small fluctuations. Let's expand our Lagrangian out in terms of these fields. Here is our Euclidean Lagrangian once more for reference:

$$\begin{aligned}
\mathcal{L}_E &= \bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2\Big)\phi(x) + \Big[\frac{\lambda}{2}|\phi(x)|^2-\mu\Big]|\phi(x)|^2\\
&\equiv \bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2\Big)\phi(x) + V_{eff}(\phi(x))
\end{aligned}$$

For the first term, the kinetic part, we expand by direct substitution. First, let's look at:

$$\begin{aligned}
\nabla^2\phi(x) &= \nabla^2\sqrt{\rho_0+\delta\rho(x)}e^{i\theta(x)}\\
&= e^{i\theta(x)}\nabla^2\sqrt{\rho_0+\delta\rho(x)} + \sqrt{\rho_0+\delta\rho(x)}\nabla^2e^{i\theta(x)} + 2\nabla e^{i\theta(x)}\cdot\nabla\sqrt{\rho_0+\delta\rho(x)}\\
&= e^{i\theta(x)}\left\{\frac{\nabla^2\delta\rho(x)}{2\sqrt{\rho_0+\delta\rho(x)}} - \frac{\nabla\delta\rho(x)\cdot\nabla\delta\rho(x)}{4\big(\sqrt{\rho_0+\delta\rho(x)}\,\big)^3} + i\frac{\nabla\theta(x)\cdot\nabla\delta\rho(x)}{\sqrt{\rho_0+\delta\rho(x)}} + [i\nabla^2\theta(x)-(\nabla\theta(x))^2]\sqrt{\rho_0+\delta\rho(x)}\right\}
\end{aligned}$$

[11] Note the convergence factor in the 2^{nd} line below. This sets our momentum scale.

[12] Notice the bold inputs: time dependence isn't present.

[13] As an aside, we could also try this method out for lower dimensional systems. We'd find a power law decay for systems with $D \leq 2$ space-time dimensions. This indicates there is no long range order, a continuous symmetry can not be broken and thus there are no Goldstone bosons in these lower dimensional systems. These results are the statement of the famous Mermin-Wagner-Coleman theorem.

Thus, to second order:

$$\begin{aligned}\bar{\phi}(x)\nabla^2\phi(x) &= \frac{\nabla^2\delta\rho(x)}{2} - \frac{\nabla\delta\rho(x)\cdot\nabla\delta\rho(x)}{4\big(\rho_0+\delta\rho(x)\big)} + i\nabla\theta(x)\cdot\nabla\delta\rho(x) + [i\nabla^2\theta(x) - (\nabla\theta(x))^2][\rho_0+\delta\rho(x)] \\ &\approx \frac{\nabla^2\delta\rho(x)}{2} - \frac{\nabla\delta\rho(x)\cdot\nabla\delta\rho(x)}{4\rho_0} + i\nabla\theta(x)\cdot\nabla\delta\rho(x) + i\nabla^2\theta(x)[\rho_0+\delta\rho(x)] - (\nabla\theta(x))^2\rho_0\end{aligned}$$

Quickly notice that if we do a quick integration by parts, and invoke our boundary conditions, that the following terms drop out:

$$i\nabla\theta(x)\cdot\nabla\delta\rho(x) + i\nabla^2\theta(x)\delta\rho(x) \to 0$$

In a similar manner:

$$\begin{aligned}\bar{\phi}(x)\partial_\tau\phi(x) &= \sqrt{\rho_0+\delta\rho(x)}\left(\frac{\partial_\tau\delta\rho(x)}{2\sqrt{\rho_0+\delta\rho(x)}} + i\sqrt{\rho_0+\delta\rho(x)}\partial_\tau\theta(x)\right) \\ &= \frac{\partial_\tau\delta\rho(x)}{2} + i\Big(\rho_0+\delta\rho(x)\Big)\partial_\tau\theta(x)\end{aligned}$$

We finally have:

$$\begin{aligned}\bar{\phi}(x)\Big(\hbar\partial_\tau - \frac{\hbar^2}{2m}\vec{\nabla}^2\Big)\phi(x) \approx& \frac{1}{2}\Big[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2\Big]\delta\rho(x) + i\rho_0\Big[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2\Big]\theta(x) \\ &+ i\delta\rho(x)\partial_\tau\theta(x) + \frac{\hbar^2}{2m}\Big[\frac{\nabla\delta\rho(x)\cdot\nabla\delta\rho(x)}{4\rho_0} + (\nabla\theta(x))^2\rho_0\Big]\end{aligned}$$

