10
Observables and Propagators

In earlier chapters we have considered the properties of several of field theories that describe physical systems isolated from the outside world. However, the only way to investigate the properties of a physical system we must interact with it. Thus, we must consider physical systems that, somehow, are coupled to external fields (or sources).

We will have two situations in mind. In one case, we will look at the problem of the interaction of states of an isolated system, i.e. a scattering problem. In this case we prepare states (or wave packets) which are sufficiently far apart so that their mutual interactions can be neglected. The prototype is the scattering of particles off a target or each other in a particle accelerator experiment. Here the physical properties are encoded in a suitable set of cross sections.

In the second case, we will imagine that we want to understand that properties of a large system "from outside", and we will consider the role of small external perturbations. Here we will develop a general approach, known as linear response theory. We will couch our results in terms of a suitable set of susceptibilities. This is the typical situation of interest in many experiments in condensed matter physics.

In both cases, all quantities of physical interest will be derived from a suitably defined correlation function or propagator. Our task will be twofold. First we will determine the general expected properties of the propagators (in particular, their analytic properties). Second, we will show that their analytic properties largely determine the behavior of cross sections and susceptibilities.

10.1 The propagator in classical electrodynamics

In a classical field theory, such as Classical Electrodynamics, we can investigate the properties of the electromagnetic field by considering the effect of a set of well localized external sources. These can be *electric charges* or, more generally, some well defined distribution of *electric currents*. The result is familiar to us: the external currents set up a radiation field that propagates in space, a propagating electromagnetic field. In Maxwell's electrodynamics these effects are described by the Maxwell equations, the equations of motion of the electromagnetic field generated by of a current distribution $j^\mu(x)$

$$\partial^2 A^\mu(x) = j^\mu(x) \tag{10.1}$$

where we have imposed the Lorentz gauge condition, $\partial_\mu A^\mu = 0$, and hence the equation of motion reduces to the wave equation.

In Classical Electrodynamics the solutions to this equation are found using the *Green function* $G(x, x')$,

$$\partial_x^2 G(x, x') = \delta^4(x - x') \tag{10.2}$$

which satisfies the *boundary condition* (or, rather, initial condition) that

$$G(x, x') = 0, \quad \text{if} \quad x_0 < x_0' \tag{10.3}$$

This is the *retarded Green function*, that we will denote by $G_R(x-x')$, which vanishes for events in the past, $x_0 < x_0'$.

The wave equation in the presence of a set of currents $j^\mu(x)$ is an inhomogeneous partial differential equation. For times in the remote past, before any currents were present, there should be no electromagnetic field present. The choice of retarded boundary conditions guarantees that the system obeys *causality*. The solution to the inhomogeneous partial differential equation is, as usual, the sum of an *arbitrary* solution of the homogeneous equation, A^μ_{in}, that represents a preexisting electromagnetic field, and a *particular* solution of the inhomogeneous field equation. We write the general solution in the form

$$A^\mu(x) = A^\mu_{in}(x) + \int d^4x' G_R(x - x') j^\mu(x') \tag{10.4}$$

where $A^\mu_{in}(x)$ is a solution of the wave equation in free space (in the absence of sources),

$$\partial^2 A^\mu_{in}(x) = 0 \tag{10.5}$$

Thus, all we need to know is the retarded Green function, $G_R(x-x')$. Notice that the choice of retarded boundary conditions ensures that, for $x_0 < x_0'$, $A^\mu(x) = A^\mu_{in}(x)$, since $G_R(x - x') = 0$ for $x_0 < x_0'$.

Let us solve for the Green function $G(x, x')$. This is most easily done in terms of the Fourier transform of $G(x, x')$

$$G(x, x') = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x - x')} \widetilde{G}(p) \tag{10.6}$$

where $p \equiv p^\mu$ and $p \cdot x \equiv p_0 x_0 - \boldsymbol{p} \cdot \boldsymbol{x}$. It is easy to check that, formally, the Fourier transform $\widetilde{G}(p)$ should be given by

$$\widetilde{G}(p) = -\frac{1}{p^2} \tag{10.7}$$

Now we have two problems. One is that $\widetilde{G}(p)$ has a singularity at $p^2 \equiv p_0^2 - \boldsymbol{p}^2 = 0$, i.e. at the eigenfrequencies of the normal modes of the free electromagnetic field $p_0 = \pm|\boldsymbol{p}|$. Thus, the integral is ill-defined, and some definition must be specified on what to do with the singularity. The other problem is that this function $G(x, x')$ does not satisfy, at least not in any obvious way, the boundary conditions.

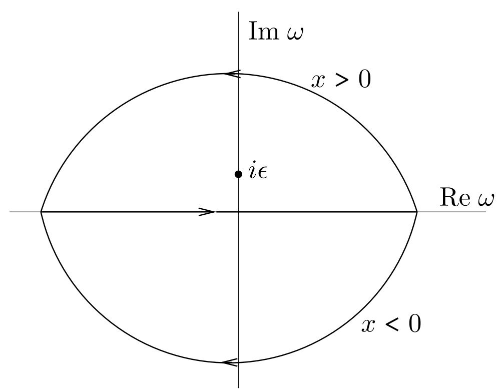

Figure 10.1 Contour in the complex plane that defines the function $\Theta(x)$.

We will solve both problems simultaneously. Let us define the *retarded Green function* by

$$G_R(x, x') \equiv \Theta(x_0 - x_0') G(x, x') \tag{10.8}$$

where $\Theta(x)$ is the step (or Heaviside) function

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{10.9}$$

Therefore, $G_R(x, x')$ vanishes for $x_0 < x_0'$ and is equal to $G(x - x')$ for $x_0 > x_0'$.

The step function $\Theta(x)$ has the formal integral representation, a Fourier transform,

$$\Theta(x) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega x}}{\omega - i\epsilon} \tag{10.10}$$

where the integral is interpreted as an integral over the contours shown in Fig. 10.1. Thus, when closing the contour on the lower half-plane, as in the case when $x < 0$ shown in Fig.10.1, the closed contour does not contain a pole, and the integral vanishes by the Cauchy Theorem. Conversely, for $x > 0$ we close the contour on a large arc in the upper half plane, and pick up a contribution from the enclosed pole equal to the residue, $e^{-\epsilon x}$ which converges to 1 as $\epsilon \to 0^+$. Notice that the integral on the large arc in the upper half plane converges to zero, for arcs with radius $R \to \infty$, only if $x > 0$.

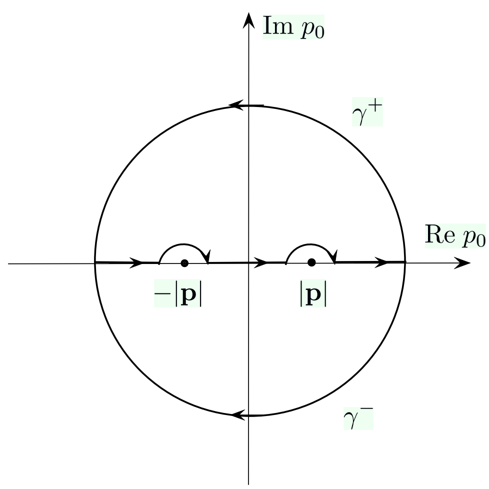

Figure 10.2 Contour in the complex plane that defines retarded Green function.

We define the retarded Green function, $G_R(x - x')$, by the following expression,

$$G_R(x - x') = - \lim_{\epsilon \to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip \cdot (x - x')}}{(p_0 + i\epsilon)^2 - \boldsymbol{p}^2} \tag{10.11}$$

which satisfies of the requirements.

We can also define the *advanced Green* function, $G_A(x-x')$, that vanishes in the future, but not in the past,

$$G_A(x-x') = 0, \qquad \text{for} \quad x_0 - x_0' > 0 \tag{10.12}$$

by changing the sign of ϵ,

$$G_A(x-x') = -\lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{(p_0 - i\epsilon)^2 - \boldsymbol{p}^2} \tag{10.13}$$

Now the poles are on the upper half-plane, and the integral vanishes for $x_0 - x_0' > 0$. However, for $x_0 - x_0' < 0$ we pick up the poles when we close on the lower half-plane, and this contribution does not vanish.

There are still two other possible choices of contours, such as the one shown in Fig.10.3. In this case, when we close on the contour γ^-, in the

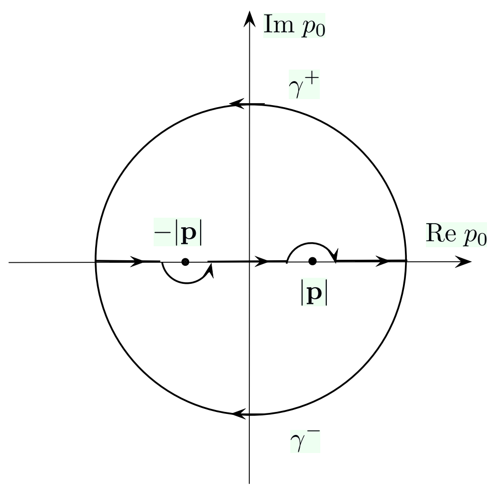

Figure 10.3 Contour in the complex plane that defines Feynman propagator, or time ordered Green function.

lower half-plane, we pick up the contribution of the positive frequency pole at $+|\boldsymbol{p}|$. The resulting frequency integral is

$$\oint_{\gamma^-} \frac{dp_0}{2\pi} \frac{e^{-ip\cdot(x-x')}}{p_0^2 - \boldsymbol{p}^2 + i\epsilon} = -\frac{i}{2|\boldsymbol{p}|} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}') - i|\boldsymbol{p}|(x_0 - x_0')}, \qquad \text{for } x_0 > x_0' \tag{10.14}$$

In the opposite case, we close on the contour γ^+ in the upper half-plane, which encloses the negative frequency pole at $-|\boldsymbol{p}|$. The integral now becomes

$$\oint_{\gamma^+} \frac{dp_0}{2\pi i} \frac{e^{-ip\cdot(x-x')}}{p_0^2 - \boldsymbol{p}^2 + i\epsilon} = \frac{i}{-2|\boldsymbol{p}|} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}') - i|\boldsymbol{p}|(x_0 - x_0')}, \qquad \text{for } x_0 < x_0' \tag{10.15}$$

With this choice, the contour integral yields the *time-ordered Green function*

$$\begin{aligned} G_F(x-x') &= -\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{p^2 + i\epsilon} \\ &= i\int \frac{d^3p}{(2\pi)^3 2|\boldsymbol{p}|} \Big\{ \Theta(x_0 - x_0')\, e^{-i|\boldsymbol{p}|(x_0 - x_0') + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \\ &\qquad + \Theta(x_0' - x_0) e^{+i|\boldsymbol{p}|(x_0 - x_0') - i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \Big\} \end{aligned} \tag{10.16}$$

which is known as the *Feynman propagator*, and bears a close formal resemblance to the mode expansions of free field theory. We will see below that the integration measure is Lorentz invariant. Also notice that this Feynman *propagator* or *Green function* propagates the *positive frequency* modes *forward* in time and the *negative frequency* modes backward in time. The alternative choice of contour simply yields the negative of $G_F(x-x')$.

10.2 The propagator in non-relativistic Quantum Mechanics

In non-relativistic quantum mechanics, the evolution of quantum states is governed by the Schrödinger Equation

$$(i\hbar\partial_t - H)\psi = 0 \tag{10.17}$$

Let $H = H_0 + V$, and V be some position and time-dependent potential that vanishes (*very slowly*) both in the remote past ($t \to -\infty$) and in the remote future ($t \to +\infty$). In this case, the eigenstates of the system are, in both limits, eigenstates of H_0. If $V(x,t)$ varies slowly with time, the states of H evolve smoothly, or adiabatically. Thus, we are describing *scattering processes* between free particle states. Let $F(\boldsymbol{x}', t'|\boldsymbol{x}, t)$ denote the amplitude

$$F(\boldsymbol{x}', t'|\boldsymbol{x}, t) \equiv \langle \boldsymbol{x}', t'|\boldsymbol{x}, t\rangle \tag{10.18}$$

We have already introduced this amplitude when we discussed the path integral picture of Quantum Mechanics.

Let us suppose that, at some time t, the system is in the state $|\psi(t)\rangle =$

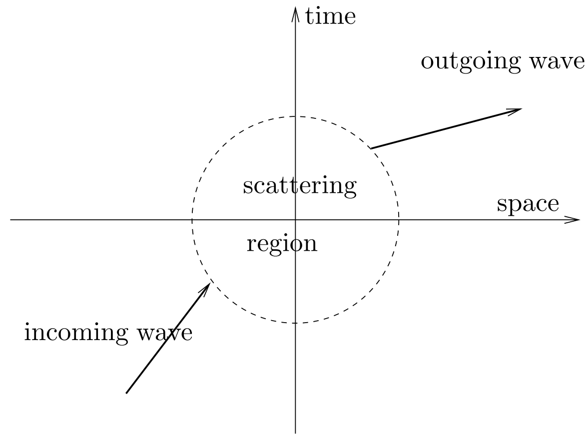

Figure 10.4 A scattering process.

