Homework 6 Solutions

Matthew O'Brien

Total Points: 100

Question 1 (Path-Integral Quantization of the Free Electromagnetic Field)

1. [4/40] The path integral for a $U(1)$ gauge field A_μ coupled to a classical source J_μ is

$$\mathcal{Z}[J] = \int \mathcal{D}A_\mu(x)\, e^{iS[A,J]}, \tag{1.1.1a}$$

$$S[A,J] = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu\right). \tag{1.1.1b}$$

However, this expression "over-counts" physically equivalent field configurations which are related by a gauge transformation. To remedy this problem, we use the Faddeev-Popov construction by defining

$$\Delta^{-1}[A_\mu] = \int \mathcal{D}U\, \delta\left(g(A_\mu^U)\right), \tag{1.1.2}$$

where $\Delta[A_\mu]$ is the Faddeev-Popov determinant—to be discussed in the next part—$A_\mu^U = U \cdot A_\mu$ is the result of applying the gauge group transformation $U \in G = U(1)$ on the gauge field A_μ, and $g(A_\mu)$ is a gauge-fixing function, which, in the Feynman-'t Hooft family of gauges, takes the form

$$g(A_\mu) = \partial^\mu A_\mu(x) - \omega(x), \tag{1.1.3}$$

for an arbitrary scalar function $\omega(x)$; gauge-fixing will be achieved by integrating over $\omega(x)$. Therefore, the path integral which correctly weights gauge-equivalent configurations is

$$\mathcal{Z}[J] = \int \mathcal{D}A_\mu(x)\, \Delta[A_\mu] \int \mathcal{D}U\, \delta\left(g(A_\mu^U)\right) e^{iS[A,J]}. \tag{1.1.4}$$

2. [4/40] By a change of variables, we can write the Faddeev-Popov determinant in the form

$$\Delta^{-1}[A_\mu] = \int \mathcal{D}g \det\left|\frac{\delta U}{\delta g}\right| \delta(g) = \left|\frac{\delta U}{\delta g}\right|_{g=0}. \tag{1.2.1}$$

For the Maxwell gauge field, the general group element is $U(x) = e^{i\phi(x)}$, and the transformed field is

$$A_\mu^U(x) = A_\mu(x) + \partial_\mu\phi(x). \tag{1.2.2}$$

Therefore, the gauge-fixing function applied to the transformed field is

$$g(A_\mu^U) = \partial^\mu A_\mu(x) + \partial^2\phi(x) - \omega(x), \tag{1.2.3}$$

and hence, the Faddeev-Popov determinant is

$$\begin{aligned}\Delta[A_\mu] &= \det\left|\frac{\delta g(A_\mu^U(x))}{\delta\phi(y)}\frac{\partial\phi(y)}{\partial U(y)}\right|_{g=0} \\ &= \det\left|\partial^2\delta(x-y)\right| \\ &= \det\left(\partial^2\right),\end{aligned} \tag{1.2.4}$$

which is manifestly independent of the gauge field configuration A_μ.

3. **[5/40]** The path integral then takes the form

$$\mathcal{Z}[J] = \det\left(\partial^2\right) \int \mathcal{D}A_\mu(x) \int \mathcal{D}U\, \delta\left(g(A_\mu^U)\right) e^{iS[A,J]}. \tag{1.3.1}$$

Consider a fixed, yet arbitrary element $U' \in G$. Under a gauge transformation by U' this partition function becomes

$$\mathcal{Z}'[J] = \det\left(\partial^2\right) \int \mathcal{D}A_\mu^{U'}(x) \int \mathcal{D}U\, \delta\left(g(A_\mu^{U'U})\right) e^{iS[A^{U'},J]}. \tag{1.3.2}$$

Therefore, we wish to show that this expression is actually equivalent to the original partition function. First, we consider the action: If the gauge transformation has the form of Eq. (1.2.2),

$$\int \mathrm{d}^4x\, \mathcal{L} \longrightarrow \int \mathrm{d}^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu + J_\mu \partial^\mu \phi\right], \tag{1.3.3}$$

since $F_{\mu\nu}$ is gauge invariant. Since the current is conserved, we can integrate the last term by parts to see

$$\int \mathrm{d}^4x\, J_\mu \partial^\mu \phi = -\int \mathrm{d}^4x\, \phi \partial_\mu J^\mu = 0. \tag{1.3.4}$$

Therefore, the action is gauge invariant as long as the sources are locally conserved. Next, we consider the Haar measure term. Using the property $\mathcal{D}U = \mathcal{D}(U'U)$ for any $U' \in G$, we can write

$$\begin{aligned} \int \mathcal{D}U\, \delta\left(g(A_\mu^{U'U})\right) &= \int \mathcal{D}(U'U)\, \delta\left(g(A_\mu^{U'U})\right) \\ &= \int \mathcal{D}U''\, \delta\left(g(A_\mu^{U''})\right) = \int \mathcal{D}U\, \delta\left(g(A_\mu^{U})\right), \end{aligned} \tag{1.3.5}$$

where we defined $U'' = U'U$, and used the fact that U'' is a "dummy" integration variable. Therefore, this term is also gauge invariant. Finally, we note that as far as the path integration measure is concerned, the gauge transformation amounts to a shift of coordinates

$$\mathcal{D}A_\mu^{U'} = \mathcal{D}(A_\mu + \partial_\mu \phi) = \mathcal{D}A_\mu, \tag{1.3.6}$$

since $\phi(x)$ is a fixed function. To convince yourself this is true, think about how we evaluated the path integral for a scalar field by defining $\varphi(x) = \bar{\varphi}(x) + \xi(x)$, where $\bar{\varphi}(x)$ was the solution of the classical equation of motion in the presence of sources. Therefore, we have shown that

$$\mathcal{Z}'[J] = \det\left(\partial^2\right) \int \mathcal{D}A_\mu(x) \int \mathcal{D}U\, \delta\left(g(A_\mu^U)\right) e^{iS[A,J]} = \mathcal{Z}[J], \tag{1.3.7}$$

that is, the partition function is gauge invariant.

4. **[6/40]** We can then exploit this gauge invariance by performing the transformation

$$A_\mu^U \longrightarrow A_\mu^{U^{-1}U} = A_\mu, \tag{1.4.1}$$

to factorize the Haar measure:

$$\mathcal{Z}[J] = \det\left(\partial^2\right) \left[\int \mathcal{D}U\right] \int \mathcal{D}A_\mu(x)\, \delta\left(g(A_\mu)\right) e^{iS[A,J]}. \tag{1.4.2}$$

This only yields an (infinite) normalization constant, which we will ignore from hereon. Then, to obtain the Feynman-'t Hooft gauge-fixed path integral, we perform a functional integral over configurations of the scalar function $\omega(x)$ which appear in $g(A_\mu)$, with a Gaussian weight, which eliminates the delta function:

$$\begin{aligned}\mathcal{Z}_\alpha[J] &= \det\left(\partial^2\right) \int \mathcal{D}\omega(x)\, \exp\left(-i\int \mathrm{d}^4x \frac{\omega^2}{2\alpha}\right) \int \mathcal{D}A_\mu(x)\, \delta\left(\partial^\mu A_\mu - \omega\right) e^{iS[A,J]} \\ &= \det\left(\partial^2\right) \int \mathcal{D}A_\mu(x) \exp\left[i\int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right)\right].\end{aligned} \tag{1.4.3}$$

We can now work with the gauge-fixed Lagrangian

$$\begin{aligned}\mathcal{L}_\alpha &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \\ &= -\frac{1}{2}g^{\mu\alpha}g^{\nu\beta}\Big(\partial_\mu A_\nu \partial_\alpha A_\beta - \partial_\mu A_\nu \partial_\beta A_\alpha\Big) - \frac{1}{2\alpha}(\partial^\mu A_\mu)(\partial^\nu A_\nu) + J_\mu A^\mu,\end{aligned} \tag{1.4.4}$$

which, after integrating by parts, becomes

$$\begin{aligned}\mathcal{L}_\alpha &= -\frac{1}{2}g^{\mu\alpha}g^{\nu\beta}\Big(- A_\nu \partial_\mu\partial_\alpha A_\beta + A_\nu \partial_\mu \partial_\beta A_\alpha\Big) + \frac{1}{2\alpha}A_\mu \partial^\mu\partial^\nu A_\nu + J_\mu A^\mu \\ &= \frac{1}{2}A_\mu\left[g^{\mu\nu}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\nu\right]A_\nu + J_\mu A^\mu.\end{aligned} \tag{1.4.5}$$

It follows that the Euler-Lagrange equation of motion for A_μ is

$$\left[g^{\mu\nu}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\nu\right]A_\nu = -J^\mu. \tag{1.4.6}$$

This equation can be inverted to write the the solution in terms of a Green function/propagator

$$\begin{aligned}A_\mu(x) &= -\int \mathrm{d}^4y\, \langle x| \left[g^{\mu\nu}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\nu\right]^{-1} |y\rangle\, J^\nu(y) \\ &\doteq \int \mathrm{d}^4y\, D^F_{\mu\nu}(x-y)J^\nu(y),\end{aligned} \tag{1.4.7}$$

from which we can identify that $D^F_{\mu\nu}$ satisfies the inhomogeneous partial differential equation

$$-\left[g^{\mu\lambda}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\lambda\right]D^F_{\lambda\nu}(x-y) = g^\mu{}_\nu\delta^{(4)}(x-y). \tag{1.4.8}$$