Now let's look at what the effective potential looks like:

$$\begin{aligned}V_{eff}(\sqrt{\rho_0+\delta\rho(x)}) &= \Big[\frac{\lambda}{2}(\rho_0+\delta\rho(x)) - \mu\Big](\rho_0+\delta\rho(x)) \\ &= \frac{\lambda}{2}(\rho_0+\delta\rho(x))(\rho_0+\delta\rho(x)) - \mu(\rho_0+\delta\rho(x)) \\ &= \frac{\lambda}{2}(\rho_0^2 + 2\rho_0\delta\rho(x) + (\delta\rho(x))^2) - \mu(\rho_0+\delta\rho(x)) \\ &= \frac{\lambda}{2}\left(\Big(\frac{\mu}{\lambda}\Big)^2 + 2\frac{\mu}{\lambda}\delta\rho(x) + (\delta\rho(x))^2\right) - \mu\Big(\frac{\mu}{\lambda} + \delta\rho(x)\Big) \\ &\to \frac{\lambda}{2}(\delta\rho(x))^2\end{aligned}$$

The right arrow is just dropping constants from our Lagrangian. Now to make some arguments on how the linear terms in $\delta\rho(x)$ fall out in the kinetic expression. The linear term is:

$$\frac{1}{2}\Big[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2\Big]\delta\rho(x)$$

and is being integrated, so we can integrate by parts to move the derivatives to the constant on the other side, and use the boundary conditions to throw this term out.

In a similar fashion the term linear in θ falls out:

$$i\rho_0\Big[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2\Big]\theta(x) \to 0$$

Where the right arrow indicates under integration by parts. Thus, our Lagrangian reduces to:

$$\mathcal{L}_E(\delta\rho,\theta) \to i\delta\rho(x)\partial_\tau\theta(x) + \frac{\hbar^2}{2m}\left[\frac{\nabla\delta\rho(x)\cdot\nabla\delta\rho(x)}{4\rho_0} + (\nabla\theta(x))^2\rho_0\right] + \frac{\lambda}{2}\big(\delta\rho(x)\big)^2$$

So our path integral:

$$Z = \int \mathcal{D}\delta\rho\mathcal{D}\theta e^{-S_E(\delta\rho,\theta)}$$

has the following effective action:

$$\begin{aligned}
S_E &= \int d\tau \int d^3\boldsymbol{x} \Bigg[i\delta\rho(x)\partial_\tau\theta(x) + \frac{\hbar^2}{2m}\Big[\frac{\nabla\delta\rho(x)\cdot\nabla\delta\rho(x)}{4\rho_0} + (\nabla\theta(x))^2\rho_0\Big] + \frac{\lambda}{2}(\delta\rho(x))^2 \Bigg] \\
&= \int d\tau \int d^3\boldsymbol{x} \Bigg[i\delta\rho(x)\partial_\tau\theta(x) + \frac{\hbar^2}{2m}\Big[\frac{-\delta\rho(x)\nabla^2\delta\rho(x)}{4\rho_0} + (\nabla\theta(x))^2\rho_0\Big] + \frac{\lambda}{2}(\delta\rho(x))^2 \Bigg] \qquad (5.2) \\
&= \int d\tau \int d^3\boldsymbol{x} \Bigg[\delta\rho(x)\Big[\frac{\lambda}{2} - \frac{\hbar^2}{2m}\frac{\nabla^2}{4\rho_0}\Big]\delta\rho(x) + i\delta\rho(x)\partial_\tau\theta(x) - \frac{\hbar^2}{2m}\rho_0\theta(x)\nabla^2\theta(x) \Bigg]
\end{aligned}$$

