Physics 582, Fall Semester 2024
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Problem Set No. 5:

Due Date: Sunday November 17, 2024, 9:00 pm US Central Time

1 Grassmann Variables

1. Let a and a^* be a pair of Grassmann variables. Let $g(a^*)$ be an "analytic function" of a single Grassmann variable a^*, i.e.

$$g(a^*) = g_0 + g_1 a^* \tag{1}$$

and let $f(a)$ be another such function. Show that the inner product $\langle f|g\rangle$ defined by

$$\langle f|g\rangle = \int da^* da \; e^{-a^* a} f^*(a) g(a^*) \tag{2}$$

implies that

$$\langle f|g\rangle = \bar{f}_0 g_0 + \bar{f}_1 g_1 \tag{3}$$

where $\bar{x}$ stands for the complex conjugate of x.

2. Show that

$$(Af)(a^*) = \int d\alpha^* d\alpha A(a^*, \alpha) f(\alpha^*) e^{-\alpha^* \alpha} = g(a^*) \tag{4}$$

is equivalent to

$$\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} A_{00} & A_{10} \\ A_{01} & A_{11} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \tag{5}$$

and that

$$(A\ B)\,(a^*, a) = \int d\alpha^* d\alpha \;\; e^{-\alpha^* \alpha} \; A(a^*, \alpha) \; B(\alpha^*, a) = C(a^*, a) \tag{6}$$

is equivalent to the standard definition of the product of two 2×2 matrices.

3. Show that the operators $\hat{a}^*$ and $\hat{a}$, defined by

$$\hat{a}^* f(\alpha^*) = a^* f(a^*) \qquad \hat{a} f(a^*) = \frac{d}{da^*} f(a^*) \tag{7}$$

satisfy canonical anticommutation relations,i.e. $\hat{a}^* \; \hat{a}^* = \hat{a} \; \hat{a} = 0$ and $\{\hat{a}^*, \hat{a}\} = 1$.

4. Show that, if $\{\xi_j\}$ is a set of N Grassmann variables ($j = 1, \ldots, N$), then

$$\mathcal{Z} = \int \prod_{j=1}^{N} d\xi_j^* d\xi_j \ \exp\{-\sum_{k,l=1}^{N} \xi_k^* M_{kl} \xi_l\} = \det M \tag{8}$$

2 Dirac Fermions

The Lagrangian density $\mathcal{L}$ for the free massive Dirac field in 4-dimensional Minkowski space is

$$\mathcal{L} = \bar{\psi} \left(i\partial\!\!\!/ - m \right) \psi \tag{9}$$

1. Consider the path integral for a free Dirac field in four space-time dimensions, coupled to a set of Grassmann sources $\bar{\eta}_\alpha(x)$ and $\eta_\alpha(x)$. Derive an expression for this generating function in terms of the sources and a Fermion determinant. Do *not* compute the determinant.

2. Use the results of previous question to show that the Feynman propagator of the Dirac theory is given by

$$S_F^{\alpha\beta}(x-y) = \langle x, \alpha | \frac{1}{i\partial\!\!\!/ - m} | y, \beta \rangle \tag{10}$$

3. Use the results of the first part of this problem to derive an expression for the four point function

$$S_F^{(4)}(x_1, x_2, x_3, x_4)_{\alpha,\beta,\gamma,\delta} = \langle 0 | \psi_\alpha(x_1) \psi_\beta(x_2) \bar{\psi}_\gamma(x_3) \bar{\psi}_\delta(x_4) | 0 \rangle \tag{11}$$

in terms of products of propagators. Beware of the signs!!!!!!

3 Functional Determinants and the Casimir Effect

In this problem we are going to consider a free scalar field $\phi(x,t)$ in $1+1$ space-time dimensions. The Lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 \tag{12}$$

where $x \equiv (x,t)$. Consider the case in which the total length of the system along the space coordinate is equal to L and assume *periodic boundary conditions*, i.e.

$$\phi(x,t) = \phi(x+L,t) \tag{13}$$

for all times t.

1. Calculate the *classical* value of the ground state energy of the system with the boundary conditions specified above.

2. Use path integral methods to derive a formal expression for the total ground state energy density (i.e. energy per unit length). This formula should contain a determinant which you should not compute for the moment.