5. **[8/40]** To determine the tensorial structure of the propagator, we need to evaluate the matrix part of the operator inverse in Eq. (1.4.7). By noting that $(\partial_\mu\partial_\nu)^2 = (\partial^2)^2$, we make an *ansatz* which is consistent with this Lorentz index structure:

$$\left[g^{\mu\nu}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\nu\right]^{-1} = A(\partial^2)g_{\mu\nu} + B(\partial^2)\partial_\mu\partial_\nu, \tag{1.5.1}$$

where $A(\partial^2)$ and $B(\partial^2)$ are coefficient functions to be determined. Throughout, we will blithely assume differential operators commute; that is, we restrict our attention to a space of sufficiently smooth functions. Multiplying both sides by the original differential operator, we have

$$\begin{aligned} g^{\mu}{}_{\nu} &= \left[g^{\mu\lambda}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\lambda\right]\left[A(\partial^2)g_{\lambda\nu} + B(\partial^2)\partial_\lambda\partial_\nu\right] \\ &= A(\partial^2)g^{\mu}{}_{\nu}\partial^2 - A(\partial^2)\left(\frac{\alpha-1}{\alpha}\right)\partial^\mu\partial_\nu + B(\partial^2)\left[\partial^2\partial^\mu\partial_\nu - \frac{\alpha-1}{\alpha}\partial^\mu\partial^2\partial_\nu\right]. \end{aligned} \tag{1.5.2}$$

Therefore, by comparing coefficients, we identify that

$$A(\partial^2) = \frac{1}{\partial^2}, \tag{1.5.3a}$$

$$B(\partial^2)\frac{\partial^2}{\alpha}\partial^\mu\partial_\nu = A(\partial^2)\left(\frac{\alpha-1}{\alpha}\right)\partial^\mu\partial_\nu. \tag{1.5.3b}$$

Finally, this implies that

$$\left[g^{\mu\nu}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\nu\right]^{-1} = \left[g_{\mu\nu} + (\alpha-1)\frac{\partial_\mu\partial_\nu}{\partial^2}\right]\frac{1}{\partial^2}, \tag{1.5.4}$$

and hence, the propagator takes the form

$$D^F_{\mu\nu}(x-y) = -\langle x|\left[g_{\mu\nu} + (\alpha-1)\frac{\partial_\mu\partial_\nu}{\partial^2}\right]\frac{1}{\partial^2}|y\rangle\,. \tag{1.5.5}$$

By identifying the propagator of the free massless scalar field

$$G^{(0)}(x-y) = \langle x|\frac{1}{\partial^2}|y\rangle\,, \tag{1.5.6}$$

we see that the gauge field propagator can be written as

$$D^F_{\mu\nu}(x-y) = -\left[g_{\mu\nu} + (\alpha-1)\frac{\partial_\mu\partial_\nu}{\partial^2}\right]G^{(0)}(x-y) = g_{\mu\nu}D_F(x-y) + \ldots, \tag{1.5.7}$$

where $D_F(x-y) = -G^{(0)}(x-y)$. A good way of convincing yourself that this relation holds is to write the propagator in momentum space. We evaluated the scalar field propagator in Homework 4, so we know that

$$D^F_{\mu\nu}(x-y) = g_{\mu\nu}\frac{i}{4\pi^2 s^2} + \ldots, \tag{1.5.8}$$

where $s^2 = (x_\mu - y_\mu)(x^\mu - y^\mu)$.

6. **[8/40]** We now return to the path integral. We perform the functional integral by the usual method and write

$$A_\mu = \tilde{A}_\mu + \xi_\mu, \tag{1.6.1}$$

where $\tilde{A}_\mu$ is given by (1.4.7), and ξ_μ represents fluctuations around this classical solution. Therefore denoting the differential operator by $\hat{O}^{\mu\nu}$, we can simplify the Lagrangian to

$$\begin{aligned} \mathcal{L}_\alpha &= \frac{1}{2}\tilde{A}_\mu\hat{O}^{\mu\nu}\tilde{A}_\nu + \frac{1}{2}\xi_\mu\hat{O}^{\mu\nu}\tilde{A}_\nu + \frac{1}{2}\tilde{A}_\mu\hat{O}^{\mu\nu}\xi_\nu + \frac{1}{2}\xi_\mu\hat{O}^{\mu\nu}\xi_\nu + J_\mu\tilde{A}^\mu + J_\mu\xi^\mu \\ &= \frac{1}{2}\xi_\mu\hat{O}^{\mu\nu}\xi_\nu + \frac{1}{2}J_\mu\tilde{A}^\mu, \end{aligned} \tag{1.6.2}$$

and hence, the path integral becomes

$$\begin{aligned}\mathcal{Z}_\alpha[J] &= \det\left(\partial^2\right)\int \mathcal{D}A_\mu(x)\, e^{i\int \mathrm{d}^4x\mathcal{L}_\alpha} \\ &= \det\left(\partial^2\right)\det\left[g^{\mu\nu}\partial^2 - \frac{\alpha-1}{\alpha}\partial^\mu\partial^\nu\right]^{-1/2} \\ &\qquad \times \exp\left[\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y J_\mu(x)D_F^{\mu\nu}(x-y)J_\nu(y)\right].\end{aligned} \tag{1.6.3}$$

First, note that the functional determinant in the prefactor which depends on α simply goes into the normalization of the path integral, so we can ignore it. Then, to see that the exponent is independent of α, we can use the form of $D_F^{\mu\nu}$ from Eq. (1.5.7) to simplify the exponent, and integrate by parts in x,

$$\begin{aligned}&\int \mathrm{d}^4x\mathrm{d}^4y J_\mu(x)J_\nu(y)D_F^{\mu\nu}(x-y) \\ &\quad= -\int \mathrm{d}^4x\mathrm{d}^4y J_\mu(x)J_\nu(y)\left[g^{\mu\nu} + (\alpha-1)\frac{\partial_x^\mu\partial_x^\nu}{\partial_x^2}\right]G^{(0)}(x-y) \\ &\quad= -\int \mathrm{d}^4x\mathrm{d}^4y J_\mu(x)G^{(0)}(x-y)J^\mu(y) \\ &\qquad\quad + (\alpha-1)\int \mathrm{d}^4x\mathrm{d}^4y(\partial_x^\mu J_\mu(x))\frac{J_\nu(y)\partial_x^\nu}{\partial_x^2}G^{(0)}(x-y) \\ &\quad= -\int \mathrm{d}^4x\mathrm{d}^4y J_\mu(x)G^{(0)}(x-y)J^\mu(y),\end{aligned} \tag{1.6.4}$$

where the notation ∂_x^μ denotes that the derivative is acting on the x variable and not y. Therefore, we have an expression which is now manifestly independent of α. Notably, this implies that any physical (gauge invariant) correlation function will be independent of α.

7. **[5/40]** The propagator in momentum space can be directly read off from Eq. (1.4.7) by writing

$$\begin{aligned}D_{\mu\nu}^F(x-y) &= \left[g_{\mu\nu} + (\alpha-1)\frac{\partial_\mu\partial_\nu}{\partial^2}\right]\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2+i\varepsilon} \\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\left[g_{\mu\nu} + (\alpha-1)\frac{p_\mu p_\nu}{p^2}\right]\frac{e^{-ip\cdot(x-y)}}{p^2+i\varepsilon},\end{aligned} \tag{1.7.1}$$

and hence,

$$D_{\mu\nu}^F(p) = \left[g_{\mu\nu} + (\alpha-1)\frac{p_\mu p_\nu}{p^2}\right]\frac{1}{p^2+i\varepsilon}, \tag{1.7.2}$$

where we have specified the $i\varepsilon$ prescription so that $D_{\mu\nu}^F(p)$ corresponds to the Feynman propagator.