$|\boldsymbol{x}, t\rangle$. At some time $t' > t$, the state of the system is $|\psi, t'\rangle$, which is obtained by solving the Schrödinger equation

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle \tag{10.19}$$

where H is generally time-dependent. The formal solution of this equation is

$$|\psi(t')\rangle = T\, e^{-\frac{i}{\hbar}\int_t^{t'} dt'' H(t'')}\, |\psi(t)\rangle \tag{10.20}$$

where T is the time ordering symbol, i.e.

$$T\, e^{\frac{-i}{\hbar}\int_t^{t'} dt'' H(t'')} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \int_t^{t'} dt_1 \ldots \int_t^{t_{n-1}} dt_n H(t_1)\ldots H(t_n) \tag{10.21}$$

Thus, the amplitude $F(\boldsymbol{x}', t'|\boldsymbol{x}, t)$ is

$$F(\boldsymbol{x}', t'|\boldsymbol{x}t) = \langle \boldsymbol{x}', t'|\boldsymbol{x}, t\rangle \equiv \langle \boldsymbol{x}'|\psi(t')\rangle \tag{10.22}$$

Hence, if the initial state is $|\psi(t)\rangle = |\boldsymbol{x}\rangle$, then we get

$$F(\boldsymbol{x}', t'|\boldsymbol{x}, t) = \langle \boldsymbol{x}'|T\, e^{-\frac{i}{\hbar}\int_t^{t'} dt'' H(t'')}\, |\boldsymbol{x}\rangle \tag{10.23}$$

For an *arbitrary* initial state $|\psi(t)\rangle$ we get

$$\langle \boldsymbol{x}'|\psi(t')\rangle = \langle \boldsymbol{x}'|\ e^{-\frac{i}{\hbar}\int_t^{t'} dt'' H(t'')}\ |\psi(t)\rangle \tag{10.24}$$

Since the states $\{|\boldsymbol{x}\rangle\}$ are complete, we have the completeness relation

$$1 = \int d\boldsymbol{x} |\boldsymbol{x}\rangle\langle \boldsymbol{x}| \tag{10.25}$$

which allows us to write

$$\langle \boldsymbol{x}'|\psi(t')\rangle = \int_{-\infty}^{+\infty} d\boldsymbol{x} \langle \boldsymbol{x}'|\ e^{-\frac{i}{\hbar}\int_t^{t'} dt'' H(t'')}\ |\boldsymbol{x}\rangle\langle \boldsymbol{x}|\psi(t)\rangle \tag{10.26}$$

so, we get

$$\psi(\boldsymbol{x}', t') = \int_{-\infty}^{+\infty} d\boldsymbol{x} \langle \boldsymbol{x}', t'|\boldsymbol{x}, t\rangle \psi(\boldsymbol{x}, t) \tag{10.27}$$

In other words, the amplitude $F(\boldsymbol{x}', t'|\boldsymbol{x}, t)$ is the kernel of the time evolution for arbitrary states. The amplitude $F(\boldsymbol{x}', t'|\boldsymbol{x}, t)$ is known as the *Schwinger function.*

The initial state $|\psi(t)\rangle$ and the final state $|\psi(t')\rangle$ are connected by the *evolution operator* $U(t', t)$

$$U(t', t) \equiv T\ e^{-\frac{i}{\hbar}\int_t^{t'} dt'' H(t'')} \tag{10.28}$$

which is *unitary* since, as a result of the hermiticity of the Hamiltonian,

$$\begin{aligned} U^\dagger(t', t) &= T\ e^{\frac{i}{\hbar}\int_t^{t'} dt'' H(t'')} \\ &= T\ e^{-\frac{i}{\hbar}\int_{t'}^{t} dt'' H(t'')} = U(t, t') \end{aligned} \tag{10.29}$$

By definition, $U(t, t')$ is the *inverse* of $U(t', t)$ since it evolves the states *backwards* in time. In addition, the operator $U(t', t)$ obeys the initial condition

$$\lim_{t' \to t} U(t', t) = I \tag{10.30}$$

where I is the identity operator. If the Hamiltonian is time-independent, the evolution operator is

$$U(t',t) = T\, e^{-\frac{i}{\hbar}\int_t^{t'} dt'' H(t'')} = e^{-\frac{i}{\hbar}H(t'-t)} \tag{10.31}$$

where the last step holds only for a time-independent Hamiltonian.

These ideas will also allow us to introduce the *Scattering Matrix* (or *S-Matrix*). If $\psi_i(\boldsymbol{x},t)$ is some initial state and $\psi_f(\boldsymbol{x}',t')$ is some final state, the matrix elements of the S-matrix between states ψ_i and ψ_f, S_{fi} are obtained by evolving the state ψ_i up to time t', and projecting it onto the state ψ_f. Namely

$$S_{fi} = \lim_{t'\to+\infty}\lim_{t\to-\infty}\int d\boldsymbol{x}\int d\boldsymbol{x}'\psi_f^*(\boldsymbol{x}',t')\langle\boldsymbol{x}',t'|\boldsymbol{x},t\rangle\psi_i(\boldsymbol{x},t) \tag{10.32}$$

Let us define the *Green function* or *propagator* $G(\boldsymbol{x}',t'|\boldsymbol{x},t)$

$$G(\boldsymbol{x}',t'|\boldsymbol{x},t) \equiv -\frac{i}{\hbar}\Theta(t'-t)\langle\boldsymbol{x}',t'|\boldsymbol{x},t\rangle = -\frac{i}{\hbar}\Theta(t'-t)F(\boldsymbol{x},t'|\boldsymbol{x},t) \tag{10.33}$$

It satisfies the equation

$$(i\hbar\partial_{t'} - H)\, G(\boldsymbol{x}',t'|\boldsymbol{x},t) = \delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t') \tag{10.34}$$

with the boundary condition

$$G(\boldsymbol{x}',t'|\boldsymbol{x},t) = 0, \quad \text{if } t' < t \tag{10.35}$$

Hence, $G(\boldsymbol{x}',t'|\boldsymbol{x},t)$ is known as the *retarded Schrödinger propagator*.

In terms of G, the S-matrix is given by (recall that $t' > t$)

$$S_{fi} = i\lim_{t'\to+\infty}\lim_{t\to-\infty}\int d\boldsymbol{x}\int d\boldsymbol{x}'\psi_f^*(\boldsymbol{x}',t')G(\boldsymbol{x}',t'|\boldsymbol{x},t)\psi_i(\boldsymbol{x},t) \tag{10.36}$$

Let us consider now the case of a free particle with Hamiltonian H_0 that is coupled to an external perturbation represented by a potential $V(\boldsymbol{x},t)$. The free Green function, G_0, satisfies the equation

$$(i\hbar\partial_{t'} - H_0)\, G_0 = \delta(\boldsymbol{x}'-\boldsymbol{x})\delta(t'-t) \tag{10.37}$$

G_0 can be regarded as the matrix elements of the following operator

$$G_0(\boldsymbol{x}',t'|\boldsymbol{x},t) = \langle\boldsymbol{x}',t'|(i\hbar\partial_t - H)^{-1}|\boldsymbol{x},t\rangle \tag{10.38}$$

Clearly, G satisfies the same equation but with the full H,

$$(i\hbar\partial_t - H)\, G = 1 \tag{10.39}$$

Hence, we can write

$$[(i\hbar\partial_t - H_0) - V]\, G = 1 \tag{10.40}$$

By using the definition of G_0, we get the operator equation

$$(G_0^{-1} - V)G = 1 \tag{10.41}$$

Thus, G satisfies the integral equation

$$G(\boldsymbol{x}', t'|\boldsymbol{x}, t) = G_0(\boldsymbol{x}', t'|\boldsymbol{x}, t) + \int d\boldsymbol{x}'' \int dt'' G_0(\boldsymbol{x}', t'|\boldsymbol{x}'', t'')V(\boldsymbol{x}'', t'')G(\boldsymbol{x}'', t''|\boldsymbol{x}, t) \tag{10.42}$$

which is known as the *Dyson Equation*. It has the formal operator solution

$$G^{-1} = G_0^{-1} - V \tag{10.43}$$

The integral equation can be solved by an iterative procedure, which amounts to a perturbative expansion in powers of V. The result is the *Born series.* Using an obvious matrix notation we get

$$G = G_0 + G_0VG_0 + G_0VG_0VG_0 + \dots \tag{10.44}$$

We will represent this series by a set of *diagrams*. Let us consider the first term to which we assign the diagram of Fig.10.5. The oriented arrow ranging

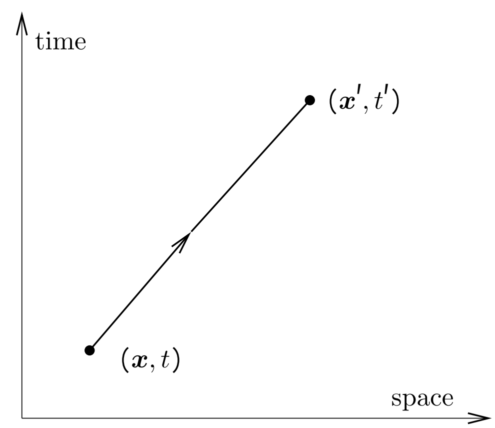

Figure 10.5 The zeroth order term.

from $(\boldsymbol{x}, t)$ to $(\boldsymbol{x}', t')$ represents the unperturbed propagator $G_0(\boldsymbol{x}'t'|\boldsymbol{x}t)$. Because of the causal boundary conditions obeyed by G_0 it can only propagate forward in time. The second term, of the series, the *Born approximation*, $\delta G^{(1)}$

$$\delta G^{(1)}(\boldsymbol{x}', t'|\boldsymbol{x}, t) = \int dx' \int dt'' G_0(\boldsymbol{x}', t'|\boldsymbol{x}'', t'')V(\boldsymbol{x}'', t'')G_0(\boldsymbol{x}'', t''|\boldsymbol{x}, t) \tag{10.45}$$

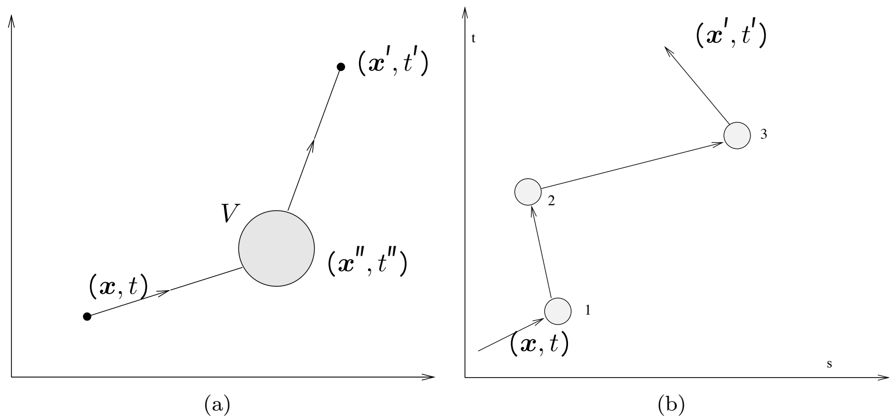

Figure 10.6 a) The first order term: the Born Approximation. b) A multiple scattering process.

is represented by the diagram of Fig.10.6a, where the shaded circle represents the action of the potential V. In general we get a diagram of the form of Fig.10.6b, which represents a multiple scattering process. Notice that all the contributions propagate strictly forward in time.

Let us compute the propagator $G_0(\boldsymbol{x}'t'|\boldsymbol{x}t)$ for a free spinless particle in three-dimensional space. The Hamiltonian H is just, $H = -\frac{\hbar^2}{2m}\bigtriangledown^2$. Thus, G_0 obeys the equation

$$\left(i\hbar\partial_{t'} + \frac{\hbar^2}{2m}\bigtriangledown_x^2\right)G_0(\boldsymbol{x}'t'|\boldsymbol{x}t) = \delta^3(\boldsymbol{x}' - \boldsymbol{x})\delta(t' - t) \tag{10.46}$$

with *causal* boundary conditions, i.e. $G(\boldsymbol{x}'t'|\boldsymbol{x}t) = 0$ for $t' < t$. Given the symmetries of this very simple system we can Fourier expand G_0.

$$G_0(\boldsymbol{x}'t'|\boldsymbol{x}t) = \int \frac{d^3p}{(2\pi)^3}\int \frac{d\omega}{2\pi}\,\widetilde{G}_0(\boldsymbol{p},\omega)e^{-\frac{i}{\hbar}\boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})+i\omega(t'-t)} \tag{10.47}$$

By direct substitution, we find that $\widetilde{G}_0(\boldsymbol{p},\omega)$ is given by

$$\widetilde{G}_0(\boldsymbol{p},\omega) = \frac{1}{\hbar\omega - \frac{\boldsymbol{p}^2}{2m}} \tag{10.48}$$

Notice that, once again, $\widetilde{G}_0(\boldsymbol{p},\omega)$ has a pole at $\hbar\omega = \frac{\boldsymbol{p}^2}{2m}$, on the real frequency axis, which is the dispersion law, or the "mass shell" condition.

Apart from being singular, this "solution" does not obey the causal boundary condition. We will enforce the boundary condition by deforming the integration contour in the complex *frequency* plane. Following our previous discussion on the Green function for classical electrodynamics, we move the pole by an infinitesimal positive amount ϵ into the upper-half of the complex frequency plane. We write the *retarded* propagator as

$$G_0^{ret}(\boldsymbol{p},\omega) = \frac{1}{\hbar\omega - \frac{\boldsymbol{p}^2}{2m} - i\epsilon} \tag{10.49}$$

and we will take the limit $\epsilon \to 0^+$ at the end of our calculations.

The frequency integral is equal to

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(t'-t)}}{\hbar\omega - \frac{\boldsymbol{p}^2}{2m} - i\epsilon} = \frac{i}{\hbar}\Theta(t'-t)e^{\frac{i}{\hbar}\frac{\boldsymbol{p}^2}{2m}(t'-t)} \tag{10.50}$$

Hence, the Green function $G_0(\boldsymbol{x}',t'|\boldsymbol{x},t)$ is

$$G_0(\boldsymbol{x}',t'|\boldsymbol{x},t) = \frac{i}{\hbar}\Theta(t'-t)\int \frac{d^3p}{(2\pi)^3} e^{\frac{i}{\hbar}\frac{\boldsymbol{p}^2}{2m}(t'-t) - \frac{i}{\hbar}\boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})} \tag{10.51}$$

Let $\psi_{\boldsymbol{p}}(\boldsymbol{x},t)$ denote the wave functions for the *stationary states* $|\boldsymbol{p}\rangle$,

$$\psi_{\boldsymbol{p}}(\boldsymbol{x},t) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x} + \frac{i}{\hbar}E(\boldsymbol{p})t} \tag{10.52}$$

where $E(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m}$. We see that $G_0(\boldsymbol{x}',t'|\boldsymbol{x},t)$ can be written in the form

$$G_0(\boldsymbol{x}',t'|\boldsymbol{x},t) = \frac{i}{\hbar}\ \Theta(t'-t)\ \int \frac{d^3p}{(2\pi)^3}\ \psi_{\boldsymbol{p}}(\boldsymbol{x}',t')\psi_{\boldsymbol{p}}^*(\boldsymbol{x},t) \tag{10.53}$$

In general, if the Hamiltonian has a complete set of stationary states $\{|n\rangle\}$ with wave functions $\psi_n(x)$ and eigenvalues E_n, the Green function is

$$G_0^{\rm ret}(\boldsymbol{x}',t'|\boldsymbol{x},t) = \frac{i}{\hbar}\Theta(t'-t)\sum_n \psi_n(\boldsymbol{x}',t')\psi_n^*(\boldsymbol{x},t) \tag{10.54}$$

where

$$\psi_n(\boldsymbol{x},t) = \psi_n(\boldsymbol{x})e^{-\frac{i}{\hbar}E_n t} \tag{10.55}$$

If the system is isolated, the Hamiltonian is time-independent, and the Green function is a function of $t' - t$. In this case, it is convenient to consider the Fourier transform

$$G_0^{\text{ret}}(\boldsymbol{x}', \boldsymbol{x}; t' - t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, G_0^{\text{ret}}(\boldsymbol{x}', \boldsymbol{x}; \omega)\, e^{i\omega(t' - t)/\hbar} \tag{10.56}$$

where we have to pick the correct integration contour so that G_0^{ret} is retarded.