We have seen this sort of thing before and we know what to do: we decouple the fields by defining a new reference with $\delta\rho(x) = \delta\rho_0(x) + \xi(x)$. Let's call $i\partial_\tau\theta(x) = J(x)$, $\hat{A} = \Big[\frac{\lambda}{2} - \frac{\hbar^2}{2m}\frac{\nabla^2}{4\rho_0}\Big]$, and its Green function $G_A(\boldsymbol{x}-\boldsymbol{x'})$, then we simply choose:

$$\delta\rho_0(x) = -\frac{1}{2}\int d^3\boldsymbol{x'}G_A(\boldsymbol{x}-\boldsymbol{x}\,')J(\tau,\boldsymbol{x}\,')$$

and we can see:

$$\begin{aligned}
S_E(\delta\rho,\theta) &= \int d\tau \int d^3\boldsymbol{x} \Bigg[\delta\rho(x)\Big[\frac{\lambda}{2} - \frac{\hbar^2}{2m}\frac{\nabla^2}{4\rho_0}\Big]\delta\rho(x) + i\delta\rho(x)\partial_\tau\theta(x) - \frac{\hbar^2}{2m}\rho_0\theta(x)\nabla^2\theta(x) \Bigg] \\
\to S_E(\xi,\theta) &= \int d\tau \int d^3\boldsymbol{x} \Bigg[\xi(x)\Big[\frac{\lambda}{2} - \frac{\hbar^2}{2m}\frac{\nabla^2}{4\rho_0}\Big]\xi(x) + \frac{1}{4}\int d^3\boldsymbol{x'}\partial_\tau\theta(\tau,\boldsymbol{x})G_A(\boldsymbol{x}-\boldsymbol{x}\,')\partial_\tau\theta(\tau,\boldsymbol{x}\,') - \frac{\hbar^2}{2m}\rho_0\theta(x)\nabla^2\theta(x) \Bigg]
\end{aligned}$$

We are almost done! We use the invariance of our integration measure to rewrite our partition function, and then integrate out ξ, which will just be a normalization constant:

$$\begin{aligned}
Z &= \int \mathcal{D}\delta\rho\mathcal{D}\theta e^{-S_E(\delta\rho,\theta)} \\
&= \int \mathcal{D}\xi\mathcal{D}\theta e^{-S_E(\xi,\theta)} \\
&= \mathcal{N}\int \mathcal{D}\theta e^{-S_E(\theta)}
\end{aligned}$$

The effective action for the phase variable is:

$$S_E(\theta) = \int d\tau \int d^3\boldsymbol{x} \Bigg[-\frac{\hbar^2}{2m}\rho_0\theta(x)\nabla^2\theta(x) + \frac{1}{4}\int d^3\boldsymbol{x'}\partial_\tau\theta(\tau,\boldsymbol{x})G_A(\boldsymbol{x}-\boldsymbol{x}\,')\partial_\tau\theta(\tau,\boldsymbol{x}\,') \Bigg]$$

We see the kinetic term is non-local as it stands. Now that was a lot of work and really puts the techniques we have learned to the test! The reward was worth it however.

5. **3/20:** Now we consider a slowly varying configurations of our field $\theta(\tau,\boldsymbol{x})$. We can resolve the issue of non-locality if we look at the Fourier transform of our Green function:

$$\begin{aligned}
G_A(\boldsymbol{x}-\boldsymbol{x'}) &= 8m\rho_0\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x'})}}{\hbar^2|\boldsymbol{p}|^2 + 4m\lambda\rho_0} \\
&\approx 8m\rho_0\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x'})}}{4m\lambda\rho_0} \\
&= \frac{2}{\lambda}\delta(\boldsymbol{x}-\boldsymbol{x'})
\end{aligned}$$

Thus, our action becomes:

$$S_E(\theta) = \int d\tau \int d^3\boldsymbol{x} \Bigg[\frac{\rho_0\hbar^2}{2m}\Big(\nabla\theta(x)\Big)^2 + \frac{1}{2\lambda}\Big(\partial_\tau\theta(x)\Big)^2 + \dots \Bigg]$$

After Wick rotating back, we have:

$$\begin{aligned}
S(\theta) &= \int d^4x \left[\frac{\rho_0\hbar^2}{2m}\Big(\nabla\theta(x)\Big)^2 - \frac{1}{2\lambda}\Big(\partial_t\theta(x)\Big)^2 + \dots \right] \\
&= \int d^4x\ \theta(x)\Big[-\frac{\rho_0\hbar^2}{2m}\nabla^2 + \frac{1}{2\lambda}\partial_t^2 + \dots \Big]\theta(x) \\
\left(\theta \to \sqrt{\frac{m}{\rho_0\hbar^2}}\theta\right) &\to \int d^4x\ \theta(x)\frac{1}{2}\Big[-\nabla^2 + \frac{m}{\hbar^2\rho_0\lambda}\partial_t^2 + \dots \Big]\theta(x) \\
&= \int d^4x\ \theta(x)\frac{1}{2}\Big[-\nabla^2 + \frac{1}{v^2}\partial_t^2 + \dots \Big]\theta(x)
\end{aligned}$$