Question 2 (One-Dimensional Quantum Heisenberg Antiferromagnet)

We begin by summarizing the key results from Homework 3:

The Hamiltonian for a one-dimensional Heisenberg antiferromagnet in the linear spin wave approximation (in the limit of an infinite chain) is

$$H = E_0 + \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}q}{2\pi} \omega_q \Big[n_c(q) + n_d(q) \Big], \tag{2.0.1}$$

where $n_c = c^\dagger c$, $n_d = d^\dagger d$ are the Bogoliubov number operators, and $\omega_q = 2JS|\sin(q)|$. The operators satisfy the commutation relations

$$[c(q), c^\dagger(q')] = [d(q), d^\dagger(q')] = (2\pi)\delta(q - q'). \tag{2.0.2}$$

The Bogoliubov operators are related to the Holstein-Primakoff operators $a(q)$ and $b(q)$ by the relations

$$a(q) = \cosh\theta(q)\, c(q) - \sinh\theta(q)\, d^\dagger(q), \tag{2.0.3a}$$

$$b(q) = \cosh\theta(q)\, d(q) - \sinh\theta(q)\, c^\dagger(q), \tag{2.0.3b}$$

$$\theta(q) = \frac{1}{2} \operatorname{arctanh} \cos(q). \tag{2.0.3c}$$

In the linear spin wave approximation, the position space bosonic operators are related to the spin operators as

$$S^+(j) = \sqrt{2S} a(j), \qquad S^-(j) = \sqrt{2S} a^\dagger(j), \qquad S_3(j) = S - n_a(j), \qquad \text{even sites}, \tag{2.0.4a}$$

$$S^+(j) = \sqrt{2S} b^\dagger(j), \qquad S^-(j) = \sqrt{2S} b(j), \qquad S_3(j) = -S + n_b(j), \qquad \text{odd sites}, \tag{2.0.4b}$$

where the bosonic operators satisfy the commutation relations

$$[a(j), a^\dagger(k)] = \delta_{jk}, \qquad [b(j), b^\dagger(k)] = \delta_{jk}. \tag{2.0.5}$$

Note that since the reduced Brillouin zone of the antiferromagnet is $[-\pi/2, \pi/2]$, the commutation relations in position and momentum space are only consistent if the respective operators are related by the Fourier transform

$$a(q) = \sqrt{2} \sum_{j\,\text{even}} e^{iqj} a(j), \qquad a(j) = \sqrt{2} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}q}{2\pi} e^{-iqj} a(q), \tag{2.0.6}$$

and similarly for the b operators.

1. [5/5] Consider the propagator

$$\begin{aligned} D_{ij}(n, t|n', t') &= -i \langle \text{gnd}| T S_i(n,t) S_j(n',t') |\text{gnd}\rangle \\ &= -i \Big[\Theta(t - t') \langle \text{gnd}| S_i(n,t) S_j(n',t') |\text{gnd}\rangle \\ &\qquad + \Theta(t' - t) \langle \text{gnd}| S_j(n',t') S_i(n,t) |\text{gnd}\rangle \Big]. \end{aligned} \tag{2.1.1}$$

In momentum and frequency space,

$$
\begin{aligned}
D_{ij}(p,\omega) &= \sum_n \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega t} e^{iqn} D_{ij}(n,t|0,0) \\
&= -i \sum_n e^{iqn} \left[\int_0^{\infty} \mathrm{d}t\, e^{-i\omega t} \langle \mathrm{gnd}| S_i(n,t) S_j(0) |\mathrm{gnd}\rangle \right. \\
&\qquad \left. + \int_{-\infty}^{0} \mathrm{d}t\, e^{-i\omega t} \langle \mathrm{gnd}| S_j(0) S_i(n,t) |\mathrm{gnd}\rangle \right]
\end{aligned} \tag{2.1.2}
$$

However, we must ensure the inverse Fourier transform reproduces the correct causal response. To do so, we observe that the exponential in the first term converges correctly as $t \to \infty$ as long as $\operatorname{Im}\omega < 0$, while for the second term we must have $\operatorname{Im}\omega > 0$. Therefore, we must define the Fourier transform as

$$
\begin{aligned}
D_{ij}(p,\omega) = -i \sum_n e^{iqn} \lim_{\varepsilon \to 0^+} &\left[\int_0^{\infty} \mathrm{d}t\, e^{-i(\omega - i\varepsilon)t} \langle \mathrm{gnd}| S_i(n,t) S_j(0) |\mathrm{gnd}\rangle \right. \\
&\left. + \int_{-\infty}^{0} \mathrm{d}t\, e^{-i(\omega + i\varepsilon)t} \langle \mathrm{gnd}| S_j(0) S_i(n,t) |\mathrm{gnd}\rangle \right].
\end{aligned} \tag{2.1.3}
$$

This is equivalent to deforming the frequency integration contour into the complex plane for the inverse transform. Note that we would have had to use the opposite signs for the $i\varepsilon$ prescription if we had started from a Fourier transform using the $e^{i\omega t}$ convention. We could also have come to the above conclusion by noting that the Heaviside step function can be written as a Fourier transform

$$
\Theta(t-t') = \int \frac{\mathrm{d}z}{2\pi i} \frac{e^{iz(t-t')}}{z - i\varepsilon}. \tag{2.1.4}
$$

By shifting the pole of the integrand into the upper half plane, we ensure that there is an unambiguous causal response; $\Theta(t-t') = 0$ for $t < t'$ and $\Theta(t-t') = 1$ for $t > t'$.

2. **[5++]** We now consider an external applied magnetic field represented by the interaction Hamiltonian

$$
H_{\mathrm{ext}} = \sum_n B_k(n,t) S_k(n,t). \tag{2.2.1}
$$

Linear response theory tells us that the magnetic field will change the vacuum expectation value of the spin components by

$$
\begin{aligned}
\delta \langle \mathrm{gnd}| S_j(n,t) |\mathrm{gnd}\rangle &= i \int_{-\infty}^{t} \mathrm{d}t' \langle \mathrm{gnd}| [H_{\mathrm{ext}}(t'), S_j(n,t)] |\mathrm{gnd}\rangle \\
&= i \int_{-\infty}^{t} \mathrm{d}t' \sum_{n'} \langle \mathrm{gnd}| [S_k(n',t'), S_j(n,t)] |\mathrm{gnd}\rangle B_k(n',t') \\
&\doteq \int_{-\infty}^{\infty} \mathrm{d}t' \sum_{n'} \chi_{jk}(n,t|n',t') B_k(n',t'),
\end{aligned} \tag{2.2.2}
$$

where χ_{jk} is the magnetic susceptibility tensor

$$
\chi_{jk}(n,t|n',t') = -i \langle \mathrm{gnd}| [S_j(n,t), S_k(n',t')] |\mathrm{gnd}\rangle \Theta(t-t') = D^R_{jk}(n,t|n',t'), \tag{2.2.3}
$$

which corresponds identically to a retarded correlation function.

3. [**10**++] We now use the results from part 1 to obtain the propagators in the spin-wave approximation.

(a) Consider the longitudinal function

$$\begin{aligned} D_{33}(n,t|n',t') &= -i\Theta(t-t')\langle\text{gnd}|\, S_3(n,t)S_3(n',t')\,|\text{gnd}\rangle \\ &\quad - i\Theta(t'-t)\langle\text{gnd}|\, S_3(n',t')S_3(n,t)\,|\text{gnd}\rangle \\ &= -i\int \frac{\mathrm{d}z}{2\pi i} e^{iz(t-t')}\left[\frac{1}{z-i\varepsilon}\langle\text{gnd}|\, S_3(n,t)S_3(n',t')\,|\text{gnd}\rangle \right. \\ &\quad \left. -\frac{1}{z+i\varepsilon}\langle\text{gnd}|\, S_3(n',t')S_3(n,t)\,|\text{gnd}\rangle\right]. \end{aligned} \tag{2.3.1}$$

We need to consider several cases separately, depending on whether n and n' are even or odd lattice sites. We begin by considering the case where n and n' are both sites on the even sublattice. Then, using the definitions above,

$$\begin{aligned} &\langle\text{gnd}|\, S_3(n,t)S_3(n',t')\,|\text{gnd}\rangle = \\ &\quad = \langle\text{gnd}|\left[S - a^\dagger(n,t)a(n,t)\right]\left[S - a^\dagger(n',t')a(n',t')\right]|\text{gnd}\rangle \\ &\quad = S^2 - 2S\langle\text{gnd}|\, a^\dagger a\,|\text{gnd}\rangle + \langle\text{gnd}|\, a^\dagger(n,t)a(n,t)a^\dagger(n',t')a(n',t')\,|\text{gnd}\rangle, \end{aligned} \tag{2.3.2}$$

where we have used the fact that the ground state has sublattice translational invariance and is time independent. The quadratic term is

$$\begin{aligned} \langle\text{gnd}|\, a^\dagger a\,|\text{gnd}\rangle &= 2\int \frac{\mathrm{d}p}{2\pi}\frac{\mathrm{d}p'}{2\pi}\langle 0|\left[\cosh\theta(p)\, c^\dagger(p) - \sinh\theta(p)\, d(p)\right] \\ &\qquad \times \left[\cosh\theta(p')\, c(p') - \sinh\theta(p')\, d^\dagger(p')\right]|0\rangle \\ &= 2\int \frac{\mathrm{d}p}{2\pi}\sinh^2\theta(p), \end{aligned} \tag{2.3.3}$$

where $|\text{gnd}\rangle = |0\rangle$ is the state with no c or d Bogoliubov quasiparticles. This is just some (infinite) number, which we denote by $\langle N\rangle$. Since the c and d quasiparticles are non-interacting, we can use Wick's theorem to simplify the quartic term, where normal ordering should be understood with respect to the ground state $|\text{gnd}\rangle = |0\rangle$:

$$\begin{aligned} &\langle\text{gnd}|\, a^\dagger(n,t)a(n,t)a^\dagger(n',t')a(n',t')\,|\text{gnd}\rangle = \\ &\quad = \langle N\rangle^2 + \langle\text{gnd}|\, a^\dagger(n,t)a^\dagger(n',t')\,|\text{gnd}\rangle\langle\text{gnd}|\, a(n,t)a(n',t')\,|\text{gnd}\rangle \\ &\qquad + \langle\text{gnd}|\, a^\dagger(n,t)a(n',t')\,|\text{gnd}\rangle\langle\text{gnd}|\, a(n,t)a^\dagger(n',t')\,|\text{gnd}\rangle. \end{aligned} \tag{2.3.4}$$