Quite explicitly, we find

$$G_0^{\text{ret}}(\boldsymbol{x}', \boldsymbol{x}; \omega) = \lim_{\epsilon \to 0^+} \sum_n \frac{\psi_n(\boldsymbol{x}')\psi_n^*(\boldsymbol{x})}{\hbar\omega - E_n - i\epsilon} \tag{10.57}$$

Here too, the denominators in this equation have zeros on the real frequency axis as $\epsilon \to 0^+$. Thus, the Green function is a series of *distributions*. In the limit $\epsilon \to 0^+$ we can write

$$\lim_{\epsilon \to 0^+} \frac{1}{x - i\epsilon} = \mathcal{P}\frac{1}{x} + i\pi\delta(x) \tag{10.58}$$

where $\mathcal{P}$ denotes the principal value, and $\delta(x)$ is the Dirac delta function. Using these results, we can write the following expressions for the real and imaginary parts of the Green function

$$\begin{aligned} \text{Re}\, G_0^{\text{ret}}(\boldsymbol{x}', \boldsymbol{x}; \omega) &= \sum_n \mathcal{P}\frac{\psi_n(\boldsymbol{x}')\psi_n^*(\boldsymbol{x})}{\hbar\omega - E_n} \\ \text{Im}\, G_0^{\text{ret}}(\boldsymbol{x}', \boldsymbol{x}; \omega) &= \pi \sum_n \psi_n(\boldsymbol{x}')\psi_n^*(\boldsymbol{x})\delta(\hbar\omega - E_n) \end{aligned} \tag{10.59}$$

We can use these results to write an expression for the *density of states* $\rho(\omega)$

$$\rho(\omega) = \sum_n \delta(\hbar\omega - E_n) \tag{10.60}$$

in terms of the retarded Green function of the form

$$\rho(\omega) = \frac{1}{\pi}\text{Im} \int d\boldsymbol{x}\; G_0^{\text{ret}}(\boldsymbol{x}, \boldsymbol{x}; \omega) \tag{10.61}$$

In other words, the spectral density is determined by the imaginary part of the Green function. Below we will find similar relationships for other quantities of physical interest.

Let us close by noting that there is a close connection between the Green function in Quantum Mechanics and the kernel of the diffusion equation. In d-dimensional space the Green function is

$$G_0(\boldsymbol{x}' - \boldsymbol{x}, t' - t) = -\frac{i}{\hbar}\Theta(t' - t) \int \frac{d^d p}{(2\pi)^d} e^{-\frac{i}{\hbar}\frac{\boldsymbol{p}^2}{2m}(t' - t) + \frac{i}{\hbar}\boldsymbol{p} \cdot (\boldsymbol{x}' - \boldsymbol{x})} \tag{10.62}$$

By completing squares inside the exponent

$$\frac{\boldsymbol{p}^2}{2m}(t'-t)-\boldsymbol{p}'\cdot(\boldsymbol{x}'-\boldsymbol{x}) = \frac{(t'-t)}{2m}\left[\boldsymbol{p}-2m\boldsymbol{p}\cdot\frac{\boldsymbol{x}'-\boldsymbol{x}}{(t'-t)}\right]$$
$$= \left(\frac{t'-t}{2m}\right)\left[\boldsymbol{p}-\left(\frac{\boldsymbol{x}'-\boldsymbol{x}}{t'-t}\right)m\right]^2-\frac{m|\boldsymbol{x}'-\boldsymbol{x}|^2}{2|t'-t|} \tag{10.63}$$

we can write

$$G_0(\boldsymbol{x}'-\boldsymbol{x},t'-t) =$$
$$-\frac{i}{\hbar}\Theta\left(t'-t\right)\int\frac{d^dp}{(2\pi)^d}\,e^{-\frac{i}{\hbar}\left(\frac{t'-t}{2m}\right)\left(\boldsymbol{p}-\left(\frac{\boldsymbol{x}'-\boldsymbol{x}}{t'-t}\right)m\right)^2+\frac{i}{2\hbar}m\frac{|\boldsymbol{x}'-\boldsymbol{x}|^2}{|t'-t|}} \tag{10.64}$$

After a straightforward integration, we find that G_0 is equal to

$$G_0(\boldsymbol{x}'-\boldsymbol{x},t'-t) = -\frac{i}{\hbar}\Theta(t'-t)\,\left(\frac{m\hbar}{2\pi(t'-t)}\right)^{d/2}e^{\frac{im}{2\hbar}\frac{|\boldsymbol{x}'-\boldsymbol{x}|^2}{(t'-t)}} \tag{10.65}$$

This formula is strongly reminiscent of the kernel for the Heat Equation (or Diffusion Equation) that was discussed in an earlier chapter,

$$\partial_\tau\psi = D\,\nabla^2\,\psi \tag{10.66}$$

where D is the diffusion constant and τ is the diffusion time. Indeed, after an analytic continuation to *imaginary time*, $t\to i\tau$, the Schrödinger equation becomes a diffusion equation with a diffusion constant $D=\frac{\hbar}{2m}$. The Green function (or *Heat Kernel*) for the Heat Equation is

$$K(\boldsymbol{x}'-\boldsymbol{x},\tau'-\tau) = \;\Theta(\tau'-\tau)\frac{1}{(4\pi D\tau)^{d/2}}\,e^{-\frac{|\boldsymbol{x}'-\boldsymbol{x}|^2}{4D(\tau'-\tau)}} \tag{10.67}$$

which, of course, agrees with the analytic continuation of the Feynman propagator $G_0(\boldsymbol{x}'-\boldsymbol{x},t'-t)$. This connection between quantum mechanics and diffusion processes is central to the path-integral picture of quantum mechanics.

10.3 Analytic properties of the propagators of free relativistic fields

10.3.1 Properties of the propagator of the real scalar field

The propagator for a free real relativistic field

$$G^{(0)}(x-x') = i\langle 0|T\phi(x)\phi(x')|0\rangle \tag{10.68}$$

is the solution of the partial differential equation

$$(\partial^2 + m_0^2)\, G^{(0)}(x-x') = \delta^4(x-x') \tag{10.69}$$

which we have discussed and solved before. $G^{(0)}(x-x')$ can be calculated by the usual Fourier expansion methods,

$$G^{(0)}(x-x') = \int \frac{d^4p}{(2\pi)^4}\, \widetilde{G}^{(0)}(p)\, e^{-ip\cdot(x-x')} \tag{10.70}$$

where $\widetilde{G}^{(0)}(p)$ is given by

$$\widetilde{G}^{(0)}(p) = \frac{-1}{p^2-m^2} \tag{10.71}$$

where $p^2 = p_\mu p^\mu$.

Once again, we will have to give a prescription for going around the poles of $\widetilde{G}_0(p)$ which yields the correct boundary conditions. We will adopt the same conventions that we used in section 10.1. For the Feynman (or time-ordered) propagator we shift the denominator by $i\epsilon$,

$$\widetilde{G}^{(0)}(p) = \frac{-1}{p^2-m^2+i\epsilon} \tag{10.72}$$

The poles of $\widetilde{G}^{(0)}(p)$ are located at $p_0 = \pm\sqrt{\boldsymbol{p}^2+m^2} \mp i\epsilon$. Thus, the positive frequency pole in the lower half-plane and the negative-frequency pole is in the upper half-plane.

We will use the integration paths shown in Fig.10.3. For $x_0 > x_0'$ we close the integration contour on the path γ^- and we pick up the contribution from the positive frequency pole at $+\sqrt{\boldsymbol{p}^2+m^2} - i\epsilon$. Thus, for $x_0 > x_0'$ we find the result

$$\oint_{\gamma^+} \frac{dp_0}{2\pi}\, \frac{e^{-ip_0(x_0-x_0')}}{p_0^2-(\boldsymbol{p}^2+m^2)+i\epsilon} \xrightarrow[\epsilon\to 0^+]{} -i\frac{e^{-i\sqrt{\boldsymbol{p}^2+m^2}\left(x_0-x_0'\right)}}{2\sqrt{\boldsymbol{p}^2+m^2}} \tag{10.73}$$

Similarly, for $x_0 < x_0'$ we close the integration contour on the path γ^+ on

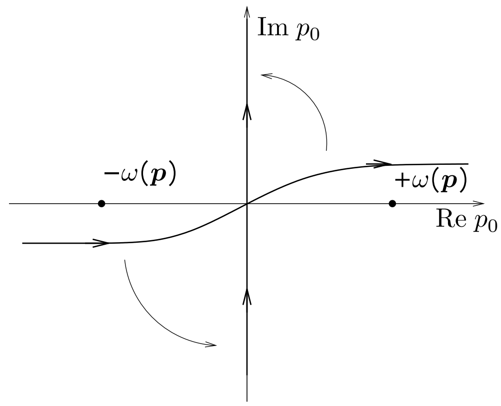

Figure 10.7 Wick Rotation.

the upper half plane where we pick up the contribution form the negative frequency pole at $-\sqrt{\boldsymbol{p}^2 + m^2} + i\epsilon$. Thus, for $x_0 < x_0'$ we obtain

$$\oint_{\gamma^-} \frac{dp_0}{2\pi} \frac{e^{-ip_0(x_0 - x_0')}}{p_0^2 - (\boldsymbol{p}^2 + m^2) + i\epsilon} \xrightarrow[\epsilon \to 0^+]{} i\frac{e^{i\sqrt{\boldsymbol{p}^2 + m^2}(x_0 - x_0')}}{-2\sqrt{\boldsymbol{p}^2 + m_0^2}} \tag{10.74}$$

By collecting terms we get

$$\begin{aligned} G^{(0)}(x - x') =& i\Theta\left(x_0 - x_0'\right) \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})} e^{+i\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{x}') - i\omega(\boldsymbol{p})\left(x_0 - x_0'\right)} \\ &+ i\Theta(x_0' - x_0) \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})} e^{-i\boldsymbol{p} \cdot \left(\boldsymbol{x} - \boldsymbol{x}'\right) + i\omega(\boldsymbol{p})(x_0 - x_0')} \end{aligned} \tag{10.75}$$

This result shows that $G^{(0)}(x - x')$ does satisfy the required boundary condition. It also shows that the positive frequency components of the field propagate forwards in time while the negative frequency components propagate backwards in time.

The simplest way to compute $G_0(x - x')$ is by means of an analytic continuation (or Wick rotation) to imaginary time, $x_0 \to ix_4$. This amounts to a rotation of the integration contour from the *real* p_0 axis to the *imaginary* p_0 axis, namely $p_0 \to ip_4$, as shown in Fig.10.7. The Wick-rotated or Euclidean correlation function $G_0^E(x - x')$, the Euclidean propagator that we calculated before, is given by

$$G^{(0)}(x-x') = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{p^2+m^2} \equiv -iG_0^E(x,x') \tag{10.76}$$

where $p^2 = -\sum_{i=1}^4 p_i p_i$ and $p\cdot x = -\sum_{i=1}^4 p_i x_i$.

The time-ordered, or Feynman, propagator does not obey causality since it does not vanish for space-like separated events, $s^2 < 0$. We can define a *causal*, or *retarded*, propagator that obeys the causal boundary condition, i.e. $G^{(0)}(x-x') = 0$ except inside the *forward* light-cone. Similarly, we can also define an *advanced* propagator which vanishes outside the *backward* light-cone. We will discuss only the retarded propagator $G^{(0)}_{\text{ret}}(x-x')$.

The retarded propagator is defined by computing the frequency integral on the path shown in the Fig.10.2 (upon the replacement $|\boldsymbol{p}| \to \omega(\boldsymbol{p})$). The retarded propagator $G^{(0)}_{\text{ret}}(x-x')$ is given by

$$\begin{aligned} G^{(0)}_{\text{ret}}(x-x') &= -\int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\int_{-\infty}^{+\infty}\frac{dp_0}{2\pi}\frac{e^{-ip_0(x_0-x_0')}}{p_0^2-\omega^2(\boldsymbol{p})} \\ &\equiv -i\int\frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\int_{-\infty}^{+\infty}\frac{dp_0}{2\pi i}e^{-ip_0(x_0-x_0')} \\ &\quad\times\left[\frac{1}{p_0-\omega(\boldsymbol{p})+i\epsilon}-\frac{1}{p_0+\omega(\boldsymbol{p})+i\epsilon}\right] \end{aligned} \tag{10.77}$$

Hence, we get

$$\begin{aligned} G^{(0)}_{\text{ret}}(x-x') &= -i\Theta(x_0-x_0')\int\frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})} \\ &\times\left[e^{i\omega(p)(x_0-x_0')-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}-e^{-i\omega(\boldsymbol{p})(x_0-x_0')+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\right] \end{aligned} \tag{10.78}$$

The integral over the momentum variables is just the quantity $i\Delta(x-x')$ that we have encountered before, c.f. Eq. (4.90),

$$i\Delta(x-x') = \left[\phi(x),\phi(x')\right] \equiv \langle 0|\left[\phi(x),\phi(x')\right]|0\rangle \tag{10.79}$$

which allows us to write

$$G^{(0)}_{\text{ret}}(x-x') = \Theta(x_0-x_0')\Delta(x-x') = -i\Theta(x_0-x_0')\langle 0|\left[\phi(x),\phi(x')\right]|0\rangle \tag{10.80}$$

10.3.2 Properties of the propagator of the Dirac field

The same line of argument we have used here for the scalar field can be used for the Dirac field. The vacuum state of the Dirac theory is defined by filling up all negative energy states. We now imagine that an electron is propagating in free space, and then an external potential is adiabatically switched on. If the potential is not too strong, we can still describe its effects by means of a Born series of multiple scattering processes, shown in Fig.10.8a.

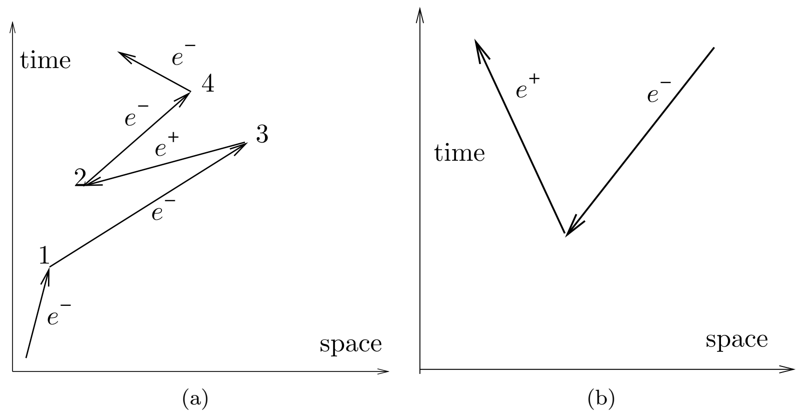

Figure 10.8 a) Scattering processes of a Dirac field. b) A pair creation process.

The electron scatters off the potential at 1 and as a result it propagates up to 3. If the potential has the correct matrix elements, at 3 the positive energy state may turn into a negative energy state. In general this won't be allowed since all negative energy states are filled, unless a negative energy state got emptied (by the action of the potential) *before the electron became scattered into that state*. Indeed, the potential can *create an electron-positron pair* out of the vacuum as in the process shown in Fig.10.8b.

Thus, it can remove an electron from an occupied negative energy state and promote it into a previously empty positive energy state. Hence, if the time t lies between t_2 and t_3, there are *three* different states propagating in the system (see Fig.10.9):

1. A positive energy state that disappears at **3**
2. A negative energy state that propagates *backwards* in time from **3** to **2**
3. A positive energy state that appeared at **2**.

