

Physics 582, Fall Semester 2023
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Problem Set No. 4:

Path Integrals in Quantum Mechanics and in Quantum Field Theory

Due Date: Friday October 20, 2023, 9:00 pm US Central Time

1 Path Integral for a particle in a double well potential.

Consider a particle with coordinate q , mass m moving in the one-dimensional double well potential $V(q)$

$$V(q) = \lambda (q^2 - q_0^2)^2 \quad (1)$$

In this problem you will use the path integral methods, *in imaginary time*, that were discussed in class to calculate the matrix element,

$$\langle q_0, \frac{\tau}{2} | -q_0, -\frac{\tau}{2} \rangle = \langle q_0 | e^{-H\tau/\hbar} | -q_0 \rangle \quad (2)$$

to *leading order* in the semiclassical expansion, in the limit $\tau \rightarrow \infty$.

1. Write down the expression of the imaginary time path integral that is appropriate for this problem. Write an explicit expression for the Euclidean Lagrangian (i.e. the Lagrangian in imaginary time). How does it differ from the Lagrangian in real time? Make sure that you specify the initial and final conditions. Do not calculate anything yet!
2. Derive the Euler-Lagrange equation for this problem (always in imaginary time). Compare it with the equation of motion in real time. Find the explicit solution for the trajectory (in imaginary time) that satisfies the initial and final conditions. Is the solution unique? Explain. What is the physical interpretation of this trajectory and of the amplitude?
Hint: Your equation of motion in imaginary time looks funny. A simple way to solve for the trajectory that you need in this problem is to think of this equation of motion as if imaginary time was real time, then to find the analog of the classical energy and to use the conservation of energy to find the trajectory.
3. Compute the imaginary time action for the trajectory you found above.

- Expand around the solution you found above. Write a formal expression of the amplitude to leading order. Find an explicit expression for the operator that enters in the fluctuation determinant. **Do not compute the determinant.**

2 Path Integral for a charged particle moving on a plane in a perpendicular magnetic field.

Consider a particle of mass m and charge $-e$ moving on a plane in the presence of an external uniform magnetic field perpendicular to the plane and with strength B . Let $\mathbf{r} = (x_1, x_2)$ and $\mathbf{p} = (p_1, p_2)$ represent the coordinates and momentum of the particle. The Hamiltonian of the system is

$$H(q, p) = \frac{1}{2m} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 \quad (3)$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential. In the gauge $\vec{\nabla} \cdot \mathbf{A}(\mathbf{r}) = 0$, the vector potential is given by

$$A_1(\mathbf{r}) = -\frac{B}{2}x_2, \quad A_2(\mathbf{r}) = \frac{B}{2}x_1 \quad (4)$$

- Derive a *path integral formula* for the transition amplitude of the process in which the particle returns to its initial location \mathbf{r}_0 at time t_f having left that point at t_i i.e.,

$$\langle \mathbf{r}_0, t_f | \mathbf{r}_0, t_i \rangle \quad (5)$$

where \mathbf{r}_0 is an arbitrary point of the plane and $|t_f - t_i| \rightarrow \infty$.

- Consider now the “ultra-quantum” limit $m \rightarrow 0$. Find the form of the action S in this limit for a path which begins and ends at \mathbf{r}_0 . Find a geometric interpretation for this formula. (Hint: at some point you may have to use Stokes theorem).
- Suppose that the particle actually is not on a two dimensional plane but is instead restricted to be on the surface of a *sphere* of large radius $R \rightarrow \infty$, and that there is a uniform magnetic field normal to the surface of the sphere. Strictly speaking this requires to have a magnetic monopole inside the sphere. Although the associated vector potential reduces to the expression given in Eq.(4) only locally, i.e. on a plane locally tangent to the sphere at given point. Question: are there any ambiguities involved in the evaluation of the expression for the action of the particle?. Think of the regions enclosed by the path and left outside of the path. What condition should satisfy the field strength B , so that the *amplitude* $e^{\frac{i}{\hbar}S}$ is free from any ambiguities?. How is this condition related to the flux quantization? Explain your arguments.

3 Path Integrals for a Scalar Field Theory

Consider the problem of a one-component charged (i.e. complex) free scalar field $\phi(x)$ in $D = 4$ space-time dimensions. The action of this theory coupled to complex sources $J(x)$ is

$$\mathcal{S} = \int d^4x \left[\partial_\mu \phi(x)^* \partial^\mu \phi(x) - m^2 \phi(x)^* \phi(x) - J(x)^* \phi(x) - J(x) \phi(x)^* \right] \quad (6)$$

1. Find an explicit formula for the *vacuum persistence amplitude* ${}_J \langle 0|0 \rangle_J$ for this theory in the form of a path integral in Minkowski space. Find the form of the path integral in Euclidean space (i.e., imaginary time).
2. Evaluate, using the methods discussed in class, the path integral you just wrote down, for a general set of complex sources $J(x)$, both in Minkowski and in Euclidean space-time.
3. Use the formulas you just derived to compute the following *two-point functions* (in the limit $J \rightarrow 0$)

$$G_2(x - x') = \langle 0 | T \phi(x) \phi^*(x') | 0 \rangle \quad (7)$$

$$G_2^*(x - x') = \langle 0 | T \phi^*(x) \phi(x') | 0 \rangle \quad (8)$$

$$G_2'(x - x') = \langle 0 | T \phi(x) \phi(x') | 0 \rangle \quad (9)$$

$$G_2'^*(x - x') = \langle 0 | T \phi^*(x) \phi^*(x') | 0 \rangle \quad (10)$$

Here x and x' are two arbitrary point in space-time. Write your solutions in terms of a Green function. Do not calculate the Green function yet.

4. Find the equation that is satisfied by the Green function in both Euclidean and Minkowski space-times. Solve the equation for the Euclidean case and find the solution in Minkowski space-time by analytic continuation. Discuss the different asymptotic behaviors of the two-point functions in both Euclidean and Minkowski space-times.
5. Use the results you derived in parts 1 and 2 of this problem to find explicit expressions for the following *four-point functions*

$$G_4^a(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2)^* \phi(x_3) \phi(x_4) | 0 \rangle \quad (11)$$

$$G_4'(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle \quad (12)$$

$$G_4'^*(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2)^* \phi(x_3)^* \phi(x_4)^* | 0 \rangle \quad (13)$$

$$G_4^b(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2) \phi(x_3)^* \phi(x_4) | 0 \rangle \quad (14)$$

$$G_4^c(x_1, x_2, x_3, x_4) = \langle 0 | T \phi(x_1)^* \phi(x_2) \phi(x_3) \phi(x_4)^* | 0 \rangle \quad (15)$$

in terms of the two point functions. Find relations among these four point functions.