We then substitute the Bogoliubov transformation. Since the Hamiltonian is diagonal in the c and d mode basis, the Heisenberg time evolution of these operators is particularly simple:

$$c(p,t) = c(p)e^{-i\omega_p t}, \qquad c^\dagger(p,t) = c^\dagger(p)e^{i\omega_p t}, \tag{2.3.5a}$$

$$d(p,t) = d(p)e^{-i\omega_p t}, \qquad d^\dagger(p,t) = d^\dagger(p)e^{i\omega_p t}. \tag{2.3.5b}$$

Therefore,

$$\begin{aligned}
\langle \text{gnd}| \, a^\dagger(n,t) a^\dagger(n',t') \, |\text{gnd}\rangle &= 2 \int \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{d}p'}{2\pi} \, \langle 0| \Big[\cosh\theta(p) \, c^\dagger(p,t) - \sinh\theta(p) \, d(p,t) \Big] \\
&\quad \times \Big[\cosh\theta(p') \, c^\dagger(p',t') - \sinh\theta(p') \, d(p',t') \Big] \, |0\rangle \, e^{ipn} e^{ip'n'} \\
&= 0, \qquad (2.3.6a) \\
\langle \text{gnd}| \, a(n,t) a(n',t') \, |\text{gnd}\rangle &= 0, \qquad (2.3.6b) \\
\langle \text{gnd}| \, a^\dagger(n,t) a(n',t') \, |\text{gnd}\rangle &= 2 \int \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{d}p'}{2\pi} \, \langle 0| \Big[\cosh\theta(p) \, c^\dagger(p,t) - \sinh\theta(p) \, d(p,t) \Big] \\
&\quad \times \Big[\cosh\theta(p') \, c(p',t') - \sinh\theta(p') \, d^\dagger(p',t') \Big] \, |0\rangle \, e^{ipn} e^{-ip'n'} \\
&= 2 \int \frac{\mathrm{d}p}{2\pi} \sinh^2\theta(p) e^{-i\omega_p(t-t')} e^{-ip(n-n')}, \qquad (2.3.6c) \\
\langle \text{gnd}| \, a(n,t) a^\dagger(n',t') \, |\text{gnd}\rangle &= 2 \int \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{d}p'}{2\pi} \, \langle 0| \Big[\cosh\theta(p) \, c(p,t) - \sinh\theta(p) \, d^\dagger(p,t) \Big] \\
&\quad \times \Big[\cosh\theta(p') \, c^\dagger(p',t') - \sinh\theta(p') \, d(p',t') \Big] \, |0\rangle \, e^{-ipn} e^{ip'n'} \\
&= 2 \int \frac{\mathrm{d}p}{2\pi} \cosh^2\theta(p) e^{-i\omega_p(t-t')} e^{-ip(n-n')}. \qquad (2.3.6d)
\end{aligned}$$

It is clear that the matrix element for $t < t'$ will simply reverse the roles of n and n', and t and t'.

Therefore, putting all of the above together, we find

$$\begin{aligned}
D_{33}(n,t|n',t') = \int \frac{\mathrm{d}z}{2\pi} \frac{\mathrm{d}p}{2\pi} e^{iz(t-t')} e^{-ip(n-n')} \Bigg[&- i(S - \langle N\rangle)^2 (2\pi)^2 \delta(z)\delta(p) \\
&+ 4 \int \frac{\mathrm{d}p'}{2\pi} e^{-ip'(n-n')} \cosh^2\theta(p) \sinh^2\theta(p') \qquad (2.3.7) \\
&\times \left(\frac{e^{i(\omega_p + \omega_{p'})(t-t')}}{z + i\varepsilon} - \frac{e^{-i(\omega_p + \omega_{p'})(t-t')}}{z - i\varepsilon} \right) \Bigg].
\end{aligned}$$

To simplify this further, we split the integral up into three parts: for the first part, we let $z = \omega$, for the second, $\omega = z + \omega_p + \omega_{p'}$, and for the third, $\omega = z - \omega_p - \omega_{p'}$. Therefore,

$$\begin{aligned}
D_{33}(n,t|n',t') = \int \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{d}p}{2\pi} e^{i\omega(t-t')} e^{-ip(n-n')} \Bigg[&- i(S - \langle N\rangle)^2 (2\pi)^2 \delta(\omega)\delta(p) \\
&+ 4 \int \frac{\mathrm{d}p'}{2\pi} e^{-ip'(n-n')} \cosh^2\theta(p) \sinh^2\theta(p') \qquad (2.3.8) \\
&\times \left(\frac{1}{\omega - \omega_p - \omega_{p'} + i\varepsilon} - \frac{1}{\omega + \omega_p + \omega_{p'} - i\varepsilon} \right) \Bigg]
\end{aligned}$$

To isolate the momentum space Fourier transform, in the second term we let $q =$

$p+p'$ and $k=(p-p')/2$ (which ensures the Jacobian is equal to 1),

$$\begin{aligned}D_{33}(n,t|n',t') &= \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}q}{2\pi} e^{i\omega(t-t')}e^{-iq(n-n')}\Bigg[-i(S-\langle N\rangle)^2(2\pi)^2\delta(\omega)\delta(q) \\ &\quad + 4\int\frac{\mathrm{d}k}{2\pi}\cosh^2\theta(q/2+k)\sinh^2\theta(q/2-k) \\ &\quad \times\left(\frac{1}{\omega-\omega_{k+q/2}-\omega_{k-q/2}+i\varepsilon}-\frac{1}{\omega+\omega_{k+q/2}+\omega_{k-q/2}-i\varepsilon}\right)\Bigg] \\ &\doteq \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}q}{2\pi} e^{i\omega(t-t')}e^{-iq(n-n')}D_{33}(q,\omega),\end{aligned} \tag{2.3.9}$$

where $D_{33}(q,\omega)$ is the propagator in momentum and frequency space. We note here that it is clear from the structure of the Bogoliubov transformation and the derivation above that this result will be essentially identical for the case where n and n' are both on the *odd* sublattice. Then, using the relation for $\theta(p)$ given above, and the identities

$$\cosh^2\left(\frac{\operatorname{arctanh} x}{2}\right)=\frac{1+\sqrt{1-x^2}}{2\sqrt{1-x^2}},\qquad \sinh^2\left(\frac{\operatorname{arctanh} x}{2}\right)=\frac{1-\sqrt{1-x^2}}{2\sqrt{1-x^2}}, \tag{2.3.10}$$

we find that the propagator in momentum and frequency space corresponding to S_3 and $S_3(n')$ on the same sublattice is

$$\begin{aligned}D_{33}(p,\omega) &= -i(S-\langle N\rangle)^2(2\pi)^2\delta(\omega)\delta(p) \\ &\quad+\int\frac{\mathrm{d}k}{2\pi}\frac{1+|\sin(k+p/2)|}{|\sin(k+p/2)|}\frac{1-|\sin(k-p/2)|}{|\sin(k-p/2)|} \\ &\quad\times\left(\frac{1}{\omega-\omega_{k+p/2}-\omega_{k-p/2}+i\varepsilon}-\frac{1}{\omega+\omega_{k+p/2}+\omega_{k-p/2}-i\varepsilon}\right).\end{aligned} \tag{2.3.11}$$

Next, for n even and n' odd, we have a matrix element like

$$\begin{aligned}&\langle\mathrm{gnd}|\, S_3(n,t)S_3(n',t')\,|\mathrm{gnd}\rangle = \\ &\quad= \langle\mathrm{gnd}|\left[S-a^\dagger(n,t)a(n,t)\right]\left[-S+b^\dagger(n',t')b(n',t')\right]|\mathrm{gnd}\rangle \\ &\quad= -S^2+S\,\langle\mathrm{gnd}|\,a^\dagger(n,t)a(n,t)\,|\mathrm{gnd}\rangle+S\,\langle\mathrm{gnd}|\,b^\dagger(n',t')b(n',t')\,|\mathrm{gnd}\rangle \\ &\qquad-\langle\mathrm{gnd}|\,a^\dagger(n,t)a(n,t)b^\dagger(n',t')b(n',t')\,|\mathrm{gnd}\rangle.\end{aligned} \tag{2.3.12}$$