An alternative interpretation is that an electron-positron pair was created at **2** and that the positron annihilated with the original electron at **3**. This

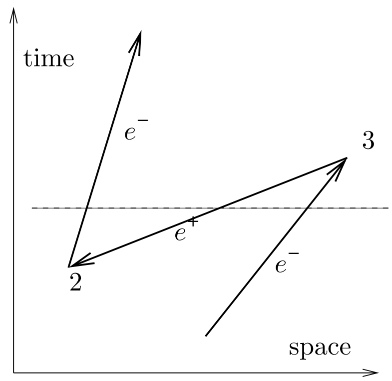

Figure 10.9 An intermediate state with an electron-positron pair.

process clearly shows that the Dirac theory is a *quantum field theory*, and cannot be described within the framework of quantum mechanics with a fixed number of particles, as in the non-relativistic case. Thus, the Fock space description is essential to the relativistic case.

These arguments suggest that we may want to seek a propagator that propagates positive energy states forward in time while negative energy states propagate backwards in time. This is the Feynman, or time-ordered, propagator, $S_F^{\alpha\alpha'}(x-x')$. It is straightforward to see that these requirements are met by the following expression

$$S_F^{\alpha\alpha'}(x-x') = -i\langle 0|T\psi_\alpha(x)\bar{\psi}_{\alpha'}(x')|0\rangle \tag{10.81}$$

which satisfies the equation of motion

$$(i\partial\!\!\!/ - m)S_F(x-x') = \delta^4(x-x') \tag{10.82}$$

The same methods that we used for the scalar field yield the solution (dropping the spinor indices)

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{p\!\!\!/ + m}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-x')} \tag{10.83}$$

where an $i\epsilon$ has been introduced in order to get the correct boundary conditions. We have also shown that $S_F(x-x')$ satisfies

$$S_F(x-x') = (i\partial\!\!\!/ + m)\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{p^2-m^2+i\epsilon} = -(i\partial\!\!\!/ + m)G^{(0)}(x-x') \tag{10.84}$$

where $G^{(0)}(x-x')$ is the Feynman or time-ordered propagator for the free massive scalar field. The $i\epsilon$ prescription ensures that positive energy states propagate forward in time, and that negative energy states propagate backward in time.

10.4 The propagator of the non-relativistic electron gas

Let us now discuss the propagator, or *one-particle Green function*, for a non-relativistic free electron gas at finite density and at zero temperature. It is defined in the usual way

$$G_0^{\alpha\alpha'}(x, x') = -i\langle \text{gnd}|T\psi_\alpha(x)\psi^\dagger_{\alpha'}(x')|\text{gnd}\rangle \tag{10.85}$$

where α and α' are spin indices. This propagator can be used to compute a number of quantities of physical interest. For example, the average *electron density* $\langle \hat{n}(x)\rangle$ is

$$\langle n(\boldsymbol{x})\rangle = \langle \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})\rangle = -i \lim_{\boldsymbol{x}'\to\boldsymbol{x}} \lim_{x_0'\to x_0} \text{tr}\, G(x, x') \tag{10.86}$$

Likewise the *current density* $\langle \boldsymbol{j}(\boldsymbol{x})\rangle$ is

$$\begin{aligned}\langle \boldsymbol{j}(\boldsymbol{x})\rangle &= \frac{\hbar}{2mi}\text{tr}\, \langle \psi^\dagger(\boldsymbol{x})\left(\nabla_x\psi(\boldsymbol{x})\right) - \left(\nabla_x\psi^\dagger(\boldsymbol{x})\right)\psi(\boldsymbol{x})\rangle \\ &= -\frac{1}{m}\lim_{\boldsymbol{x}'\to\boldsymbol{x}} \lim_{x_0'\to x_0} \left(\nabla_{\boldsymbol{x}} - \nabla_{\boldsymbol{x}'}\right)\text{tr}\, G(x, x')\end{aligned} \tag{10.87}$$

and the *magnetization density* $\boldsymbol{M}(\boldsymbol{x}) = \langle \psi^\dagger_\alpha(\boldsymbol{x})\boldsymbol{\sigma}_{\alpha\beta}\psi_\beta(\boldsymbol{x})\rangle$,

$$\boldsymbol{M}(\boldsymbol{x}) = -i \lim_{\boldsymbol{x}'\to\boldsymbol{x}} \lim_{x_0'\to x_0} \text{tr}\left[G(x, x')\boldsymbol{\sigma}\right] \tag{10.88}$$

Let us compute $G(x, x')$ by the standard method of Fourier transforms,

$$\begin{aligned}G(x, x') &= \int \frac{d^3p}{(2\pi)^3}\int\frac{d\omega}{2\pi}\,\widetilde{G}(\boldsymbol{p}, \omega)\, e^{i\left[\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}') - \omega(t-t')\right]} \\ &\equiv \int \frac{d^4p}{(2\pi)^4}\,\widetilde{G}(p)\, e^{ip\cdot(x-x')}\end{aligned} \tag{10.89}$$

where $p_0 = \omega$ and $t = x_0$. Let $E(\boldsymbol{p})$ be the single particle energies measured from the chemical potential μ, $E(\boldsymbol{p}) = \frac{p^2}{2m} - \mu$. Thus, we get

$$G(\boldsymbol{x}, t) = -i \int \frac{d^3p}{(2\pi)^3} \Theta(t) \langle \text{gnd} | \psi(\boldsymbol{p}) \psi^\dagger(\boldsymbol{p}) | \text{gnd} \rangle e^{i\,(\boldsymbol{p} \cdot \boldsymbol{x} - E(\boldsymbol{p})\,t)} + i \int \frac{d^3p}{(2\pi)^3} \Theta(-t) \langle \text{gnd} | \psi^\dagger(\boldsymbol{p}) \psi(\boldsymbol{p}) | \text{gnd} \rangle e^{i\,(\boldsymbol{p} \cdot \boldsymbol{x} - E(\boldsymbol{p}) t)} \tag{10.90}$$

Recall that $|\text{gnd}\rangle$ is the state in which all negative energy states, with $E(\boldsymbol{p}) < 0$ or, equivalently, $\epsilon(p) < \mu$, are filled. Thus, the Green function becomes

$$G(\boldsymbol{x}, t) = -i \int \frac{d^3p}{(2\pi)^3} \left[\Theta(t) \left(1 - n(\boldsymbol{p})\right) - \Theta(-t) n(\boldsymbol{p}) \right] e^{i\,(\boldsymbol{p} \cdot \boldsymbol{x} - E(\boldsymbol{p}) t)} \tag{10.91}$$

where $n(\boldsymbol{p}) = \langle \text{gnd} | \psi^\dagger(\boldsymbol{p}) \psi(\boldsymbol{p}) | \text{gnd} \rangle$ is the Fermi-Dirac distribution (or Fermi function in short) at zero temperature,

$$n(\boldsymbol{p}) = \begin{cases} 1 & |\boldsymbol{p}| \le p_F \\ 0 & \text{otherwise} \end{cases} \tag{10.92}$$

We can write the Fourier transform $G(\boldsymbol{p}, \omega)$ in the form

$$G(\boldsymbol{p}, \omega) = -i \left\{ \Theta(|\boldsymbol{p}| - p_F) \int_0^\infty e^{i(\omega - E(\boldsymbol{p}))t} dt - \Theta(p_F - |\boldsymbol{p}|) \int_0^\infty dt e^{-i(\omega - E(\boldsymbol{p}))t} \right\} \tag{10.93}$$

The integrals in this expression define distributions of the form

$$\int_0^\infty dt\, e^{ist} = \lim_{\epsilon \to 0^+} \int_0^\infty dt\, e^{ist - \epsilon\, t} = i \lim_{\epsilon \to 0^+} \frac{1}{s + i\epsilon} = i \left(\mathcal{P} \frac{1}{s} - i\pi \delta(s) \right) \tag{10.94}$$

where $\mathcal{P}\frac{1}{s}$ is the *principal value*

$$\mathcal{P} \frac{1}{s} = \lim_{\epsilon \to 0} \frac{s}{s^2 + \epsilon^2} \tag{10.95}$$

and $\delta(s)$ is the Dirac δ-function,

$$\delta(s) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{s^2 + \epsilon^2} \tag{10.96}$$

We can use these results to write $\widetilde{G}(\boldsymbol{p}, \omega)$ as

$$\widetilde{G}(\boldsymbol{p}, \omega) = \frac{\Theta(|\boldsymbol{p}| - p_F)}{\omega - E(\boldsymbol{p}) + i\epsilon} + \frac{\Theta(p_F - |\boldsymbol{p}|)}{\omega - E(\boldsymbol{p}) - i\epsilon} \tag{10.97}$$

where $E(\boldsymbol{p}) = \epsilon(\boldsymbol{p}) - \mu$. An equivalent, and more compact, expression is

$$\widetilde{G}(\boldsymbol{p},\omega) = \frac{1}{\omega - E(\boldsymbol{p}) + i\epsilon \,\mathrm{sign}(|\boldsymbol{p}| - p_F)} \tag{10.98}$$

We now notice that

$$\begin{aligned} \mathrm{Im}\,\widetilde{G}(\boldsymbol{p},\omega) =& -\pi\Theta(|\boldsymbol{p}| - p_F)\delta(\omega - \epsilon(\boldsymbol{p}) + \mu) + \pi\Theta(p_F - |\boldsymbol{p}|)\delta(\omega - \epsilon(\boldsymbol{p}) + \mu) \\ =& -\pi\delta(\omega - \epsilon(\boldsymbol{p}) + \mu)\Big[\Theta\left(|\boldsymbol{p}| - p_F\right) - \Theta(p_F - |\boldsymbol{p}|)\Big] \end{aligned} \tag{10.99}$$

The last identity shows that

$$\mathrm{sign}\ \mathrm{Im}\widetilde{G}(\boldsymbol{p},\omega) = -\mathrm{sign}\ \omega \tag{10.100}$$

Hence, we may also write $\widetilde{G}(\boldsymbol{p},\omega)$ as

$$\widetilde{G}(\boldsymbol{p},\omega) = \frac{1}{\omega - E(\boldsymbol{p}) + i\epsilon \ \mathrm{sign}\ \omega} \tag{10.101}$$

In this expression, $\widetilde{G}(\boldsymbol{p},\omega)$ has poles at $\omega = E(\boldsymbol{p})$ and that all the poles with $\omega > 0$ are infinitesimally shifted downwards to the lower half-plane, while the poles with $\omega < 0$ are raised upwards to the upper half-plane by the same amount. Since $E(\boldsymbol{p}) = \epsilon(\boldsymbol{p}) - \mu$, all poles with $\epsilon(\boldsymbol{p}) > \mu$ are shifted downwards, while all poles with $\epsilon(\boldsymbol{p}) < \mu$ are shifted upwards.

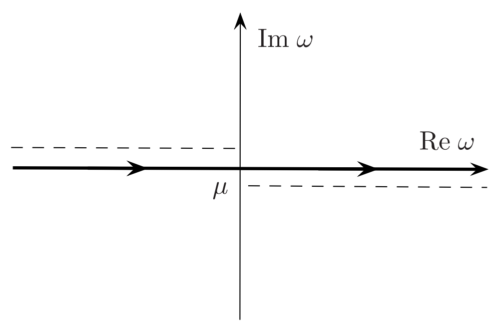

Figure 10.10 Analytic structure of the non-relativistic fermion propagator at finite chemical potential μ. The broken lines show the location of the poles for different momenta $\boldsymbol{p}$ on the upper and lower half planes.

10.5 The scattering matrix

The problems of real physical interest very rarely involve free fields. In general we have to deal with interacting fields. For the sake of definiteness we will consider a scalar field but the ideas that we will discuss have general applicability.

The field $\phi(\boldsymbol{x}, t)$ in the Heisenberg representation is related to the Schrödinger operator $\phi(\boldsymbol{x})$ through the time evolution operator generated by the Hamiltonian

$$\phi(\boldsymbol{x}, t) \equiv \phi(x) = e^{iHt}\ \phi(\boldsymbol{x}, 0)\ e^{-iHt} \tag{10.102}$$

(for $\hbar = 1$). Recall that in the Schrödinger representation the *fields are fixed* but the *states evolve* according to the Schrödinger equation

$$H|\Phi\rangle = i\partial_t|\Phi\rangle, \tag{10.103}$$

whereas, in the Heisenberg representation, the *states are fixed* but the *fields evolve* by following the equations of motion,

$$i\partial_t\phi(x) = [\phi(x), H] \tag{10.104}$$

In an interacting system, the problem is precisely how to determine the evolution operator e^{iHt}. Thus, the Heisenberg representation cannot be constructed *a priori*.

Let us assume that the Hamiltonian can be split into a sum of two terms

$$H = H_0 + H_{int} \tag{10.105}$$

where H_0 represents a system whose states are fully known to us (a problem that we know how to solve) and, in this sense are "free". H_{int} represents the interactions. For technical reasons we will have to assume that $H_{int}(t)$, as a function of time, vanishes (very smoothly) both in the remote past and in the remote future.

We now define the *Interaction Representation*. In this representation one defines the *fields* ϕ_{in} which evolve as the Heisenberg fields with Hamiltonian H_0

$$i\partial_t\phi_{\text{in}}(x) = [\phi_{\text{in}}, H_0] \tag{10.106}$$

These operators create and destroy free *incoming* states. We call these states incoming since as $t \to -\infty$ there are no interactions. In the absence of interactions, the states do not evolve but, if interactions are present, they do. There is a unitary operator $U(t)$ which governs the time evolution of the states and the S-matrix.

We want to find an operator $U(t)$ such that

$$\phi(x, t) = U^{-1}\phi_{\text{in}}(x, t)U(t) \tag{10.107}$$

where $\phi(x,t)$ is the Heisenberg field operator for the full Hamiltonian H. The operator $U(t)$ must be *unitary* and satisfy the initial condition

$$\lim_{t\to-\infty} U(t) = I \tag{10.108}$$

Since $U(t)$ is unitary and invertible, it must satisfy the condition $U^{-1}(t) = U^{\dagger}(t)$. Thus

$$\partial_t U(t) U^{-1}(t) + U(t)\partial_t U^{-1}(t) = 0 \tag{10.109}$$

Since the operators ϕ_{in} and ϕ are defined by the Heisenberg evolution equation, Eq. (10.107), we get

$$\begin{aligned} \partial_t \phi_{\text{in}} =& \partial_t U\ \phi\ U^{-1} + U\ \partial_t \phi\ U^{-1} + U\ \phi\ \partial_t U^{-1} \\ =& \partial_t U\ \phi\ U^{-1} + iU\ [H, \phi]\ U^{-1} + U\ \phi\ U^{-1}\ U\ \partial_t U^{-1} \end{aligned} \tag{10.110}$$

In other words,

$$\partial_t \phi_{\text{in}} = iU(t)[H(\phi), \phi]U^{-1}(t) + \partial_t U U^{-1} \phi_{\text{in}} + \phi_{\text{in}} U \partial_t U^{-1} \tag{10.111}$$

Similarly, the Hamiltonian must obey the identity

$$H(\phi_{\text{in}}) = U(t) H(\phi) U^{-1}(t) \tag{10.112}$$

which implies that ϕ_{in} should obey

$$\partial_t \phi_{\text{in}} = i[H(\phi_{\text{in}}), \phi_{\text{in}}] + [(\partial_t U)\ U^{-1}, \phi_{\text{in}}] \tag{10.113}$$

Since ϕ_{in} obeys the free equation of motion, Eq.(10.106), we find that the evolution operator $U(t)$ must satisfy the condition

$$[iH_{\text{int}}(\phi_{\text{in}}) + (\partial_t U)\ U^{-1}, \phi_{\text{in}}] = 0 \tag{10.114}$$

for *all* operators ϕ_{in}. Therefore the operator in the left argument of the commutator must be a c-number, i.e. it is proportional to the identity operator. However, since $\lim_{t\to-\infty} H_{\text{int}}(\phi_{\text{in}}) = 0$ and $\lim_{t\to-\infty} U(t) = I$, this c-number must be equal to zero.