This is just the action of a *massless Klein-Gordon field* with the speed of light equal to: $v^2 = \frac{\rho_0\hbar^2\lambda}{m} = \frac{\mu\hbar^2}{m}$. Indeed, we can see that the propagator for this scalar field is of the form:

$$\begin{aligned}
G(x-x') &= i\Big\langle T\big(\theta(x)\theta(x')\big)\Big\rangle \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - \frac{\omega^2}{v^2} + i\epsilon}e^{ip(x-x')} \\
&\propto \frac{1}{v^2(t-t')^2 - (\boldsymbol{x}-\boldsymbol{x}')^2}
\end{aligned}$$

Which satisfies the following equation of motion:

$$\Big[-\nabla^2 + \frac{1}{v^2}\partial_t^2\Big]\Big\langle T\big(\theta(x)\theta(x')\big)\Big\rangle = -i\delta(x-x')$$

Both of these equations are that of a *relativistic massless Klein-Gordon field.*

Note we can also determine the behavior of the following propagator in the limit of large separations and equal time:

$$\langle T\big(\phi(x)\phi^\dagger(x')\big)\rangle \approx \rho_0 e^{-\langle T[\theta(x)-\theta(x')]^2/2\rangle} \to \rho_0 + \frac{c}{|\boldsymbol{x}-\boldsymbol{x}'|^2} + \dots$$

Here c is a constant of order 1. Again we get long-ranged order! Try this calculation out in a lower space-time dimension if you have time, and see what you get.

Appendix: Derivation of Bose Coherent state integral

Let's find the path integral using the Bose coherent states:

$$|\{\phi(\boldsymbol{x})\}\rangle = e^{\int d\boldsymbol{x}\phi(\boldsymbol{x})\hat{\phi}^\dagger(\boldsymbol{x})}|0\rangle$$

With the coherent states we know that the Resolution of the Identity takes the form:

$$\mathcal{I} = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}e^{-\int d\boldsymbol{x}|\phi(\boldsymbol{x})|^2}|\{\phi\}\rangle\langle\{\phi\}|$$

From here we can repeat the general prescription of slicing up time and inserting the resolution of the identity in between each segment. Let's take: $\Delta t = \epsilon N$; what I am about to do with this should be all too familiar. Breaking the propagator up into tiny segments and inserting our resolution of our identity between each segment, we find:

$$\langle f|e^{-\frac{i}{\hbar}\hat{H}\Delta t}|i\rangle = \langle f|\big[1 - \frac{i}{\hbar}\hat{H}\Delta t\big]^N|i\rangle$$

$$= \int \prod_j^N \mathcal{D}\phi_j\mathcal{D}\bar{\phi}_j\, e^{-\sum_j\int d\boldsymbol{x}|\phi_j(\boldsymbol{x})|^2}\left[\prod_k^{N-1}\langle\{\phi_{k+1}\}|\big[1 - \frac{i}{\hbar}\hat{H}\Delta t\big]|\{\phi_k\}\rangle\right]$$

$$\times\langle f|\big[1 - i\frac{\epsilon}{\hbar}\hat{H}\big]|\{\phi_N\}\rangle\big[1 - i\frac{\epsilon}{\hbar}\hat{H}\big]\langle\{\phi_1\}|i\rangle$$

Let's suppose the Hamiltonian is a normal ordered operator, then we know that:

$$\langle\{\phi_{k+1}\}|\Big[1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})\Big]|\{\phi_k\}\rangle = \langle\{\phi_{k+1}\}|\{\phi_k\}\rangle\Big[1 - i\frac{\epsilon}{\hbar}H(\{\bar{\phi}_{k+1}\}, \{\phi_k\}\Big]$$