It is simple to see that $\langle a^\dagger a\rangle=\langle b^\dagger b\rangle$. This time,

$$\begin{aligned}&\langle\mathrm{gnd}|\,a^\dagger(n,t)a(n,t)b^\dagger(n',t')b(n',t')\,|\mathrm{gnd}\rangle = \\ &\quad=\langle N\rangle^2+\langle\mathrm{gnd}|\,a^\dagger(n,t)b^\dagger(n',t')\,|\mathrm{gnd}\rangle\langle\mathrm{gnd}|\,a(n,t)b(n',t')\,|\mathrm{gnd}\rangle \\ &\qquad+\langle\mathrm{gnd}|\,a^\dagger(n,t)b(n',t')\,|\mathrm{gnd}\rangle\langle\mathrm{gnd}|\,a(n,t)b^\dagger(n',t')\,|\mathrm{gnd}\rangle,\end{aligned} \tag{2.3.13}$$

and substituting the Bogoliubov operators:

$$\langle \text{gnd}| a^\dagger(n,t) b^\dagger(n',t') |\text{gnd}\rangle = -2\int \frac{\mathrm{d}p}{2\pi} \sinh\theta(p)\cosh\theta(p) e^{-i\omega_p(t-t')} e^{ip(n-n')}, \tag{2.3.14a}$$

$$\langle \text{gnd}| a(n,t) b(n',t') |\text{gnd}\rangle = -2\int \frac{\mathrm{d}p}{2\pi} \sinh\theta(p)\cosh\theta(p) e^{-i\omega_p(t-t')} e^{-ip(n-n')}, \tag{2.3.14b}$$

$$\langle \text{gnd}| a^\dagger(n,t) b(n',t') |\text{gnd}\rangle = \langle \text{gnd}| a(n,t) b^\dagger(n',t') |\text{gnd}\rangle = 0. \tag{2.3.14c}$$

Therefore,

$$\begin{aligned} D_{33}(n,t|n',t') = \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}p}{2\pi} e^{i\omega(t-t')} e^{-ip(n-n')} \Bigg[& i(S-\langle N\rangle)^2 (2\pi)^2 \delta(\omega)\delta(p) \\ & + \int \frac{\mathrm{d}p'}{2\pi} e^{-ip'(n-n')} \sinh 2\theta(p) \sinh 2\theta(p') \\ & \times \left(\frac{1}{\omega+\omega_p+\omega_{p'}-i\varepsilon} - \frac{1}{\omega-\omega_p-\omega_{p'}+i\varepsilon} \right) \Bigg], \end{aligned} \tag{2.3.15}$$

and hence, the propagator in momentum and frequency space for sites on opposite sublattices is

$$\begin{aligned} D_{33}(p,\omega) = & \, i(S-\langle N\rangle)^2 (2\pi)^2 \delta(\omega)\delta(p) \\ & + \int \frac{\mathrm{d}k}{2\pi} \frac{\cos(k+p/2)}{|\sin(k+p/2)|} \frac{\cos(k-p/2)}{|\sin(k-p/2)|} \\ & \times \left(\frac{1}{\omega+\omega_{k+p/2}+\omega_{k-p/2}-i\varepsilon} - \frac{1}{\omega-\omega_{k+p/2}-\omega_{k-p/2}+i\varepsilon} \right). \end{aligned} \tag{2.3.16}$$

(b) Consider the time-ordered propagator

$$\begin{aligned} D_{+-}(n,t|n',t') &= -i \langle \text{gnd}| T S^+(n,t) S^-(n',t') |\text{gnd}\rangle \\ &= -i\Theta(t-t') \langle \text{gnd}| S^+(n,t) S^-(n',t') |\text{gnd}\rangle \\ &\quad - i\Theta(t'-t) \langle \text{gnd}| S^-(n',t') S^+(n,t) |\text{gnd}\rangle \\ &= -i \int \frac{\mathrm{d}z}{2\pi i} e^{iz(t-t')} \Bigg[\frac{1}{z-i\varepsilon} \langle \text{gnd}| S^+(n,t) S^-(n',t') |\text{gnd}\rangle \\ &\qquad - \frac{1}{z+i\varepsilon} \langle \text{gnd}| S^-(n',t') S^+(n,t) |\text{gnd}\rangle \Bigg]. \end{aligned} \tag{2.3.17}$$

When n and n' are both sites on the even sublattice, using the matrix elements calculated above,

$$\langle \text{gnd}| S^+(n,t) S^-(n',t') |\text{gnd}\rangle = 4S \int \frac{\mathrm{d}p}{2\pi} \cosh^2\theta(p) e^{-i\omega_p(t-t')} e^{-ip(n-n')}, \tag{2.3.18a}$$

$$\langle \text{gnd}| S^-(n',t') S^+(n,t) |\text{gnd}\rangle = 4S \int \frac{\mathrm{d}p}{2\pi} \sinh^2\theta(p) e^{i\omega_p(t-t')} e^{-ip(n-n')}. \tag{2.3.18b}$$

Putting this together, we find that

$$\begin{aligned} D_{+-}(n,t|n',t') &= 4S \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}p}{2\pi} e^{i\omega(t-t')} e^{-ip(n-n')} \left[\frac{\sinh^2\theta(p)}{\omega-\omega_p+i\varepsilon} - \frac{\cosh^2\theta(p)}{\omega+\omega_p-i\varepsilon} \right] \\ &\doteq \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}p}{2\pi} e^{i\omega(t-t')} e^{-ip(n-n')} D_{+-}(p,\omega). \end{aligned} \tag{2.3.19}$$

Therefore, the propagator in momentum and frequency space corresponding to $S^+(n)$ and $S^-(n')$ on the same sublattice is

$$D_{+-}(p,\omega) = \frac{2S}{|\sin(p)|}\left[\frac{1-|\sin(p)|}{\omega-\omega_p+i\varepsilon} - \frac{1+|\sin(p)|}{\omega+\omega_p-i\varepsilon}\right]. \tag{2.3.20}$$

Next, when n is even and n' is odd,

$$\langle\text{gnd}| S^+(n,t)S^-(n',t')|\text{gnd}\rangle = -4S\int\frac{\mathrm{d}p}{2\pi}\cosh\theta(p)\sinh\theta(p)e^{-i\omega_p(t-t')}e^{-ip(n-n')}, \tag{2.3.21a}$$

$$\langle\text{gnd}| S^-(n',t')S^+(n,t)|\text{gnd}\rangle = -4S\int\frac{\mathrm{d}p}{2\pi}\cosh\theta(p)\sinh\theta(p)e^{i\omega_p(t-t')}e^{-ip(n-n')}. \tag{2.3.21b}$$

Therefore, the propagator in momentum and frequency space corresponding to operators on opposite sublattices is

$$D_{+-}(p,\omega) = 2S\frac{\cos(p)}{|\sin(p)|}\left[\frac{1}{\omega+\omega_p-i\varepsilon} - \frac{1}{\omega-\omega_p+i\varepsilon}\right]. \tag{2.3.22}$$

4. [**5**++] Consider the equal sublattice transverse susceptibility $\chi_{+-}(p,\omega)$. There is no need to calculate anything new to find this susceptibility. By comparing the definition of the susceptibility to the time-ordered propagators, the only differences are: i) The causal structure; since there is only $\Theta(t-t')$, all denominators will have $-i\varepsilon$ instead of $\pm i\varepsilon$. ii) The commutator introduces an additional negative between the two terms. Therefore, we can immediately write down

$$\begin{aligned}\chi_{+-}(p,\omega) &= \frac{2S}{|\sin(p)|}\left[\frac{1-|\sin(p)|}{\omega-\omega_p-i\varepsilon} - \frac{1+|\sin(p)|}{\omega+\omega_p-i\varepsilon}\right]\\ &= \frac{4S}{|\sin(p)|}\frac{\omega_p-\omega|\sin(p)|}{(\omega-i\varepsilon)^2-\omega_p^2}\\ &= \frac{4S(2JS-\omega)}{\omega^2-\omega_p^2-i\varepsilon\,\text{sign}(\omega)},\end{aligned} \tag{2.4.1}$$

where we have used $\omega_p = 2JS|\sin(p)|$. Therefore, by using the identity

$$\lim_{\varepsilon\to0+}\frac{1}{x-i\varepsilon} = \mathcal{P}\frac{1}{x} + i\pi\delta(x), \tag{2.4.2}$$

in the limit $\omega\to0^+$, we find

$$\chi_{+-}(p,\omega) \simeq 8JS^2\left[\mathcal{P}\frac{1}{\omega^2-\omega_p^2} + i\pi\delta(\omega^2-\omega_p^2)\right]. \tag{2.4.3}$$

In this form it is clear, since $\omega_p=0$ at $p=\pi$, that there is a pole at $\omega=0$ corresponding to this momentum. By comparing this result with Eq. (10.165) of the lecture notes, we can identify that the residue at this pole is

$$Z = 8JS^2. \tag{2.4.4}$$

Question 3 (Spectral Function for the Dirac Propagator)

1. [10/25] Consider the Dirac propagator

$$S^F_{\alpha\beta}(x,y) = -i \langle 0| T\psi_\alpha(x)\bar{\psi}_\beta(y) |0\rangle . \tag{3.1.1}$$

For now, let us consider the case when $x^0 > y^0$ only. Then, inserting a complete basis $\{|n\rangle\}$ as a resolution of the identity, we have

$$S^F_{\alpha\beta}(x,y) = -i \sum_n \langle 0| \psi_\alpha(x) |n\rangle \langle n| \bar{\psi}_\beta(y) |0\rangle . \tag{3.1.2}$$