We thus arrive to an operator equation for $U(t)$

$$i\partial_t U = H_{\text{int}}(\phi_{\text{in}})\ U(t) \tag{10.115}$$

The operator U governs the time evolution of the states in the Interaction Representation, since the state $|\Phi\rangle_{\text{in}}$ becomes

$$U(t)|\Phi\rangle_{\text{in}} = |\Phi(t)\rangle \tag{10.116}$$

In particular, the *outgoing states* $|\Phi\rangle_{\text{out}}$, i.e. the states at $t \to +\infty$, which are also *free states*, are related to the in-states by the operator $U(t)$ in the limit $t \to +\infty$

$$|\Phi\rangle_{\text{out}} = \lim_{t\to+\infty} U(t)|\Phi\rangle_{\text{in}} \equiv S|\Phi\rangle_{\text{in}} \tag{10.117}$$

where $S = \lim_{t\to+\infty} U(t)$ is the *S-matrix*.

The equation of motion for $U(t)$ can formally be solved using methods that we have discussed before. The solution is

$$U(t) = Te^{-i\int_{-\infty}^{t} dt' H_{\text{int}}\left(\phi_{\text{in}}(t')\right)} \tag{10.118}$$

where T is the time-ordering operator. In terms of the interaction part $\mathcal{L}_{\text{int}}$ of the Lagrangian we find that the S-matrix is given by the expression

$$S = \lim_{t\to+\infty} U(t) = Te^{-i\int_{-\infty}^{+\infty} dt' H_{\text{int}}\left(\phi_{\text{in}}(t')\right)} = T\, e^{i\int d^4x\, \mathcal{L}_{\text{int}}(\phi_{\text{in}})} \tag{10.119}$$

This result is the starting point for the computation of the S-matrix using perturbation theory in H_{int}.

10.6 Physical information contained in the S-Matrix

Let us compute transition matrix elements between arbitrary *in* and *out* states. Let $|i, \text{in}\rangle$ be the initial incoming state, and $|f, \text{out}\rangle$ be the final outgoing state. The transition probability $W_{i\to f}$ is then given by a matrix element of the S-matrix,

$$W_{i\to f} = |\langle f, \text{out}|i, \text{in}\rangle|^2 \equiv |\langle f, \text{in}|S|i, \text{in}\rangle|^2 \tag{10.120}$$

since $\langle f, \text{out}|S = \langle f, \text{in}|$. We can split S into non-interacting and interacting parts

$$S = I + iT \tag{10.121}$$

where I is the identity operator, represents the free part, and the T-matrix (not to be confused with the time-ordering symbol!) represents the interactions. In terms of the T-matrix, the transition probability is

$$W_{i\to f} = |\langle f|i\rangle + i\langle f|T|i\rangle|^2 \tag{10.122}$$

Although from now on we will discuss the case of a scalar field, the arguments can be generalized to all other problems of interest with only minor modifications.

Let us consider the situation in which the initial state $|i, \text{in}\rangle$ consists of two wave packets with only positive frequency components

$$|i, \text{in}\rangle = \int \frac{d^3p_1}{2p_1^0(2\pi)^3} \int \frac{d^3p_2}{2p_2^0(2\pi)^3} f_1(p_1) f_2(p_2) |p_1, p_2; \text{in}\rangle \tag{10.123}$$

The incoming flux is equal to $\int \frac{d^3p}{2p_0(2\pi)^3} |f(p)|^2$. Each component $|p_1, p_2; \text{in}\rangle$ will have a matrix element with the final state $|f; \text{out}\rangle$.

Since we have translation invariance, the total 4-momentum should be conserved. If we denote by P_f the momentum of the state $|f, \text{out}\rangle$, we can write the matrix element of the T-matrix as

$$\langle f|T|p_1p_2\rangle = (2\pi)^4 \delta^4(P_f - p_1 - p_2) \langle f|\mathcal{T}|p_1p_2\rangle \tag{10.124}$$

where $\mathcal{T}$ is called the reduced operator which, as we see, acts only on the energy shell, $p^2 = m^2$.

If we neglect the forward scattering contribution, the transition probability $W_{i\to f}$ becomes

$$\begin{aligned} W_{i\to f} = &\int \frac{d^3p_1}{2p_1^0(2\pi)^3} \int \frac{d^3p_2}{2p_2^0(2\pi)^3} \int \frac{d^3q_1}{2q_1^0(2\pi)^3} \int \frac{d^3q_2}{2q_2^0(2\pi)^3} \\ &\times (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)\, (2\pi)^4 \delta^4(P_f - p_1 - p_2) \\ &\times f_1^*(p_1) f_2^*(p_2) f_1(q_1) f_2(q_2) \langle f|\mathcal{T}|p_1p_2\rangle^* \langle f|\mathcal{T}|q_1q_2\rangle \end{aligned} \tag{10.125}$$

The incoming states are assumed to be sharply peaked around some momenta $\bar{p}_1$ and $\bar{p}_2$ with a spread Δp, such that we can approximate the matrix element as follows

$$\langle f|\mathcal{T}|p_1p_2\rangle \approx \langle f|\mathcal{T}|q_1q_2\rangle \approx \langle f|\mathcal{T}|\bar{p}_1\bar{p}_2\rangle \tag{10.126}$$

Under these assumptions, the form for the transition probability is

$$W_{i\to f} = \int d^4x |\tilde{f}_1(x)|^2 |\tilde{f}_2(x)|^2 (2\pi)^4 \delta^4(P_f - \bar{p}_1 - \bar{p}_2) |\langle f|\mathcal{T}|\bar{p}_1\bar{p}_2\rangle|^2 \tag{10.127}$$

where $\tilde{f}(x)$ is the Fourier transform of $f(p)$. The integrand of Eq.(10.127) is the transition probability per unit time and volume

$$\frac{dW_{i\to f}}{dtdV} = |\tilde{f}_1(x)|^2 |\tilde{f}(x)|^2 (2\pi)^4 \delta^4(P_f - \bar{p}_1 - \bar{p}_2) |\langle f|\mathcal{T}|\bar{p}_1\bar{p}_2\rangle|^2 \tag{10.128}$$

In position space, the flux is $i\int d^3x \tilde{f}^*(x) \overleftrightarrow{\partial_0} \tilde{f}(x)$. If $\tilde{f}(x)$ is sufficiently smooth, we can make the following approximation

$$i\tilde{f}^*(x)\overleftrightarrow{\partial_0}\tilde{f}(x) \approx 2\bar{p}_0|\tilde{f}(x)|^2 \tag{10.129}$$

Let us assume that particle 1 is incident in the laboratory reference frame, and that in the laboratory frame particle 2 is at rest. The density of particles in the target is

$$\frac{dn_2}{dV} = 2\bar{p}_2^0|\tilde{f}_2(x)|^2 \tag{10.130}$$

where $\bar{p}_2^0 = m_2$ since particle 2 is at rest. The incident flux is the velocity multiplied by the density of particles in the beam. Hence,

$$\Phi_{\text{in}} = \frac{|\bar{p}_1|}{|\bar{p}_0^1|}2\bar{p}_1^0|\tilde{f}_1(x)|^2 = 2|\bar{p}_1||\tilde{f}_1(x)|^2 \tag{10.131}$$

Then, the *differential cross section* $d\sigma$ is related to the transition probability by the relation

$$\frac{dW_{i\to f}}{dtdV} = \frac{dn_2}{dV}\cdot\Phi_{\text{in}}\cdot d\sigma \tag{10.132}$$

Hence

$$2m_2|\tilde{f}_2(x)|^2 2|\bar{p}_1||\tilde{f}_1(x)|^2 d\sigma = \\ |\tilde{f}_1(x)|^2|\tilde{f}_2(x)|^2(2\pi)^4\delta^4(P_f - \bar{p}_1 - \bar{p}_2)|\langle f|\mathcal{T}|\bar{p}_1\bar{p}_2\rangle|^2 \tag{10.133}$$

Therefore, the differential cross section $d\sigma$ is

$$d\sigma = (2\pi)^4\delta^4(P_f - \bar{p}_1 - \bar{p}_2)\frac{1}{4m_2|\bar{p}_1|}|\langle f|\mathcal{T}|\bar{p}_1\bar{p}_2\rangle|^2 \tag{10.134}$$

The quantity in the denominator $m_2|\bar{p}_1|$ can be written in the relativistic invariant way

$$m_2|\bar{p}| = m_2\sqrt{\bar{p}_1^{02} - m_1^2} = \left[(\bar{p}_2\cdot\bar{p}_1)^2 - m_1^2m_2^2\right]^{1/2} \tag{10.135}$$

Thus far we have not made any assumptions about the nature of the final state. If the process that we consider involves two particles going in and n particles coming out, the total differential cross-section becomes

$$d\sigma = \frac{1}{4\left[(\bar{p}_2\cdot\bar{p}_1)^2 - m_1^2m_2^2\right]^{1/2}}\int_\Delta \frac{d^3p_3}{(2\pi)^3 2p_3^0}\cdots\int_\Delta \frac{d^3p_{n+2}}{(2\pi)^3 2p_{n+2}^0} \\ \times |\langle p_3,\ldots,p_{n+2}|\mathcal{T}|p_1p_2\rangle|^2(2\pi)^4\delta(p_1 + p_2 - \sum_{i=3}^{n+2}p_i) \tag{10.136}$$

where Δ is an energy-momentum resolution. This expression shows that the central issue is to compute matrix elements of the reduced operator $\mathcal{T}$.

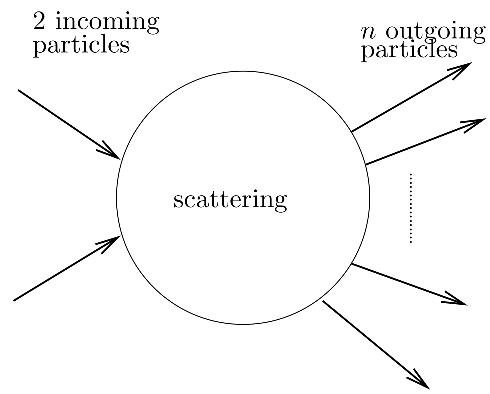

Figure 10.11 A $2 \to n$ scattering process.

10.7 Asymptotic states and the analytic properties of the propagator

We will show now that the S-matrix elements can be calculated if we know the vacuum expectation value (v.e.v.) of time-ordered products of field operators $\phi(x)$ in the Heisenberg representation. This is the Lehmann, Symanzik and Zimmermann (LSZ) approach.

In order to make this connection, it is necessary to relate the interacting fields $\phi(x)$ to a set of fields that create (or destroy) the *actual states* in the spectrum of the system, representing particle that are *far from each other*. In any scattering experiment, the initial states are sharply peaked wave packets which can be constructed to be arbitrarily close to the eigenstates of the system. Of course, the true eigenstates are plane waves and any two such states will necessarily have a non-vanishing overlap in space. But wave packets which are essentially made of just one state will not overlap if the wave packets (the "particles") are sufficiently far apart from each other at the initial time. Since they do not overlap, they do not interact. In this sense the spectrum of incoming states can be generated by a set of free fields. We will make the further assumption that the *in* and *out* states, the so-called *asymptotic states*, are created by such a set of free fields. We will denote these free fields by $\phi_{\text{in}}(x)$.

These assumptions amount to say that the states of the *fully interacting theory* are in *one-to-one correspondence* with the states of a non-interacting theory. In some loose sense, this hypothesis implies that the information that we can obtain from perturbation theory is always qualitatively correct. In other words, it is assumed that the states of the interacting theory are adiabatically connected to those of a non-interacting theory.

However, these assumptions can fail in several possible ways. A mild failure would be the appearance of *bound states* which, of course, are not present in the unperturbed theory. This situation is actually rather common and it can be remedied without too much difficulty. Two examples of this case are positronium states in QED and the collective modes of the Landau theory of the Fermi liquid.

There are however several ways in which this picture can fail in a rather serious way. One case is the situation in which the fields of the Lagrangian do not describe any of the asymptotic states of the theory. An example of this case is Quantum Chromodynamics (QCD) whose Lagrangian describes the dynamics of *quarks* and *gluons* which are not present in the asymptotic states since *quarks* are *confined* and *gluons* are *screened*. The asymptotic states of QCD are mesons, baryons, and glue-balls which are *bound states* of quarks and gluons.

Another possible failure of this hypothesis is the case in which the states created by the fields of the Lagrangian are not the true elementary excitations but rather they behave like some effective composite object of some more elementary states. In such case the true one-particle states may be *orthogonal* to the states created by the fields of the Lagrangian. This is a rather common situation in theories in $1 + 1$ dimensions whose spectrum is generated by a set of soliton-like states, which are extended objects in terms of the bare fields of the Lagrangian. Something very similar happens in the theory of the fractional quantum Hall states of two-dimensional electron fluids in high magnetic fields.

A third way in which the hypothesis may fail is that the quantum fluctuations may be so strong that the particle interpretation simply fails. We will see that this is what happens if the field theory is at a non-trivial fixed point of the Renormalization Group. In general at a fixed point most (but not all) observables develop *anomalous dimensions*. This means that the result of a scattering process is not a set of well defined particles. In this case we will see that the scattering amplitude does not have poles (which we associate with particle states) but branch cuts. Hence, in this case the conventional particle interpretation of the quantum field theory fails.

In this chapter we will not consider these very interesting situations (which will be discussed later on in other chapters) and assume instead that the adiabatic hypothesis (or scenario) actually holds. For the sake of concreteness, we will deal with scalar fields, and we will require $\phi_{\rm in}$ to be a free massive field that obeys the Klein-Gordon equation

$$\left(\partial^2 + m^2\right)\phi_{\rm in} = 0 \tag{10.137}$$

where m is the *physical mass*. In general, the physical mass is different from the mass parameter m_0 that enters in the Lagrangian of the interacting theory.

