Physics 582, Fall Semester 2023
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Problem Set No. 7/ Final Exam

Due Date: Friday, December 15, 2023; 9:00 pm US Central Time

This is a take home final exam. Your solutions must be uploaded before the deadline to the my.physics website for uploads for Physics 582. This can be done either by writing your solutions in LaTeX and sending me the pdf file, or by scanning your handwritten solutions and sending me them as a pdf file. If you choose the latter option you will have to make sure that the scanned file is clearly legible and this will require the use of a dark pen and clear handwriting. Your solutions will have to be uploaded to the my.physics website for Physics 582 (as you have done with the homeworks) before Friday December 15 9:00 pm US central Time, without exception. No solutions will be accepted after that.

In this set you will study the properties of a theory of a self-interacting ***complex*** scalar field $\phi(x)$, in perturbation theory. The Lagrangian density for this theory in four-dimensional Minkowski space-time is

$$\mathcal{L} = \partial_\mu \phi^*(x)\, \partial^\mu \phi(x) - m_0^2\, \phi^*(x)\phi(x) - \lambda \left(\phi^*(x)\, \phi(x)\right)^2 + J(x)^* \phi(x) + J(x)\phi^*(x) \tag{1}$$

Assume that $m_0^2 > 0$ and that *coupling constant* $\lambda > 0$. Here $J(x)$ are a set of (complex) sources. Here you will be asked to do calculations using both canonical (operator) and path integral methods.

1. Derive the classical equations of motion of this theory. Give an argument to show that the *classical* ground state of the theory is given by the classical field configuration $\bar{\phi}(x) = 0$.

2. Use path integral methods to derive an explicit expression for the generating function for the vacuum expectation values (v. e. v.) of time-ordered products $\mathcal{Z}_0\left[J, J^*\right]$ for the **free field theory** both in Minkowski space-time and in Euclidean space time (imaginary time). Assume that the space time has infinite extent and that the sources (and the fields) vanish at infinity. Write your answer as an expression involving the sources $J(x)$, and a suitable v. e. v. 's of a time-ordered product of two free fields.

3. Use the path integral method to calculate, at the level of the *free field theory, i.e.,* $\lambda = 0$, the following *time ordered products* of the field $\phi(x)$:

(a) $\langle 0|T\ \phi(x)\phi(x')|0\rangle$, (b) $\langle 0|T\ \phi^*(x)\phi(x')|0\rangle$, (c) $\langle 0|T\ \phi(x)\phi^*(x')|0\rangle$, and (d) $\langle 0|T\ \phi^*(x)\phi^*(x')|0\rangle$. Determine which of these expectation values are different from zero and explain why. Use this result to define a set of Feynman propagators for this system. You can give your answer in terms of the Feynman propagator for a *real* scalar field, $G_0(x,x')$, that was discussed in class.

4. Use path integral methods to derive a formula for the generating function $\mathcal{Z}[J,J^*]$ of the full **interacting theory** in terms of $\mathcal{Z}_0[J,J^*]$ and of the interaction part of the Lagrangian. Use this formula to derive the Feynman rules for this theory both in Minkowski space-time and in Euclidean space-time.

5. Give a general argument to prove that the vacuum diagrams cancel out from the computation of v. e. v. 's of any number of field operators.

6. Use the Feynman rules that you derived in part 4 to obtain the perturbation theory expansion in Euclidean space-time for (a) the *vacuum diagrams*, (b) the *two-point functions*, and (c) the *four-point functions*, in position space and up to and *including* all terms up to **second order** in the coupling constant λ. Assign a Feynman diagram to each term. Give an explicit formula for each term, including the multiplicity factors. Do not do the integrals!. Verify the cancellation of the vacuum diagrams.

7. Derive an expression for the two-point functions and the four-point functions in Euclidean space-time in momentum space to order λ for the two point functions and to order λ^2 for the four point functions. Draw a Feynman diagram for each term in momentum space. Do not do the integrals!.

8. Consider the free propagators in Minkowski space-time. Rotate the integration contour to the imaginary frequency axis. Explain the relation between this procedure and the analytic continuation to *imaginary time*. What domain of space-time events is actually described by this procedure? Show that the integrals that you get are equivalent to integrals in *Euclidean* space-time dimensions $D = 4$.

9. Derive an expression for the *self-energy* to first order in λ in Euclidean space-time. Use this result to get a formula for the *effective* (or *renormalized*) *mass* m to first order in λ. Does the mass get renormalized up or down? Can you give a simple explanation for this result? (hint: think of the shift of the energy levels in quantum mechanics).

10. Use a procedure analogous to the one you employed in part 9 to define an *effective* or *renormalized coupling constant* g in terms of the *bare coupling constant* λ and of the renormalized mass m, to leading order in λ.

Useful Integrals:

You will find useful to know the following integrals in Euclidean momentum space:

$$\frac{1}{A^n B^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \, \frac{x^{n-1}(1-x)^{m-1}}{(xA+(1-x)B)^{n+m}} \tag{2}$$

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{\left(p^2+2p\cdot q+m_0^2\right)^n} = \frac{1}{2} \, \frac{S_D}{(2\pi)^D} \, \frac{\Gamma(\frac{D}{2})\Gamma(n-\frac{D}{2})}{\Gamma(n)} \left(m_0^2-q^2\right)^{\frac{D}{2}-n} \tag{3}$$

where S_D is the volume of the D-dimensional unit hypersphere

$$S_D = \frac{2\,\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \tag{4}$$

and $\Gamma(s)$ is the Γ-function

$$\Gamma(s) = \int_0^\infty dt \; t^{s-1} e^{-t} \tag{5}$$

For $s \to 0$, the Γ-function has the asymptotic behavior

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{1}{s} - \gamma + O(s) \tag{6}$$

where γ is the Euler constant

$$\gamma = 0.57721... = \lim_{m\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}-\ln m\right] \tag{7}$$