3. Use the method of the ζ-function to compute the quantum correction to the ground state energy density. Consider the massless limit $m \to 0$ only. Write your answer down in the form of an *extensive* piece and a *finite-size* term which vanishes as $L \to \infty$ like A/L^η, with $\eta > 0$. Find the value of this exponent η as well as the value of the coefficient A, sign included. Note: You may have to keep a dependence on the mass in one of the two terms. Keep just the leading behavior in the small mass limit. Note: At some point of the calculation the following result may be useful:
Poisson Summation Formula:
$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dx \;\; f(x) \;\; e^{2\pi i m x} \tag{14}$$

4. If you interpret the dependence of the ground state energy on the linear size of the system L as a potential energy for the"walls" that confine the system, what can you say about the force that the zero-point fluctuations exert on these "walls".
Note: In order to speak about walls we should have used vanishing, instead of periodic boundary conditions as we have done. The calculation is somewhat more complicated in that case. This effect, i.e. a force exerted on the walls of a system by the zero point motion of a field is known as the *Casimir effect*.

4 The Weakly-Interacting Bose Gas

Consider a gas of non-relativistic Bose particles at fixed density ρ inside a very large box of linear size L in three space dimensions. Let $\phi^\dagger(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$ be a set of boson creation and annihilation operators. The second quantized Hamiltonian is

$$H = \int d^3x \; \phi^\dagger(\boldsymbol{x}) \left(\frac{\hat{\boldsymbol{p}}^2}{2m} - \mu \right) \phi(\boldsymbol{x}) + \frac{1}{2} \int d^3x \int d^3x' \hat{n}(\boldsymbol{x}) V(\boldsymbol{x}-\boldsymbol{x}') \hat{n}(\boldsymbol{x}'). \tag{15}$$

where μ is the chemical potential, $\hat{n} = \phi^\dagger \phi$ and $V(r)$ is a rotationally invariant short range interaction which we will take to be equal to

$$V(\boldsymbol{x}-\boldsymbol{x}') = \lambda \delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{16}$$

The positive constant λ is the scattering amplitude and will play the role of a coupling constant for this system.

1. Use the method of Bose coherent states to find a path integral *formula* for the partition function of this system at temperature T. Do not compute the path integral at this stage. Assume *constant* boundary conditions at spacial infinity (i.e. that the field amplitude approaches a constant value at the boundaries). Carefully specify the boundary conditions in the imaginary time dimension. Write your answer down in the form

$$\mathcal{Z} = \int \mathcal{D}\phi^* \, \mathcal{D}\phi \;\; e^{-S_E(\phi^*, \phi)} \tag{17}$$

   and give an explicit expression for the Euclidean action S_E.

2. Use the method of semiclassical quantization (i.e. the saddle point expansion) to determine the classical path at temperature T. What condition should be satisfied by $\phi(x)$ in order for it to be such a classical path?. Find the relationship between the ground state of the system at $T = 0$ and this classical path in the limit $T \to 0$. Is the solution unique?. Justify your answer.
   Hint: Think of the symmetries of the Lagrangian. This will give you an idea about the uniqueness of the classical path. You may find it convenient to write the classical path $\phi(x)$ in the form of an amplitude times a phase.

3. Compute the time-ordered Green function

$$G(x-y) = -i \, \langle \hat{T} \hat{\phi}(x) \hat{\phi}^\dagger(y) \rangle \tag{18}$$

   at $T = 0$, in the semiclassical limit. What is the asymptotic value of G in the limit of equal times and large space separation?. Give a physical interpretation of this result.

4. Consider small quantum fluctuations around the classical path found in the previous sections. Write an arbitrary (but close) configuration $\phi(x)$ in the form

$$\phi(x) = \sqrt{\rho_0 + \delta\rho(x)} \quad e^{i\theta(x)} \tag{19}$$

   Expand the action in powers of $\delta\rho$ and θ up to second order in both. Check the cancellation of the linear terms. Integrate out the density fluctuations and find an effective action for the phase variable

$$e^{-S_{\text{eff}}(\theta(x))} = \int \mathcal{D}\delta\rho \; e^{-S_E} \tag{20}$$

   which is quadratic in θ.

5. Show that, for configurations $\{\theta(x)\}$ which are slowly varying, the effective action has the form

$$S_{\text{eff}} = \int d^4x \; \frac{1}{2} K \left[(\partial_\tau \theta)^2 + v^2 \left(\boldsymbol{\nabla} \theta(x) \right)^2 \right] \tag{21}$$

and calculate the coefficients K and v. Find the analytic continuation of this expression back in real time. Show that v is the velocity of propagation of the excitations. Find the time ordered propagator of the phase field $\theta(x)$. What equation of motion does it satisfy?. Draw an analogy between this equation and the equation of motion for a relativistic massless scalar field.