As the initial state evolves in time, the particles approach each other and begin to overlap. Interactions take place and, after some time, the system evolves to some final state, consisting of a set of well defined particles, the out state. The unitary operator that connects in and out states is precisely the S-matrix of the interaction representation. The only difference here, resides in the fact that the in and out states *are not* eigenstates of some unperturbed system, but the actual eigenstates of the full theory.

This picture assumes that there is a stable vacuum state $|0\rangle$ such that the observed particles are the elementary excitations of this vacuum. The free fields $\phi_{\rm in}(x)$ are just a device to generate the spectrum and have no real connection with actual dynamics. On the other hand, the interacting field $\phi(x)$ creates not only one-particle states but also many-particle states. This is so because its equations of motion are non-linear. Hence the matrix element of $\phi(x)$ and $\phi_{\rm in}(x)$ between the vacuum $|0\rangle$ and in-one-particle states $|1\rangle$, are generally different since $\phi_{\rm in}$ creates *only* one-particle states. This must be true even as $t \to -\infty$. We state this difference by writing

$$\langle 1|\phi(x)|0\rangle = Z^{1/2}\langle 1|\phi_{\rm in}|0\rangle \tag{10.138}$$

The proportionality constant Z is known as the wave-function renormalization. If $Z \neq 1$, the operator $\phi(x)$ must have a non-zero multi-particle projection. Notice that this is only an identity of these matrix elements, and not an identity between the fields themselves.

In the interaction representation it is possible to derive a similar looking identity that originates from the fact that the unperturbed and perturbed states do not have the same normalization. It is important to stress that this approach makes the *essential* assumption that the states that are reached through perturbation theory in the interaction representation can approximate with arbitrary precision *all* of the *exact* states of the theory. This assumption is the hypothesis that the asymptotic states are generated by free fields.

The operators that create the physical asymptotic states satisfy canonical commutation relations, and the commutator of a pair of such fields is

$$\langle 0|[\phi_{\rm in}(x), \phi_{\rm in}(x')]|0\rangle = i\Delta(x - x'; m) \tag{10.139}$$

where m is the physical mass. On the other hand, the interacting fields satisfy

$$\langle 0|[\phi(x), \phi(x')]|0\rangle = \sum_n \Big[\langle 0|\phi(x)|n\rangle\langle n|\phi(x')|0\rangle - (x \leftrightarrow x')\Big] \tag{10.140}$$

where $\{|n\rangle\}$ is a complete set of physical (in) states. The operators $\phi(x)$ are related to the operator $\phi(0)$ at the origin at some time $x_0 = 0$ by

$$\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x} \tag{10.141}$$

where P_μ is the *total* 4-momentum operator. If P_n^μ is the 4-momentum of the state $|n\rangle$, we can write

$$\langle 0|[\phi(x), \phi(x')]|0\rangle = \sum_n \left[\langle 0|\phi(0)|n\rangle e^{-iP_n\cdot(x-x')}\langle n|\phi(0)|0\rangle - (x \leftrightarrow x')\right] \tag{10.142}$$

We now insert the identity

$$1 = \int d^4Q \; \delta^4(Q - P_n) \tag{10.143}$$

to get

$$\langle 0|[\phi(x), \phi(x')]|0\rangle = \int d^4Q \; \sum_n \delta^4(Q-P_n)|\langle 0|\phi(0)|n\rangle|^2 \left(e^{-iQ\cdot(x-x')} - e^{iQ\cdot(x-x')}\right) \tag{10.144}$$

We can rewrite this expression in terms of a *spectral density* $\rho(Q)$

$$\langle 0|[\phi(x), \phi(x')]|0\rangle = \int \frac{d^4Q}{(2\pi)^3}\rho(Q)\left(e^{-iQ\cdot(x-x')} - e^{iQ\cdot(x-x')}\right) \tag{10.145}$$

where $\rho(Q)$ given by

$$\rho(Q) = (2\pi)^3 \sum_n \delta^4(Q - P_n)|\langle 0|\phi(0)|n\rangle|^2 \tag{10.146}$$

Let us recall that $\Delta(x - x'; m)$ is given by

$$\begin{aligned} i\Delta(x - x'; m) &= \int \frac{d^3Q}{(2\pi)^3 2Q_0}\left(e^{-iQ\cdot(x-x')} - e^{iQ\cdot(x-x')}\right) \\ &= \int \frac{d^4Q}{(2\pi)^3}\epsilon(Q^0)\delta^4(Q^2 - m^2)e^{-iQ\cdot(x-x')} \end{aligned} \tag{10.147}$$

where $\epsilon(Q^0) = \text{sign}(Q^0)$. Thus we can write

$$\langle 0|[\phi(x), \phi(x')]|0\rangle = \int \frac{d^4Q}{(2\pi)^3} \rho(Q)\epsilon(Q^0) e^{-iQ\cdot(x-x')} \tag{10.148}$$

Since $\rho(Q)$ is Lorentz invariant, by construction it can only be a real and positive function of Q^2

$$\rho(Q) = \sigma(Q^2) > 0 \tag{10.149}$$

Hence

$$\langle 0|[\phi(x), \phi(x')]|0\rangle = \int \frac{d^4Q}{(2\pi)^3} \sigma(Q^2)\epsilon(Q^0) e^{-iQ\cdot(x-x')} \tag{10.150}$$

We will now rewrite this expression in the form of an integral over the spectrum. Let us insert the identity

$$1 = \int_0^\infty d\mu^2 \delta(Q^2 - \mu^2) \tag{10.151}$$

where μ^2 is a spectral parameter, to obtain

$$\begin{aligned}\langle 0|[\phi(x), \phi(x')]|0\rangle &= \int \frac{d^4Q}{(2\pi)^3} \left[\int_0^\infty d\mu^2 \delta(Q^2 - \mu^2)\right] \sigma(Q^2)\epsilon(Q_0) e^{-iQ\cdot(x-x')} \\ &= \int_0^\infty d\mu^2 \sigma(\mu^2) \left[\int \frac{d^4Q}{(2\pi)^3} \epsilon(Q_0)\delta(Q^2 - \mu^2) e^{-iQ\cdot(x-x')}\right]\end{aligned} \tag{10.152}$$

where $\sigma(\mu^2)$ is

$$\sigma(\mu^2) = (2\pi)^3 \sum_n \delta(P_n^2 - \mu^2) |\langle 0|\phi(0)|n\rangle|^2 \tag{10.153}$$

Thus, we can write the v. e. v. of the commutator of the interacting fields as

$$\langle 0|[\phi(x), \phi(x')]|0\rangle = i \int_0^\infty d\mu^2 \sigma(\mu^2) \Delta(x - x'; \mu^2) \tag{10.154}$$

If we assume that the theory has a *physical particle* with mass m, i.e. one-particle states of mass m, we can write the final expression

$$-i\langle 0|[\phi(x), \phi(x')|0\rangle = Z\Delta(x - x'; m^2) + \int_{m_1^2}^\infty d\mu^2 \sigma(\mu^2) \Delta(x - x'; \mu^2) \tag{10.155}$$

where the first term represents the one-particle states and the integral represents the continuum of multi-particle states with a threshold at m_1. In other words, if there is a stable particle with mass m, the spectral function must have a δ-function at $\mu^2 = m^2$ with strength Z, the spectral weight of the one-particle state.

Since the field $\phi(x)$ obeys equal-time canonical commutation relations with the canonical momentum $\Pi(x) = \partial_0 \phi(x)$, we get

$$-i\langle 0|[\Pi(\boldsymbol{x}, x_0), \phi(\boldsymbol{x}', x_0)]|0\rangle = Z \lim_{x_0' \to x_0} \partial_0 \Delta(x - x'; m^2) + \int_{m_1^2}^{\infty} d\mu^2 \sigma(\mu^2) \lim_{x_0' \to x_0} \partial_0 \Delta(x - x'; \mu^2) \tag{10.156}$$

On the other hand, the free field commutator $\Delta(x - x'; m^2)$ obeys the initial condition

$$\lim_{x_0' \to x_0} \partial_0 \Delta(x - x'; m^2) = \lim_{x_0' \to x_0} [\Pi(x), \phi(x')] = -i\delta^3(\boldsymbol{x} - \boldsymbol{x}') \tag{10.157}$$

Hence, we find that the spectral function $\sigma(\mu^2)$ obeys the *spectral sum rule*

$$1 = Z + \int_{m_1^2}^{\infty} d\mu^2 \sigma(\mu^2) \tag{10.158}$$

Since $\sigma(\mu^2) > 0$ we find that $0 \le Z \le 1$. The lower end of the integration range, the threshold for multi-particle production m_1^2 is equal to $4m^2$ since, at least, we must create two elementary excitations.

A similar analysis can be done for the Feynman (time-ordered) propagator

$$G_F(x - x'; m) = -i\langle 0|T\phi(x)\phi(x')|0\rangle \tag{10.159}$$

which has the *spectral representation*

$$G_F(x - x'; m) = ZG_0(x - x'; m) + \int_{m_1^2}^{\infty} d\mu^2 \sigma(\mu^2) G_0(x - x'; \mu^2) \tag{10.160}$$

This decomposition is known as the *Lehmann representation*.

In Eq.(10.160) $G_0(x - x'; m^2)$ is the Feynman propagator for a free field

$$G_0(x - x') = -\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x - x')}}{p^2 - m^2 + i\epsilon} \tag{10.161}$$

In the limit $\epsilon \to 0^+$, the poles of the integrand can be manipulated to give

$$\lim_{\epsilon \to 0^+} \frac{1}{p^2 - m^2 + i\epsilon} = \mathcal{P}\frac{1}{p^2 - m^2} - i\pi\delta(p^2 - m^2) \tag{10.162}$$

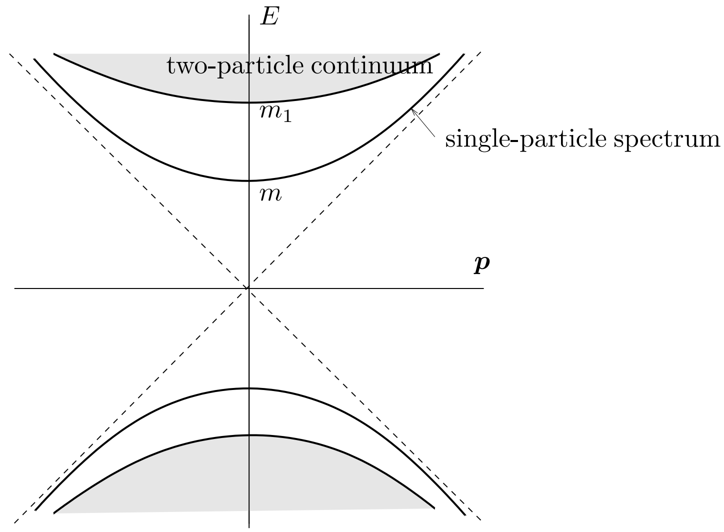

Figure 10.12 Spectrum of the propagator.

Using this identity we get that, in momentum space, the propagator is

$$G_F(p;m) = -\frac{Z}{p^2 - m^2 + i\epsilon} - \int_{m_1}^{\infty} d\mu^2 \frac{\sigma(\mu^2)}{p^2 - \mu^2 + i\epsilon} \tag{10.163}$$

Its imaginary part is given by

$$\text{Im } G_F(p;m) = \pi Z \, \delta(p^2 - m^2) + \pi \int_{m_1^2}^{\infty} \sigma(\mu^2)\delta(p^2 - \mu^2) \tag{10.164}$$

Hence

$$\frac{1}{\pi}\text{Im } G_F(p;m) = Z \, \delta(p^2 - m^2) + \sigma(p^2)\Theta(p^2 - m_1^2) \tag{10.165}$$

Once the imaginary part is known, the real part is found through the *Kramers-Krönig* or *dispersion relation*.

$$\text{Re } G_F(p, m^2) = \frac{1}{\pi} \mathcal{P} \int_0^{\infty} d\mu^2 \, \frac{\text{Im } G_F(p, \mu^2)}{\mu^2 - p^2 - i\epsilon} \tag{10.166}$$

We see that, in general, there are two contributions to Im $G_F(p;m)$. The first term is the contribution from the single particle states. In addition, then is a smooth contribution (the second term) which results form multi-particle production. While the single-particle states contribute with *an isolated pole* (a *δ-function* in the imaginary part), the multiple particle states (or continuum) are represented by a *branch cut*.

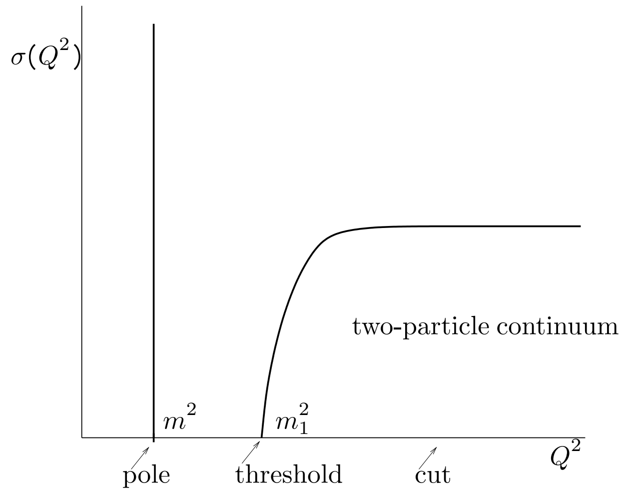

Figure 10.13 The analytic structure of the propagator is encoded in the spectral density $\sigma(Q^2)$.

There is a simple and natural physical interpretation of these results. If the incoming state has $Q^2 < m^2$ it cannot propagate since the allowed value is at least, m^2 (the physical mass). If $Q^2 > m_1^2$, and if there are no bound states, the incoming state can *decay* into *at least* two single particle states. Hence $m_1^2 = 4m^2$. These states should form a continuum since given the initial momentum P_i there are many multi-particle states with the same total momentum. Thus, those processes are incoherent. Notice that without interactions, the incoming state would not have been able to decay into several single particle states.

We should stress that the propagators of all the theories that we have discussed have the same type of analytic structure that we have discussed here.

10.8 The S-matrix and the expectation value of time-ordered products

We are now in position to find the connection between S-matrix elements and v. e. v. of time-ordered fields. For simplicity we will keep in mind the case of scalar fields but the results are easily generalizable. The actual derivation is rather lengthy and unilluminating. We will discuss its meaning and refer to standard textbooks for details.

Let's assume that we want to evaluate the matrix element

$$\langle p_1, \ldots, p_n; \text{out}|q_1, \ldots, q_m; \text{in}\rangle = \langle p_1, \ldots, p_n|S|q_1, \ldots, q_m\rangle \tag{10.167}$$

We will assume that all incoming and outgoing momenta are different. This matrix element is given by the *reduction formula*

$$\begin{aligned}\langle p_1, \ldots, p_n|S|q_1, \ldots, q_m\rangle =& \frac{i}{Z^{(n+m)/2}} \int d^4y_1 \ldots d^4y_n d^4x_1 \ldots d^4x_n \\ \times \exp\left[i(\sum_{\ell=1}^{n} p_\ell \cdot y_\ell - \sum_{k=1}^{m} q_k \cdot x_k)\right] & \prod_{\ell=1}^{n} \left(\partial_{y_\ell}^2 + m^2\right) \prod_{k=1}^{m} \left(\partial_{x_k}^2 + m^2\right) \\ & \times \langle 0|T\left(\phi(y_1)\ldots\phi(y_n)\phi(x_1)\ldots\phi(x_m)\right)|0\rangle\end{aligned} \tag{10.168}$$

where m^2 is the physical mass and the external momenta p and q are on the mass shell, $p^2 = q^2 = m^2$.