For a (space-time) translationally invariant theory, we can shift the fermion fields to the origin with the use of the momentum operator:

$$\begin{aligned} S^F_{\alpha\beta}(x,y) &= -i \sum_n \langle 0| e^{iP\cdot x}\psi_\alpha(0)e^{-iP\cdot x} |n\rangle \langle n| e^{iP\cdot y}\bar{\psi}_\beta(0)e^{-iP\cdot y} |0\rangle \\ &= -i \sum_n e^{-ip_n\cdot(x-y)} \langle 0| \psi_\alpha(0) |n\rangle \langle n| \bar{\psi}_\beta(0) |0\rangle , \end{aligned} \tag{3.1.3}$$

where p_n is the eigenvalue of the momentum operator corresponding to the state $|n\rangle$. We can then insert the identity:

$$\begin{aligned} S^F_{\alpha\beta}(x,y) &= -i \int \mathrm{d}^4q \sum_n \delta^{(4)}(q-p_n)e^{-iq\cdot(x-y)} \langle 0| \psi_\alpha(0) |n\rangle \langle n| \bar{\psi}_\beta(0) |0\rangle \\ &\doteq -i \int \frac{\mathrm{d}^4q}{(2\pi)^3} A_{\alpha\beta}(q)e^{-iq\cdot(x-y)}, \end{aligned} \tag{3.1.4}$$

where we have defined

$$A_{\alpha\beta}(q) = (2\pi)^3 \sum_n \delta^{(4)}(q-p_n) \langle 0| \psi_\alpha(0) |n\rangle \langle n| \bar{\psi}_\beta(0) |0\rangle . \tag{3.1.5}$$

Inserting another identity, we obtain

$$S^F_{\alpha\beta}(x,y) = -i \int_0^\infty \mathrm{d}\mu^2 \int \frac{\mathrm{d}^4q}{(2\pi)^3}\delta(\mu^2-q^2)A_{\alpha\beta}(q)e^{-iq\cdot(x-y)}. \tag{3.1.6}$$

We can manipulate this expression by first integrating over q^0 to eliminate the δ function. However, since the spectrum of the Dirac theory is positive definite, all eigenvalues p_n have component $p^0_n \geq 0$. Therefore, the matrix elements $A_{\alpha\beta}(q) \sim \Theta(q^0)$, and hence,

$$\begin{aligned} S^F_{\alpha\beta}(x,y) &= -i \int_0^\infty \mathrm{d}\mu^2 \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \int_0^\infty \mathrm{d}q^0\delta(\mu^2-q^2)A_{\alpha\beta}(q)e^{-iq\cdot(x-y)} \\ &= -i \int_0^\infty \mathrm{d}\mu^2 \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{2E(\boldsymbol{q})} A_{\alpha\beta}(q)e^{-iq\cdot(x-y)}\bigg|_{q^0=E(\boldsymbol{q})}, \end{aligned} \tag{3.1.7}$$

where $E(\boldsymbol{q}) = \sqrt{\boldsymbol{q}^2+\mu^2}$. We can then re-introduce the q^0 integration variable using the Cauchy residue formula. Since we are assuming that $x^0 > y^0$, the exponential factor will converge in the lower half complex q^0 plane. Therefore, we want to express the integrand as a contour integral enclosing a single pole in the LHP:

$$\begin{aligned} \frac{1}{2E(\boldsymbol{q})} A_{\alpha\beta}(q)e^{-iq\cdot(x-y)}\bigg|_{q^0=E(\boldsymbol{q})} &= -\oint_{\gamma_-} \frac{\mathrm{d}q^0}{2\pi i} \frac{A_{\alpha\beta}(q)e^{-iq\cdot(x-y)}}{q^2-\mu^2+i\varepsilon} \\ &= i\int_{-\infty}^\infty \frac{\mathrm{d}q^0}{2\pi} \frac{A_{\alpha\beta}(q)e^{-iq\cdot(x-y)}}{q^2-\mu^2+i\varepsilon}, \end{aligned} \tag{3.1.8}$$

where γ_- is the negatively-oriented (clockwise) semicircular contour enclosing the pole at $q^0 = E(\boldsymbol{q}) - i\varepsilon$, and in the last line we have used Jordan's lemma to express the contour integral as an integral over the real axis. Therefore, we have found that

$$S^F_{\alpha\beta}(x,y) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \int_0^\infty \mathrm{d}\mu^2 \frac{A_{\alpha\beta}(q)}{q^2 - \mu^2 + i\varepsilon} e^{-iq\cdot(x-y)}. \tag{3.1.9}$$

From this expression we can directly invert the Fourier transform to find

$$S^F_{\alpha\beta}(p) = \int_0^\infty \mathrm{d}\mu^2 \frac{\rho_{\alpha\beta}(p)}{p^2 - \mu^2 + i\varepsilon}, \tag{3.1.10a}$$

$$\rho_{\alpha\beta}(p) = (2\pi)^3 \sum_n \delta^{(4)}(p - p_n) \langle 0| \psi_\alpha(0) |n\rangle \langle n| \bar{\psi}_\beta(0) |0\rangle . \tag{3.1.10b}$$

2. [15/25] To observe how the spectral function transforms under a parity transformation $\mathcal{P}$, we can insert the identity $I = \mathcal{P}^{-1}\mathcal{P}$ into Eq. (3.1.10b). Then, using the fact that the vacuum is parity invariant, and that the spinors transform as $\mathcal{P}\psi(x_0, \boldsymbol{x})\mathcal{P}^{-1} = \gamma^0\psi(x_0, -\boldsymbol{x})$,

$$\begin{aligned} \rho_{\alpha\beta}(p) &= (2\pi)^3 \sum_n \delta^{(4)}(p - p_n) \langle 0| \mathcal{P}^{-1}\mathcal{P}\psi_\alpha(0)\mathcal{P}^{-1}\mathcal{P} |n\rangle \langle n| \mathcal{P}^{-1}\mathcal{P}\bar{\psi}_\beta(0)\mathcal{P}^{-1}\mathcal{P} |0\rangle \\ &= (2\pi)^3 \sum_n \delta^{(4)}(p - p_n) \langle 0| \gamma^0_{\alpha\lambda}\psi_\lambda(0)\mathcal{P} |n\rangle \langle n| \mathcal{P}^{-1}\bar{\psi}_\sigma(0)\gamma^0_{\sigma\beta} |0\rangle . \end{aligned} \tag{3.2.1}$$

We can expand the matrix elements in any complete basis, so we are free to switch to the basis related to the original one by a parity transformation: $\{\mathcal{P}^{-1} |n\rangle\}$. This has the effect of taking each $p_n \to (p_n^0, -\boldsymbol{p}_n)$, so that

$$\rho_{\alpha\beta}(p) = \gamma^0_{\alpha\lambda}\rho_{\lambda\sigma}(p^0, -\boldsymbol{p})\gamma^0_{\sigma\beta}. \tag{3.2.2}$$

Recall that the gamma matrices form a complete basis for the space of 4×4 matrices [see Eq. (2.139) of the notes]. Then, note that: i) p^μ is the only 4-vector present in this problem, and ii) $(\not{p})^2 = p_\mu p^\mu = p^2$. This implies that, since $\rho_{\alpha\beta}$ has no spacetime indices, the most general structure we can write down which is consistent with Lorentz covariance is

$$\begin{aligned} \rho_{\alpha\beta}(p) &= \rho_1(p^2)p^\mu\Gamma^V_\mu + \rho_2(p^2)\Gamma^S + \rho_3(p^2)p^\mu\Gamma^A_\mu + \rho_4(p^2)\Gamma^P + \rho_5(p^2)p^\mu p^\nu\Gamma^T_{\mu\nu} \\ &= \rho_1(p^2)\not{p}_{\alpha\beta} + \rho_2(p^2)\delta_{\alpha\beta} + \rho_3(p^2)(\not{p}\gamma^5)_{\alpha\beta} + \rho_4(p^2)\gamma^5_{\alpha\beta}, \end{aligned} \tag{3.2.3}$$

where we have used the fact that $\Gamma^T_{\mu\nu}$ is an antisymmetric tensor. Then, the parity transformation property of $\rho_{\alpha\beta}(p)$ we derived above implies

$$\begin{aligned} \rho(p) &= \rho_1(p^2)\gamma^0(p^0\gamma^0 + (-\boldsymbol{p})\cdot\boldsymbol{\gamma})\gamma^0 + \rho_2(p^2)\gamma^0\gamma^0 \\ &\qquad + \rho_3(p^2)\gamma^0(p^0\gamma^0 + (-\boldsymbol{p})\cdot\boldsymbol{\gamma})\gamma^5\gamma^0 + \rho_4(p^2)\gamma^0\gamma^5\gamma^0 \\ &= \rho_1(p^2)\not{p} + \rho_2(p^2) - \rho_3(p^2)\not{p}\gamma^5 - \rho_4(p^2)\gamma^5. \end{aligned} \tag{3.2.4}$$

