Physics 582, Fall Semester 2021
Professor Eduardo Fradkin

Problem Set No. 6
Propagators and Correlation Functions
Due Date: Saturday December 4, 2021, 9:00 pm, US Central Time

1 Path-Integral Quantization of the Free Electromagnetic Field.

In this problem you are asked to use the path-integral quantization of the free electromagnetic (Maxwell) gauge field, in the *Feynman-'t Hooft* family of gauges, with gauge fixing parameter α (see the Lecture notes for definitions).

1. Derive an explicit expression for the path-integral for a Maxwell gauge field, in the Feynman-'t Hooft family of gauges, coupled to a set of classical (*i.e.* not quantized) currents $J_\mu(x)$.

2. Derive an explicit formula for the Faddeev-Popov determinant for this family of gauges, and show that for a Maxwell field it does not depend on the gauge field configuration.

3. Show that the explicit expression for the path integral does not depend on the choice of gauge if the currents are conserved.

4. Use your results from the previous parts to derive an inhomogeneous partial differential equation obeyed by the propagator in Minkowski spacetime in the Feynman-'t Hooft family of gauges.

5. Show that the expression that you derived in part contains a Lorentz covariant term of the form

$$D^F_{\mu\nu}(x, x') = g_{\mu\nu}\ D_F(x, x') + \dots \tag{1}$$

where $\dots$ are the other terms. Find a relation between $D_F(x, x')$ and the Feynman propagator for a free massless real scalar field. Use this relationship to calculate an explicit formula for the dependence of $D_F(x, x')$ on the spacetime coordinates.

6. Show the explicit dependence of the path integral on the propagator you computed above. Show that this expression is independent of the gauge fixing parameter α.

7. Find an expression, *in momentum space*, for the *Feynman propagator* for the free electromagnetic field $D^F_{\mu\nu}(x,x')$

$$D^F_{\mu\nu}(x,x') = -i\langle 0|T\ A_\mu(x)\ A_\nu(x')|0\rangle \tag{2}$$

Make sure to specify your $i\epsilon$ prescription.

2 Propagators and Correlation Functions for the One-Dimensional Quantum Heisenberg Antiferromagnet.

In problem I of set 3 you studied the one dimensional quantum Heisenberg antiferromagnet in the spin wave (or semiclassical) approximation. In that problem you constructed the ground state and the single particle excitations, the spin waves. In this problem you will study the response of that system to an external space and time dependent magnetic field. This external field will be represented by an additional term to the Hamiltonian of the form

$$H_{\text{ext}} = \sum_{n=-N/2+1}^{N/2} B_k(n,t)\cdot \hat{S}_k(n,t) \tag{3}$$

which we will regard as a perturbation.

1. Consider for the moment that the external field has been switched off. Derive an expression for the following propagators

   (a)
   $$D_{33}(nt,n't') = -i\langle \text{gnd}|T\ \hat{S}_3(n,t)\hat{S}_3(n',t')|\text{gnd}\rangle \tag{4}$$

   (b)
   $$D_{+-}(nt,n't') = -i\langle \text{gnd}|T\ \hat{S}^+(n,t)\hat{S}^-(n',t')|\text{gnd}\rangle \tag{5}$$

   in momentum and frequency space. Be very careful and very explicit in the way you treat the poles of these propagators. Show that your choice of frequency integration contour yields a propagator which satisfies the correct boundary conditions.

2. Use Linear Response Theory to derive an expression for the *magnetic susceptibilities* χ_{33} and χ_{+-} (in position space) of this system in in terms of correlation (or retarded) functions of this system.

3. Use Wick's theorem to find an expression for the corresponding *time-ordered* functions in the *spin-wave approximation* in momentum and frequency space.

4. Use the results of the previous sections to show that $\chi_{+-}(p,\omega)$ has, in the limit $\omega\to 0$, a pole at $p=\pi$. Calculate the residue of this pole. The residue is the *square* of the order parameter of the system in this approximation.

3 Spectral Function for the Dirac Propagator

1. Derive a formal expression for the spectral function $\rho_{\alpha\alpha'}(p)$ of the Feynman propagator for the Dirac theory
$$S_F^{\alpha\alpha'}(p) = \int_0^\infty dm'^2 \, \frac{\rho_{\alpha\alpha'}(p)}{p^2 - m'^2 + i\epsilon} \tag{6}$$
in terms of matrix elements of the field operators. Recall that the Dirac propagator is
$$S_F^{\alpha\alpha'}(x, x') = -i\langle 0|T \, \psi_\alpha(x) \, \bar{\psi}_{\alpha'}(x')|0\rangle \tag{7}$$

2. Show that, if the vacuum is invariant under parity $\mathcal{P}$,
$$\mathcal{P}\psi(x)\mathcal{P}^{-1} \equiv \gamma_0\psi \tag{8}$$
$\rho_{\alpha\alpha'}$ has the simpler form
$$\rho_{\alpha\alpha'} = \rho_1(p^2) \, \not{p}_{\alpha\alpha'} + \rho_2(p^2) \, \delta_{\alpha\alpha'} \tag{9}$$
Compute explicitly $\rho_1(p^2)$ and $\rho_2(p^2)$ for the free Dirac theory.

4 Wick's Theorem

This is an exercise on the of Wick's theorem on a specific theory, the 3-component free scalar field $\phi_a(x)$, with $a = 1, 2, 3$, with the global $O(3)$ invariance $\phi_a(x) \to O_{ab}\phi_b(x)$, where O_{ab} is an arbitrary 3×3 rotation matrix.

1. Use *symmetry arguments* to determine which of the following v.e.v. are non zero (sums over repeated indices is implied)
   (a) $\langle 0|T\phi_a(x)\phi_a(x')|0\rangle$
   (b) $\langle 0|T\phi_a(x)\phi_b(x')\phi_b(x'')|0\rangle$
   (c) $\langle 0|T\phi_a(x)\phi_b(x')\phi_b(x'')\phi_a(x''')|0\rangle$
2. Use Wick's theorem to find expressions for the v.e.v. of (b) and (c) in (1) in terms of the v.e.v. of (a).

5 Reduction Formulas

In this problem you will consider a theory of a complex scalar field $\phi(x)$ ("pions" $\pi^\pm$) coupled to the quantized electromagnetic field $A_\mu(x)$. Consider the process $\gamma \to \pi^+ + \pi^-$ (*pair creation*) with a photon of 4-momentum p_i and polarization α in the initial *in* state and pions of momenta p_+ and p_- in the final state. The S-matrix element is
$$\langle p_+ p_-|\hat{S}|p_i, \alpha\rangle \tag{10}$$
Find a *reduction formula* which relates this matrix element to v.e.v. of a set of time-ordered fields for this particular process.