Let us consider for example the $2 \to 2$ process

$$\langle p_1, p_2; \text{out}|q_1, q_2; \text{in}\rangle = \langle p_1 p_2, \text{in}|S|q_1, q_2; \text{in}\rangle = \langle p_1 p_2, \text{out}|a_{\text{in}}^\dagger(q_1)|q_2, \text{in}\rangle \tag{10.169}$$

where $a_{\text{in}}^\dagger(q_1)$ is

$$a_{\text{in}}^\dagger(q_1) = -i \int_{\text{fixed } t} d^3x\, e^{-iq_1 \cdot x} \overleftrightarrow{\partial_0} \phi_{\text{in}}(x) \tag{10.170}$$

Hence, the matrix element is

$$\begin{aligned}\langle p_1 p_2; \text{out}|q_1 q_2; \text{in}\rangle &= -i \lim_{t\to-\infty} \int_t d^3x\, e^{-iq_1 \cdot x}\, \overleftrightarrow{\partial_0} \langle p_1 p_2; \text{out}|\phi_{\text{in}}(x)|q_2; \text{in}\rangle \\ &\equiv -i \lim_{t\to-\infty} \frac{1}{Z^{1/2}} \int_t d^3x\, e^{-iq_1 \cdot x}\, \overleftrightarrow{\partial_0} \langle p_1 p_2; \text{out}|\phi(x)|q_2; \text{in}\rangle\end{aligned} \tag{10.171}$$

where we have made the replacement of ϕ_{in} by $\frac{1}{Z^{1/2}}\phi$ *inside* the matrix element $(t \to -\infty)$.

But

$$\langle 0|\phi(x)|1\rangle = Z^{1/2}\langle 0|\phi_{\text{in}}(x)|1\rangle = Z^{1/2}\langle 0|\phi_{\text{out}}(x)|1\rangle \tag{10.172}$$

and

$$\left(\lim_{t_f\to+\infty} - \lim_{t_i\to-\infty}\right) \int d^3x\, F(\boldsymbol{x}, t) = \lim_{t_f\to+\infty} \lim_{t_i\to-\infty} \int_{t_i}^{t_f} dt\, \frac{\partial}{\partial t} \int d^3x\, F(\boldsymbol{x}, t) \tag{10.173}$$

These formulas allow us to write

$$
\begin{aligned}
&\lim_{t_f\to+\infty}\int d^3x\,\langle p_1p_2,\mathrm{out}|a^\dagger_{\mathrm{in}}(q_1)|q_2,\mathrm{in}\rangle =\\
&=\lim_{t_i\to-\infty}\int d^3x\,\langle p_1p_2,\mathrm{out}|a^\dagger_{\mathrm{in}}(q_1)|q_2,\mathrm{in}\rangle+\int d^4x\,\partial_0\langle p_1p_2,\mathrm{out}|a^\dagger_{\mathrm{in}}(q_1)|q_2,\mathrm{in}\rangle\\
&=\langle p_1p_2,\mathrm{out}|a^\dagger_{\mathrm{out}}(q_1)|q_2,\mathrm{in}\rangle
\end{aligned}
\tag{10.174}
$$

Thus, the matrix element is

$$
\begin{aligned}
\langle p_1p_2,\mathrm{out}|q_1q_2,\mathrm{in}\rangle =&\langle p_1p_2,\mathrm{out}|a^\dagger_{\mathrm{out}}(q_1)|q_2\mathrm{in}\rangle\\
&+\frac{i}{Z^{1/2}}\int d^4x\,\partial_0\left[e^{-iq_1\cdot x}\overleftrightarrow{\partial_0}\langle p_1p_2\mathrm{out}|\phi(x)|q_2\mathrm{in}\rangle\right]
\end{aligned}
\tag{10.175}
$$

The first contribution is a *disconnected term* and it is given by

$$
\begin{aligned}
&\langle p_1p_2,\mathrm{out}|a^\dagger_{\mathrm{out}}(q_1)|q_2,\mathrm{in}\rangle =\\
&\quad=(2\pi)^3 2p_1^0\delta^3(p_1-q_1)\langle p_2,\mathrm{out}|q_1,\mathrm{in}\rangle+(2\pi)^3 2p_2^0\delta^3(p_2-q_1)\langle p_1,\mathrm{out}|q_1,\mathrm{in}\rangle
\end{aligned}
\tag{10.176}
$$

Notice that q_1 is on the mass shell, $q_1^2=m^2$, and e^{-iq_1x} is a solution of the Klein-Gordon equation

$$
(\partial^2+m^2)e^{iq_1\cdot x}=0 \qquad (q_1^2=m^2) \tag{10.177}
$$

The second contribution to the matrix element can also be evaluated (for arbitrary states α and β)

$$
\begin{aligned}
&\int d^4x\,\partial_0\left[e^{-iq_1\cdot x}\overleftrightarrow{\partial_0}\langle\beta,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle\right]=\\
&=\int d^4x\left[e^{-iq_1\cdot x}\partial_0^2\langle\beta,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle+(-\partial_0^2e^{-iq_1\cdot x})\langle\beta,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle\right]\\
&=\int d^4x\left[e^{-iq_1\cdot x}\partial_0^2\langle\beta,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle+\left[\left(-\nabla^2+m^2\right)e^{-iq_1\cdot x}\right]\langle\beta,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle\right]\\
&=\int d^4x e^{-iq_1\cdot x}(\partial^2+m^2)\langle\beta,\mathrm{out}|\phi(x)|\alpha,\mathrm{in}\rangle
\end{aligned}
\tag{10.178}
$$

where we have integrated by parts. Hence, the matrix element is

$$
\begin{aligned}
&\langle p_1p_2,\mathrm{out}|q_1q_2,\mathrm{in}\rangle=\frac{i}{Z^{1/2}}\int d^4x_1e^{-iq_1\cdot x_1}(\partial^2+m^2)\langle p_1p_2,\mathrm{out}|\phi(x_1)|q_2,\mathrm{in}\rangle\\
&+(2\pi)^3 2p_1^0\delta^3(p_1-q_1)\langle p_2,\mathrm{out}|q_2,\mathrm{in}\rangle+(2\pi)^3 2p_2^0\delta^3(p_2-q_1)\langle p_1,\mathrm{out}|q_2,\mathrm{in}\rangle
\end{aligned}
\tag{10.179}
$$

The matrix element inside the integrand of the first term is equal to

$$\langle p_1 p_2, \text{out}|\phi(x_1)|q_2, \text{in}\rangle = \lim_{y_1^0 \to +\infty} \frac{i}{Z^{1/2}} \int d^3 y_1 e^{ip_1 \cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \text{out}|\phi(y_1)\phi(x_1)|q_2 \text{in}\rangle \tag{10.180}$$

where (by definition) $y_1^0 > x_1^0$. This expression is also equal to

$$\begin{aligned}\langle p_1 p_2 \text{out}|\phi(x_1)|q_2 \text{in}\rangle =& \langle p_2, \text{out}|\phi(x_1) a_{\text{in}}(p_1)|q_2, \text{in}\rangle \\ &+ \frac{i}{Z^{1/2}} \int d^4 y_1 e^{ip_1 \cdot y_1} (\partial_{y_1}^2 + m^2) \langle p_2 \text{out}|T\phi(y_1)\phi(x_1)|q_2 \text{in}\rangle\end{aligned} \tag{10.181}$$

By substituting back into the expression from the matrix element we find that the latter is equal to

$$\begin{aligned}&\langle p_1 p_2, \text{out}|q_1 q_2, \text{in}\rangle = \\ &\quad = (2\pi)^3 2p_1^0 \delta^3(p_1 - q_1) \langle p_2 \text{out}|q_2 \text{in}\rangle + (2\pi)^3 2p_2^0 \delta^3(p_2 - q_1) \langle p_1 \text{out}|q_2 \text{in}\rangle \\ &\quad + \frac{i}{Z^{1/2}} \int d^4 x_1 e^{-iq_1 \cdot x_1} (\partial_{x_1}^2 + m^2) \langle p_2 \text{out}|\phi(x_1|0, \text{in}\rangle (2\pi)^3 2q_2^0 \delta^3(q_2 - p_1) \\ &\quad + \left(\frac{1}{Z^{1/2}}\right)^2 \int d^4 x_1 d^4 y_1 e^{i(p_1 \cdot y_1 - q_1 \cdot x_1)} (\partial_{x_1}^2 + m^2)(\partial_{y_1}^2 + m^2) \\ &\qquad \times \langle p_2 \text{out}|T\phi(y_1)\phi(x_1)|q_2 \text{in}\rangle\end{aligned} \tag{10.182}$$

By iterating this process once more we obtain the *reduction formula* of Eq.(10.168) plus disconnected terms.

The reduction formula provides the connection between the on-shell S-matrix elements and v. e. v. of time ordered products. Notice that the reduction formula implies that the v.e.v. of time-ordered products of the field operators must have poles in the variables p_1^2 (where p_i is conjugate to x_i) and that the matrix element is the residue of this pole. We will see later on that this residue is the on-shell one-particle irreducible vertex function.

The reduction formula shows that all scattering data can be understood in terms of an appropriate v.e.v. of a time ordered product of field operators. The problem that we are left to solve is the computation of these v.e.v.'s. In the next chapter we will use perturbation theory to compute these expectation values.

10.9 Linear response theory

In addition to the problem of evaluating S-matrix elements, it is of interest to consider the *response* of a system to weak localized external perturbations.

These responses will tell us much about the nature of both the ground state and of the low-lying states of the system. This method is of great importance for the study of systems in condensed matter physics.

Let H be the full Hamiltonian of a system. We will consider the coupling of the system to weak external sources. Let $\hat{\mathcal{O}}(x,t)$ be a hermitian operator representing a *local observable* such as the charge density, the current density or the local magnetic moment. Let us represent the coupling to the external source by an extra term $H_{\text{ext}}(t)$ in the Hamiltonian. The total Hamiltonian now is

$$H_T = H + H_{\text{ext}} \tag{10.183}$$

If the source is adiabatically switched on and off, then the Heisenberg representation for the isolated system becomes the interaction representation for the full system. Hence, exactly as in the interaction representation, all the observables obey the Heisenberg equations of motion of the system in the absence of the external source while the states will follow the external source in their evolution.

Let $|\text{gnd}\rangle$ be the *exact* ground state (or vacuum) of the system in the absence of any external sources. The external sources perturb this ground state and cause the v.e.v. of the local observable $\hat{\mathcal{O}}(x,t)$ to change:

$$\langle \text{gnd}|\hat{\mathcal{O}}(\boldsymbol{x},t)|\text{gnd}\rangle \to \langle \text{gnd}|U^{-1}(t)\;\hat{\mathcal{O}}(\boldsymbol{x},t)\;U(t)|\text{gnd}\rangle \tag{10.184}$$

where the time evolution operator $U(t)$ is now given by

$$U(t) = T\exp\{-\frac{i}{\hbar}\int_{-\infty}^{t} dt' H_{\text{ext}}(t')\} \tag{10.185}$$

Linear response theory consists in evaluating the changes in the expectation values of the observables to *leading order* in the *external perturbation*. Thus, to leading order in the external sources, the *change* of the v.e.v. is

$$\delta\langle \text{gnd}|\hat{\mathcal{O}}(\boldsymbol{x},t)|\text{gnd}\rangle = \frac{i}{\hbar}\int_{-\infty}^{t} dt'\;\langle \text{gnd}|[H_{\text{ext}}(t'),\hat{\mathcal{O}}(\boldsymbol{x},t)]|\text{gnd}\rangle + \ldots \tag{10.186}$$

Quite generally, we will be interested in the case in which H_{ext} represents the local coupling of the system to an external source $f(\boldsymbol{x},t)$ through the observable $\hat{\mathcal{O}}(\boldsymbol{x},t)$. Thus, we will choose the perturbation $H_{\text{ext}}(t)$ to have the form

$$H_{\text{ext}}(t) = \int d^3x\; f(\boldsymbol{x},t)\;\hat{\mathcal{O}}(\boldsymbol{x},t) \tag{10.187}$$

The function $f(\boldsymbol{x},t)$ is usually called the *force*.

If the observable is normal ordered relative to the ground state of the isolated system, it must have a vanishing expectation value in the ground state, $\langle \text{gnd}|\hat{\mathcal{O}}(\boldsymbol{x},t)|\text{gnd}\rangle = 0$. Hence, the change of its expectation value will be equal to the final value and it is given by

$$\delta\langle \text{gnd}|\hat{\mathcal{O}}(\boldsymbol{x},t)|\text{gnd}\rangle = \\ = \frac{i}{\hbar}\int_{-\infty}^{t} dt' \int d^3x' \, \langle \text{gnd}|[\hat{\mathcal{O}}(\boldsymbol{x},t'), \hat{\mathcal{O}}(\boldsymbol{x},t)]|\text{gnd}\rangle \; f(\boldsymbol{x}',t') + \cdots \tag{10.188}$$

The main assumption of linear response theory is that the *response* is *proportional* to the *force*. The proportionality constant is interpreted as a *generalized susceptibility* χ. Thus, we write the change of the v.e.v. in the form

$$\delta\langle \text{gnd}|\hat{\mathcal{O}}(\boldsymbol{x},t)|\text{gnd}\rangle = \chi \cdot f \equiv \int d^3x' \int_{-\infty}^{t} dt' \; \chi(x,x') \; f(x') + \ldots \tag{10.189}$$

where $\chi(x,x')$ is the susceptibility.

Let $D^R(x,x')$ represent the *retarded correlation function* of the observable $\hat{\mathcal{O}}(\boldsymbol{x},t)$,

$$D^R(x,x') = -i\Theta(x_0 - x_0')\langle \text{gnd}|[\hat{\mathcal{O}}(x), \hat{\mathcal{O}}(x')]|\text{gnd}\rangle \tag{10.190}$$

We see that $\langle \mathcal{O}(x)\rangle$ is determined by $D^R(x,x')$ since

$$\delta\langle \text{gnd}|\mathcal{O}(x)|\text{gnd}\rangle = \frac{1}{\hbar}\int d^4x' D^R(x,x') f(x') + \ldots \tag{10.191}$$

Therefore, the responses and the susceptibilities are given by retarded correlation functions, not by time-ordered ones. However, since the retarded and time-ordered correlation functions are related by an analytic continuation, the knowledge of the latter gives the information about the former.