Therefore, the γ^5 terms of the expansion are inconsistent with parity invariance, and hence,

$$\rho_{\alpha\beta}(p) = \rho_1(p^2)\not{p}_{\alpha\beta} + \rho_2(p^2)\delta_{\alpha\beta}. \tag{3.2.5}$$

In the free Dirac theory, we have the mode expansion

$$\psi_\alpha(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{m}{p^0}\sum_\sigma \left[b_\sigma(\boldsymbol{p})u_\alpha^{(\sigma)}(p)e^{-ip\cdot x} + d_\sigma^\dagger(\boldsymbol{p})v_\alpha^{(\sigma)}(p)e^{ip\cdot x}\right]. \tag{3.2.6}$$

We can explicitly calculate the matrix elements:

$$\begin{aligned}
\langle 0|\,\psi_\alpha(0)\,|n\rangle &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{m}{p^0}\sum_\sigma \langle 0|\,b_\sigma(\boldsymbol{p})\,|n\rangle\, u_\alpha^{(\sigma)}(p) \\
&= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{m}{p^0}\sum_\sigma \langle 0|\,b_\sigma(\boldsymbol{p})b_{s_n}^\dagger(\boldsymbol{p}_n)\,|0\rangle\, u_\alpha^{(\sigma)}(p) \\
&= u_\alpha^{(s_n)}(p_n), &(3.2.7a)\\
\langle n|\,\bar\psi_\beta(0)\,|0\rangle &= \bar u_\beta^{(s_n)}(p_n), &(3.2.7b)
\end{aligned}$$

where s_n is the spin eigenvalue of the state $|n\rangle$. Therefore, the spectral function is

$$\begin{aligned}
\rho_{\alpha\beta}(p) &= (2\pi)^3\sum_n \delta^{(4)}(p-p_n)u_\alpha^{(s_n)}(p_n)\bar u_\beta^{(s_n)}(p_n) \\
&= (2\pi)^3\sum_s\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{m}{q^0}\delta^{(4)}(p-q)u_\alpha^{(s)}(q)\bar u_\beta^{(s)}(q) \\
&= (2\pi)^3\int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{m}{q^0}\delta(p^0-q^0)\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\frac{(\not{q}+m)_{\alpha\beta}}{2m} &(3.2.8)
\end{aligned}$$

where we have identified the spin sum as the projection operator $\Lambda_+(q)$, given in Eqs. (7.23) and (7.73) of the lecture notes. Then, observe that

$$\begin{aligned}
\delta(p^2-m^2)\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) &= \delta(p_0^2-[\boldsymbol{q}^2+m^2])\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \\
&= \delta(p_0^2-q_0^2)\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \\
&= \frac{1}{2q^0}\delta(p^0-q^0)\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), &(3.2.9)
\end{aligned}$$

since the physical states have $q^0 > 0$. Therefore, we find the spectral function is

$$\begin{aligned}
\rho_{\alpha\beta}(p) &= \delta(p^2-m^2)(\not{p}+m)_{\alpha\beta}, \\
\implies \rho_1(p^2) = \delta(p^2-m^2), &\qquad \rho_2(p^2) = m\delta(p^2-m^2). &(3.2.10)
\end{aligned}$$

We can check that this result yields the correct free Dirac propagator: By considering Eq. (3.1.6), we see that the functions $\rho_{1,2}(p^2)$ can be evaluated directly at $p^2 = \mu^2$ in the calculation of the propagator—as in the derivation of the Lehmann representation of the scalar field propagator—so that

$$\begin{aligned}
S^F(p) &= \int_0^\infty \mathrm{d}\mu^2\frac{(\not{p}+m)}{p^2-\mu^2+i\varepsilon}\delta(\mu^2-m^2) \\
&= \frac{\not{p}+m}{p^2-m^2+i\varepsilon}, &(3.2.11)
\end{aligned}$$

which is exactly the familiar free Dirac propagator.

Question 4 (Wick's Theorem)

Consider a 3-component free scalar field $\phi_a(x)$, with $a = 1, 2, 3$, with the global $O(3)$ invariance $\phi_a \to O_{ab}\phi_b$, where O_{ab} is an arbitrary rotation matrix.

1. **[9/15]** In a theory with $O(3)$ invariance, the time-ordered vacuum expectation values of field operators should be invariant under a symmetry transformation. Consider the following correlation functions under an arbitrary global rotation:

   (a) The 2-point function transforms as

$$\begin{aligned}\langle 0|\, T\phi_a(x)\phi_a(x')\,|0\rangle &\longrightarrow \langle 0|\, TO_{ab}\phi_b(x)O_{ac}\phi_c(x')\,|0\rangle \\ &= \langle 0|\, T\phi_b(x)O_{ab}O_{ac}\phi_c(x')\,|0\rangle \\ &= \langle 0|\, T\phi_b(x)(O^T)_{ba}O_{ac}\phi_c(x')\,|0\rangle \\ &= \langle 0|\, T\phi_b(x)\delta_{bc}\phi_c(x')\,|0\rangle \\ &= \langle 0|\, T\phi_a(x)\phi_a(x')\,|0\rangle\,, \end{aligned} \tag{4.1.1}$$

   where we have used the fact that $O^T O = I$. Therefore, the 2-point function is manifestly invariant under an arbitrary rotation, and so is non-zero.

   (b) By the same argument, the 3-point function must transform as

$$\langle 0|\, T\phi_a(x)\phi_b(x')\phi_b(x'')\,|0\rangle \longrightarrow O_{ac}\,\langle 0|\, T\phi_c(x)\phi_b(x')\phi_b(x'')\,|0\rangle\,, \tag{4.1.2}$$

   since there is one free index. In order for this expression to be invariant under the action of an arbitrary rotation O, it must vanish.

   (c) Since the 4-point function has no free indices, it must also be left invariant by the action of a rotation, just like the 2-point function:

$$\langle 0|\, T\phi_a(x)\phi_b(x')\phi_b(x'')\phi_a(x''')\,|0\rangle \longrightarrow \langle 0|\, T\phi_a(x)\phi_b(x')\phi_b(x'')\phi_a(x''')\,|0\rangle\,. \tag{4.1.3}$$

   Therefore, it is non-zero.

2. **[6/15]** As established above, we have

$$\langle 0|\, T\phi_a(x)\phi_b(x')\phi_b(x'')\,|0\rangle = 0. \tag{4.2.1}$$

   For the 4-point function, applying Wick's theorem yields

$$\begin{aligned}\langle 0|\, T\phi_a(x)\phi_b(x')\phi_b(x'')\phi_a(x''')\,|0\rangle &= \langle 0|\, T\phi_a(x)\phi_b(x')\,|0\rangle\,\langle 0|\, T\phi_b(x'')\phi_a(x''')\,|0\rangle \\ &\quad + \langle 0|\, T\phi_a(x)\phi_b(x'')\,|0\rangle\,\langle 0|\, T\phi_b(x')\phi_a(x''')\,|0\rangle \\ &\quad + \langle 0|\, T\phi_a(x)\phi_a(x''')\,|0\rangle\,\langle 0|\, T\phi_b(x')\phi_b(x'')\,|0\rangle\,. \end{aligned} \tag{4.2.2}$$

   Then, since the fields are free, we must have

$$\langle 0|\, T\phi_a(x)\phi_b(x')\,|0\rangle = \delta_{ab}\,\langle 0|\, T\phi_a(x)\phi_a(x')\,|0\rangle\,, \tag{4.2.3}$$

   without summation over the repeated indices here. Therefore, writing out the sums explicitly,

$$\begin{aligned}\langle 0|\, T\phi_a(x)\phi_b(x')\phi_b(x'')\phi_a(x''')\,|0\rangle = \sum_{a,b=1}^{3} \delta_{ab}\Big[&\langle 0|\, T\phi_a(x)\phi_a(x')\,|0\rangle\,\langle 0|\, T\phi_a(x'')\phi_a(x''')\,|0\rangle \\ &+ \langle 0|\, T\phi_a(x)\phi_a(x'')\,|0\rangle\,\langle 0|\, T\phi_a(x')\phi_a(x''')\,|0\rangle \Big] \\ + \sum_{a,b=1}^{3} &\langle 0|\, T\phi_a(x)\phi_a(x''')\,|0\rangle\,\langle 0|\, T\phi_b(x')\phi_b(x'')\,|0\rangle\,. \end{aligned} \tag{4.2.4}$$

Additionally, since the vacuum must transform as a singlet under the action of the symmetry group, the correlation function for each component ϕ_a must be identical and equal to the propagator of a single free scalar field

$$\langle 0| \, T\phi_a(x)\phi_a(x') \, |0\rangle = G^{(0)}(x - x'). \tag{4.2.5}$$

Therefore, we can also write

$$\begin{aligned} \langle 0| \, T\phi_a(x)\phi_b(x')\phi_b(x'')\phi_a(x''') \, |0\rangle &= 3G^{(0)}(x - x')G^{(0)}(x'' - x''') \\ &+ 3G^{(0)}(x - x'')G^{(0)}(x' - x''') + 9G^{(0)}(x - x''')G^{(0)}(x' - x''). \end{aligned} \tag{4.2.6}$$