Our path integral now becomes:

$$\langle f|e^{-\frac{i}{\hbar}\hat{H}\Delta t}|i\rangle = \int\Big[\prod_j^N\mathcal{D}\phi_j\mathcal{D}\bar{\phi}_j\Big]\, e^{-\sum_j^N\int d\boldsymbol{x}|\phi_j(\boldsymbol{x})|^2}e^{-\sum_k^{N-1}\int d\boldsymbol{x}\phi_{k+1}(\boldsymbol{x})\phi_k(\boldsymbol{x})}\left[\prod_k^{N-1}\Big(1 - i\frac{\epsilon}{\hbar}H(\{\bar{\phi}_{k+1}\}, \{\phi_k\})\Big)\right]$$

$$\times\langle f|\big[1 - i\frac{\epsilon}{\hbar}\hat{H}\big]|\{\phi_N\}\rangle\langle\{\phi_1\}|\big[1 - i\frac{\epsilon}{\hbar}\hbar\hat{H}\big]|i\rangle$$

Finally, let's expand the initial and final states using our coherent states; we find:

$$\langle f| = \langle f|\mathcal{I} = \int\mathcal{D}\phi_f\mathcal{D}\bar{\phi}_f e^{-\int d\boldsymbol{x}|\phi_f(\boldsymbol{x})|^2}\langle f|\{\phi_f\}\rangle\langle\{\phi_f\}| = \int\mathcal{D}\phi_f\mathcal{D}\bar{\phi}_f e^{-\int d\boldsymbol{x}|\phi_f(\boldsymbol{x})|^2}\bar{\psi}_f(\phi(\boldsymbol{x}, t_f))\langle\{\phi_f\}|$$

and,

$$|i\rangle = \mathcal{I}|i\rangle = \int\mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i e^{-\int d\boldsymbol{x}|\phi_i(\boldsymbol{x})|^2}|\{\phi_i\}\rangle\langle\{\phi_i\}|i\rangle = \int\mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i e^{-\int d\boldsymbol{x}|\phi_i(\boldsymbol{x})|^2}|\{\phi_i\}\rangle\psi_i(\bar{\phi}(\boldsymbol{x}, t_i))$$

We then see that the last term becomes:

$$\langle f|\big[1 - i\frac{\epsilon}{\hbar}\hat{H}\big]|\{\phi_N\}\rangle\langle\{\phi_1\}|\big[1 - i\frac{\epsilon}{\hbar}\hbar\hat{H}\big]|i\rangle = \int\int\mathcal{D}\phi_f\mathcal{D}\bar{\phi}_f\mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i\,\bar{\psi}_f\big(\phi(\boldsymbol{x}, t_f)\big)\psi_i\big(\bar{\phi}(\boldsymbol{x}, t_i)\big)e^{-\int d\boldsymbol{x}\Big(|\phi_f(\boldsymbol{x})|^2 + |\phi_i(\boldsymbol{x})|^2\Big)} \tag{5.3}$$

$$\times\big[1 - i\frac{\epsilon}{\hbar}H\big(\{\bar{\phi}_f\}, \{\phi_N\}\big)\big]\big[1 - i\frac{\epsilon}{\hbar}H\big(\{\bar{\phi}_1\}, \{\phi_i\}\big)\big]\langle\{\phi_f\}|\{\phi_N\}\rangle\langle\{\phi_1\}|\{\phi_i\}\rangle \tag{5.4}$$

The remaining bits are the same as the coherent state path integral derived in the chapter: we expand the overlaps, group like terms, and integrate by parts. Our final result is:

$$\int\mathcal{D}\phi\mathcal{D}\bar{\phi}\exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f}dt\bigg(\int d^dx\frac{\hbar}{i}\Big[\phi(\boldsymbol{x}, t)\partial_t\bar{\phi}(\boldsymbol{x}, t) - \bar{\phi}(\boldsymbol{x}, t)\partial_t\phi(\boldsymbol{x}, t)\Big] - H[\phi, \bar{\phi}]\bigg)\right\}$$

$$\times\bar{\psi}_f(\phi(\boldsymbol{x}, t_f))\psi_i(\bar{\phi}(\boldsymbol{x}, t_i))e^{\frac{1}{2}\int d^3x\Big(|\phi(\boldsymbol{x}, t_f)|^2 + |\phi(\boldsymbol{x}, t_i)|^2\Big)}$$