Question 5 (Reduction Formulas) [**15 points**]

Consider the pair production of π mesons by a photon

$$\gamma \longrightarrow \pi^+ + \pi^- . \tag{5.1}$$

This process is governed by a theory of a complex scalar field $\phi(x)$ coupled to the electromagnetic field $A_\mu(x)$. The mode expansion for the scalar field is

$$\phi(x) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2k_0} \left[b(\boldsymbol{k}) e^{-ik\cdot x} + d^\dagger(\boldsymbol{k}) e^{ik\cdot x} \right], \tag{5.2}$$

where $b(\boldsymbol{k})$ is the operator which annihilates a π^+ meson, while $d(\boldsymbol{k})$ annihilates a π^- meson. Here, k is on-shell, so $k^2 = m^2$, where m is the mass of the $\pi^\pm$ meson. If we work in the Coulomb gauge, then $A_0(x) = 0$, and the spatial components of the vector potential have the following mode expansion

$$\boldsymbol{A}(x) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2k_0} \sum_{\lambda=1,2} \boldsymbol{\varepsilon}_\lambda(k) \left[a_\lambda(\boldsymbol{k}) e^{-ik\cdot x} + a_\lambda^\dagger(\boldsymbol{k}) e^{ik\cdot x} \right], \tag{5.3}$$

where $\boldsymbol{\varepsilon}_\lambda \cdot \boldsymbol{\varepsilon}_{\lambda'} = \delta_{\lambda\lambda'}$. In this case, the on-shell condition reads $k^2 = 0$. We can then invert these mode expansions to obtain

$$b(\boldsymbol{k}) = i \int \mathrm{d}^3\boldsymbol{x}\, e^{ik\cdot x} \overleftrightarrow{\partial}_0 \phi(x), \tag{5.4a}$$

$$d(\boldsymbol{k}) = i \int \mathrm{d}^3\boldsymbol{x}\, e^{ik\cdot x} \overleftrightarrow{\partial}_0 \phi^\dagger(x), \tag{5.4b}$$

$$a_\alpha^\dagger(\boldsymbol{k}) = -i \int \mathrm{d}^3\boldsymbol{x}\, e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \boldsymbol{\varepsilon}_\alpha(k) \cdot \boldsymbol{A}(x). \tag{5.4c}$$

Then, we can write the amplitude for meson pair production as

$$\begin{aligned} \langle p_+, p_-; \mathrm{out} | p_i, \alpha; \mathrm{in} \rangle &= \langle p_+, p_-; \mathrm{out} |\, a_{\mathrm{in}}^\dagger(\boldsymbol{p}_i, \alpha) \,|0\rangle \\ &= -i \int \mathrm{d}^3\boldsymbol{x}\, e^{-ip_i\cdot x} \overleftrightarrow{\partial}_{x_0} \langle p_+, p_-; \mathrm{out} |\, \boldsymbol{\varepsilon}_\alpha(p_i) \cdot \boldsymbol{A}_{\mathrm{in}}(x) \,|0\rangle \\ &= -i \lim_{x^0 \to -\infty} \frac{1}{Z_3^{1/2}} \int \mathrm{d}^3\boldsymbol{x}\, e^{-ip_i\cdot x} \overleftrightarrow{\partial}_{x_0} \langle p_+, p_-; \mathrm{out} |\, \boldsymbol{\varepsilon}_\alpha(p_i) \cdot \boldsymbol{A}(x) \,|0\rangle , \end{aligned} \tag{5.5}$$

where Z_3 is the wavefunction renormalisation for the electromagnetic field. Then, using the fundamental theorem of calculus, we can re-write this as

$$\begin{aligned} \langle p_+, p_-; \mathrm{out} | p_i, \alpha; \mathrm{in} \rangle &= \langle p_+, p_-; \mathrm{out} |\, a_{\mathrm{out}}^\dagger(\boldsymbol{p}_i, \alpha) \,|0\rangle \\ &\quad + \frac{i}{Z_3^{1/2}} \int \mathrm{d}^4x\, \partial_{x_0} \left[e^{-ip_i\cdot x} \overleftrightarrow{\partial}_{x_0} \langle p_+, p_-; \mathrm{out} |\, \boldsymbol{\varepsilon}_\alpha(p_i) \cdot \boldsymbol{A}(x) \,|0\rangle \right]. \end{aligned} \tag{5.6}$$

However, since there are no photons in the “out” state, the first (disconnected) term vanishes, since the creation operator annihilates the state to its left. Therefore,

$$\langle p_+, p_-; \mathrm{out} | p_i, \alpha; \mathrm{in} \rangle = \frac{i}{Z_3^{1/2}} \int \mathrm{d}^4x \left[e^{-ip_i\cdot x} \partial_{x_0}^2 \langle \dots \rangle - (\partial_{x_0}^2 e^{-ip_i\cdot x}) \langle \dots \rangle \right]. \tag{5.7}$$

Then, since p_i is on-shell and light-like, $\partial_0^2 e^{-ip_i\cdot x} = \nabla^2 e^{-ip_i\cdot x}$, and hence,

$$\begin{aligned}\langle p_+, p_-; \text{out}|p_i, \alpha; \text{in}\rangle &= \frac{i}{Z_3^{1/2}} \int \mathrm{d}^4x \left[e^{-ip_i\cdot x}\partial_{x_0}^2 \langle \dots \rangle - (\nabla_x^2 e^{-ip_i\cdot x})\langle \dots \rangle\right] \\ &= \frac{i}{Z_3^{1/2}} \int \mathrm{d}^4x \left[e^{-ip_i\cdot x}\partial_x^2 \langle p_+, p_-; \text{out}| \boldsymbol{\varepsilon}_\alpha(p_i)\cdot \boldsymbol{A}(x)\,|0\rangle\right], \end{aligned} \tag{5.8}$$

where we have integrated the second term by parts to shift the spatial derivative onto the matrix element instead of the exponential. We can repeat a very similar process for the mesons in the out state:

$$\begin{aligned}\langle p_+, p_-; \text{out}| \boldsymbol{\varepsilon}_\alpha(p_i)\cdot \boldsymbol{A}(x)\,|0\rangle &= \langle 0|\, T b_{\text{out}}(p_+) d_{\text{out}}(p_-) \boldsymbol{\varepsilon}_\alpha(p_i)\cdot \boldsymbol{A}(x)\,|0\rangle \\ &= \lim_{y_1^0\to\infty} \lim_{y_2^0\to\infty} \left(\frac{i}{Z^{1/2}}\right)^2 \int \mathrm{d}^3\boldsymbol{y_1}\mathrm{d}^3\boldsymbol{y_2}\, e^{ip_+\cdot y_1} e^{ip_-\cdot y_2} \overleftrightarrow{\partial}_{y_1^0} \overleftrightarrow{\partial}_{y_2^0} \\ &\qquad \times \langle 0|\, T\phi(y_1)\phi^\dagger(y_2) \boldsymbol{\varepsilon}_\alpha(p_i)\cdot \boldsymbol{A}(x)\,|0\rangle\,, \end{aligned} \tag{5.9}$$

where Z is the wavefunction renormalisation for the scalar field. Then, since there are no mesons in the "in" state, we can repeat the exact same procedure as above. However, we note that $\partial_0^2 e^{ip_+\cdot y_1} = (\nabla^2 - m^2)e^{ip_+\cdot y_1}$, and similarly for p_-, so

$$\begin{aligned}&\langle p_+, p_-; \text{out}| \boldsymbol{\varepsilon}_\alpha(p_i)\cdot \boldsymbol{A}(x)\,|0\rangle = \\ &\quad = \langle 0|\, T b_{\text{in}}(p_+) d_{\text{in}}(p_-) \boldsymbol{\varepsilon}_\alpha(p_i)\cdot \boldsymbol{A}(x)\,|0\rangle + \left(\frac{i}{Z^{1/2}}\right)^2 \int \mathrm{d}^4y_1\mathrm{d}^4y_2\, e^{ip_+\cdot y_1} e^{ip_-\cdot y_2} \\ &\qquad \times (\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2)\,\langle 0|\, T\phi(y_1)\phi^\dagger(y_2)\boldsymbol{\varepsilon}_\alpha(p_i)\cdot \boldsymbol{A}(x)\,|0\rangle \end{aligned} \tag{5.10}$$

Finally, we collect everything to obtain the reduction formula

$$\begin{aligned}\langle p_+, p_-; \text{out}|p_i, \alpha; \text{in}\rangle &= \frac{i^3}{Z Z_3^{1/2}} \int \mathrm{d}^4x\mathrm{d}^4y_1\mathrm{d}^4y_2 \exp\Big[i\big(p_+\cdot y_1 + p_-\cdot y_2 - p_i\cdot x\big)\Big] \\ &\qquad \times \partial_x^2(\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2)\boldsymbol{\varepsilon}_\alpha(p_i)\cdot \langle 0|\, T\boldsymbol{A}(x)\phi(y_1)\phi^\dagger(y_2)\,|0\rangle\,. \end{aligned} \tag{5.11}$$