Let us Fourier transform the time-dependence of the ground state expectation value $\langle \mathcal{O}(\boldsymbol{x},t)\rangle$. The Fourier transform, $\langle \mathcal{O}(\boldsymbol{x},\omega)\rangle$, is given by the expression

$$\delta\langle \text{gnd}|\mathcal{O}(\boldsymbol{x},\omega)|\text{gnd}\rangle = \\ \int d^3x' \Big\{ -\frac{i}{\hbar}\int_{-\infty}^{0} d\tau \langle \text{gnd}|[\mathcal{O}(\boldsymbol{x},t), \mathcal{O}(\boldsymbol{x}',t+\tau)]|\text{gnd}\rangle \; e^{i\omega\tau}\Big\} f(\boldsymbol{x}',\omega) \tag{10.192}$$

where $f(\boldsymbol{x},\omega)$ is the Fourier transform of $f(\boldsymbol{x},t)$. Thus, the Fourier transform of the generalized susceptibility, $\chi(\boldsymbol{x},\boldsymbol{x}';\omega)$ is given by

$$\chi(\boldsymbol{x},\boldsymbol{x}';\omega) = -\frac{i}{\hbar}\int_{-\infty}^{0} d\tau \; e^{i\omega\tau} \; \langle \text{gnd}|[\mathcal{O}(\boldsymbol{x},0)\mathcal{O}(\boldsymbol{x}',\tau)]|\text{gnd}\rangle \tag{10.193}$$

which is known as the *Kubo Formula*. Hence

$$\chi(\boldsymbol{x}, \boldsymbol{x}'; \omega) = \frac{1}{\hbar} \int_{-\infty}^{+\infty} d\tau \ e^{i\omega\tau} D^R(x, x') \tag{10.194}$$

If we also Fourier transform the space dependence, we get

$$\langle \mathcal{O}(\boldsymbol{p}, \omega) \rangle = \frac{1}{\hbar} D^R(\boldsymbol{p}, \omega) f(\boldsymbol{p}, \omega) \tag{10.195}$$

The generalized susceptibility $\chi(\boldsymbol{p}, \omega)$ now becomes

$$\chi(\boldsymbol{p}, \omega) = \frac{\langle \mathcal{O}(\boldsymbol{p}, \omega) \rangle}{f(\boldsymbol{p}, \omega)} = \frac{1}{\hbar} D^R(\boldsymbol{p}, \omega) \tag{10.196}$$

In practice we will compute the *time-ordered* correlation function $D(x, x')$. If we recall our discussion about the propagator, we expect $D(\boldsymbol{p}, \omega)$ to have poles on the real frequency axis. For $D(x, x')$ to be time ordered, all the poles with $\omega < 0$ should be moved (infinitesimally) into the upper half of the complex frequency plane, while all poles with $\omega > 0$ should be moved into lower half-plane. Thus $D(\boldsymbol{p}, \omega)$ is not analytic on either half-plane. On the other hand, the retarded correlation function $D_R(\boldsymbol{p}, \omega)$ is (with our conventions for Fourier transforms) analytic in the lower half-plane. Thus, we can relate the time-ordered correlation function $D(\boldsymbol{p}, \omega)$ to the retarded correlation function $D^R(\boldsymbol{p}, \omega)$ by

$$\text{Re} D^R(\boldsymbol{p}, \omega) = \text{Re} D(\boldsymbol{p}, \omega) \equiv \hbar \text{Re} \chi(\boldsymbol{p}, \omega) \tag{10.197}$$

$$\text{Im} D^R(\boldsymbol{p}, \omega) = \text{Im} D(\boldsymbol{p}, \omega) \ \text{sign} \omega \equiv \hbar \text{Im} \chi(\boldsymbol{p}, \omega) \tag{10.198}$$

The time-ordered correlation function $D(\boldsymbol{p}, \omega)$, i.e. the propagator for the observable $\hat{\mathcal{O}}(\boldsymbol{x}, t)$, admits a spectral (or Lehmann) representation similar to that of the propagator for the relativistic scalar field of Eq.(10.160). Similarly, we can define the *spectral function* $A(\boldsymbol{p}, \omega)$ of the observable to be

$$A(\boldsymbol{p}, \omega) = \text{Im} D^{\text{ret}}(\boldsymbol{p}, \omega) \tag{10.199}$$

The relations of Eq.(10.197) and Eq.(10.198) imply that the susceptibility $\chi(\boldsymbol{p}, \omega)$ obeys the Kramers-Krönig (or dispersion) relation

$$\text{Re} \chi(p, \omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{+\infty} d\omega' \ \frac{\text{Im} \chi(\boldsymbol{p}, \omega')}{\omega' - \omega} \tag{10.200}$$

where $\mathcal{P}$ denotes the principal value of the expression.

Finally, let us recast the formulas for a general change of an arbitrary operator into a more compact form. We can apply the formulas that we derived for the interaction representation just to the part of the Hamiltonian that involves the coupling to the external sources $H_{\text{ext}}(t)$. The interaction representation S-matrix is

$$S = \lim_{t \to +\infty} U(t) = T\, e^{-\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt H_{\text{ext}}(t)} \tag{10.201}$$

Let $\langle \text{gnd}, \text{out}|\text{gnd}, \text{in}\rangle$ be the *vacuum persistence amplitude*

$$\begin{aligned}\langle \text{gnd}, \text{out}|\text{gnd}, \text{in}\rangle &= \langle \text{gnd}|S|\text{gnd}\rangle \\ &= \langle \text{gnd}|T\, e^{-\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, H_{\text{ext}}(t)}|\text{gnd}\rangle \end{aligned} \tag{10.202}$$

which is a functional of the forces (or sources) $f(\boldsymbol{x}, t)$, and that we will denote by $Z[f]$

$$Z[f] = \langle \text{gnd}|T\, e^{-\frac{i}{\hbar}\int d^4x\, f(x)\, \mathcal{O}(x)}|\text{gnd}\rangle \tag{10.203}$$

In the limit in which the sources are weak, we can expand $Z[f]$ in powers of $f(x)$ to obtain

$$\begin{aligned} Z[f] =&1 - \frac{i}{\hbar}\int d^4x f(x)\langle \text{gnd}|\mathcal{O}(x)|\text{gnd}\rangle \\ &+ \frac{1}{2!}\left(-\frac{i}{\hbar}\right)^2 \int d^4x \int d^4x' f(x) f(x')\langle \text{gnd}|T\,\mathcal{O}(x)\mathcal{O}(x')|\text{gnd}\rangle + \dots \end{aligned} \tag{10.204}$$

The second term vanishes if the operator $\mathcal{O}(\boldsymbol{x}, t)$ is normal-ordered. Within the same degree of precision, we can re-exponentiate the resulting expression and find

$$Z[f] = e^{\frac{i}{2\hbar}\int d^4x \int d^4x' f(x) K(x, x') f(x') + O(f^3)} \tag{10.205}$$

where the kernel $K(x, x')$ is the time-ordered correlation function of the observable $A(x)$,

$$K(x, x') = \frac{i}{\hbar}\langle \text{gnd}|T\mathcal{O}(x)\mathcal{O}(x')|\text{gnd}\rangle \tag{10.206}$$

In general, the observables of physical interest are, at least, bilinear functions of the fields. Thus, the kernels $K(x, x')$ represent not one-particle propagators but, in general, propagators for two or more excitations, i.e. they involve four-point functions of the field operators, or higher.

We can learn a lot from a physical system if the spectral functions of the kernels $K(x, x')$ are known. In general we expect that the spectral function will have a structure similar to that of the propagator: one (or more) delta-function contributions and a branch cut. The delta-functions are two (or more) particle bound states, which are known as the collective modes. The branch cuts originate from the two or multi-particle continuum. Examples of collective modes are plasmons (or sound waves) in electron liquids, spin waves in magnets, phase modes in superconductors and superfluids, etc.

10.10 The Kubo formula and the electrical conductivity of a metal

As an application, we will now consider the response of an electron gas to weak external electromagnetic fields $A_\mu(x)$. The formalism can be generalized easily to other systems (relativistic or not) and responses. In particular, we will discuss how to relate response functions to the electrical conductivity of a metal. More general applications can be found in the classic text by Martin (Martin, 1968).

There are three effects (and couplings) that we need to take into consideration: a) electrostatic, b) diamagnetic (or orbital) and c) paramagnetic. The electrostatic coupling is simply the coupling of the local charge density of the electron fluid to an external (scalar) potential. In this case H_{ext} given by

$$H_{\text{ext}} = \sum_{\sigma=\uparrow,\downarrow} \int d^3x \; e\, \phi(\boldsymbol{x},t)\; \psi_\sigma^\dagger(\boldsymbol{x},t)\psi_\sigma(\boldsymbol{x},t) \tag{10.207}$$

where $\phi(\boldsymbol{x},t) \equiv A_0(\boldsymbol{x},t)$ is the scalar potential (or time component) of the vector potential $A_\mu(x)$.

The diamagnetic coupling (or orbital) coupling follows from the minimal coupling to the space components of the external vector potential $\boldsymbol{A}(x)$. The kinetic energy term H_{Kin} is modified following the minimal coupling prescription to become

$$H_{\text{Kin}}(A) = \int d^3x \sum_{\sigma=\uparrow,\downarrow} \frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\boldsymbol{A}(x)\right)\psi_\sigma^\dagger(x) \cdot \left(\nabla - \frac{ie}{\hbar c}\boldsymbol{A}(x)\right)\psi_\sigma(x) \tag{10.208}$$

which can be written as a sum of two terms

$$H_{\text{Kin}}(A) = H_{\text{Kin}}(0) + H_{\text{ext}}(A) \tag{10.209}$$

where $H_{\text{Kin}}(0)$ is the Hamiltonian in the absence of the field and $H_{\text{ext}}(A)$

is the perturbation,

$$H_{\text{ext}}(A) = \int d^3x \left[\boldsymbol{J}(\boldsymbol{x},t) \cdot \boldsymbol{A}(\boldsymbol{x},t) - \frac{e^2}{2mc^2} \boldsymbol{A}^2(\boldsymbol{x},t) \sum_\sigma \psi_\sigma^\dagger(\boldsymbol{x},t)\psi_\sigma(\boldsymbol{x},t) \right] \tag{10.210}$$

Here $\boldsymbol{J}(x)$ is the gauge-invariant charge current

$$\begin{aligned} \boldsymbol{J}(x) =& \frac{ie\hbar}{2mc} \sum_\sigma \left[\psi_\sigma^\dagger(x) \nabla \psi_\sigma(x) - \nabla \psi_\sigma^\dagger(x) \psi_\sigma(x) \right] - \frac{e^2}{mc^2} \boldsymbol{A}(x) \sum_\sigma \psi_\sigma^\dagger(x)\psi_\sigma(x) \\ \equiv& \frac{ie\hbar}{2mc} \sum_\sigma \left[\psi_\sigma^\dagger(x) \boldsymbol{D} \psi_\sigma(x) - (\boldsymbol{D}\psi_\sigma(x))^\dagger \psi_\sigma(x) \right] \end{aligned} \tag{10.211}$$

where $\boldsymbol{D} = \nabla + i\frac{e}{\hbar c}\boldsymbol{A}(x)$ is the covariant derivative (in space). Clearly $\boldsymbol{J}(x)$ is the sum of the two terms, the mass current and the diamagnetic term, $\frac{e^2}{mc^2}\boldsymbol{A}^2 \sum_\sigma \psi_\sigma^\dagger \psi_\sigma$.

We can write the total perturbation, including the scalar potential A_0, in the form

$$H_{\text{ext}} = \int d^3x \left[J_\mu(x) A^\mu(x) - \frac{e^2}{2mc^2} \boldsymbol{A}^2 \sum_\sigma \psi_\sigma^\dagger \psi_\sigma \right] \tag{10.212}$$

Finally, we consider the paramagnetic coupling of the external magnetic field to the spin degrees of freedom of the system which has the form of a Zeeman interaction

$$H_{\text{ext}}^{\text{Zeeman}} = \int d^3x \; g\boldsymbol{B}(x) \cdot \sum_{\sigma,\sigma'} \psi_\sigma^\dagger(x) \boldsymbol{S}_{\sigma\sigma'} \psi_{\sigma'}(x) \tag{10.213}$$

where g is typically of the order of the Bohr magneton μ_B and $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$ for spin one-half systems.

A straightforward application of the linear response formulas derived above yields the following expression for the expectation value of the current $\langle J_\mu \rangle'$ in the presence of the perturbation

$$\langle J_\mu(x) \rangle' = \langle J_\mu(x) \rangle_{\text{gnd}} - \frac{i}{\hbar} \int_{-\infty}^{t} dt' \int d^3x' \langle \text{gnd} | [J_\nu(x'), J_\mu(x)] | \text{gnd} \rangle A_\nu(x') \tag{10.214}$$

This formula suggests that we should define the *retarded current correlation function* $\mathcal{D}_{\mu\nu}^R(x,x')$

$$\mathcal{D}_{\mu\nu}^{\text{Ret}}(x,x') = -i\Theta(x_0 - x_0') \langle \text{gnd} | [J_\mu(x), J_\nu(x')] | \text{gnd} \rangle \tag{10.215}$$

The *induced current* $\langle J_\mu \rangle_{\text{ind}}$

$$\langle J_\mu \rangle_{\text{ind}} = \langle J_\mu \rangle' - \langle j_\mu \rangle_{\text{gnd}} \tag{10.216}$$

(where j_μ is the mass current) has a very simple form in terms of the retarded correlation function $\mathcal{D}^R_{\mu\nu}(x,x')$,

$$\langle J_\mu(x)\rangle_{\rm ind} = \frac{1}{\hbar}\int d^4x' \mathcal{D}^{\rm Ret}_{\mu\nu}(x,x')A^\nu(x') - \frac{e^2}{mc^2}A_k(x)\langle n(x)\rangle\delta_{\mu k} + O(A^2) \tag{10.217}$$

Below we will show that $\langle J_\mu(x)\rangle_{\rm ind}$ is conserved i.e.$\partial^x_\mu\langle J^\mu(x)\rangle_{\rm ind} = 0$, and gauge-invariant.

We can express these results in terms of an effective action for the external electromagnetic field A_μ

$$Z_{\rm eff}[A_\mu] = \mathcal{N}e^{\displaystyle\frac{i}{2\hbar}\int d^4x \int d^4y \; A_\mu(x)\Pi^{\mu\nu}(x-y)A_\nu(y)} \tag{10.218}$$

such that

$$\langle J_\mu(x)\rangle_{\rm ind} = \frac{\hbar}{i}\frac{1}{Z_{\rm eff}[A_\mu]}\frac{\delta Z_{\rm eff}[A_\mu]}{\delta A_\mu(x)} = \int d^4y \; \Pi_{\mu\nu}(x-y)A^\nu(y) \tag{10.219}$$

By inspection we see that the effective *polarization tensor* $\Pi_{\mu\nu}(x-y)$ is related to the current correlation function $\mathcal{D}_{\mu\nu}(x-y)$,

$$\Pi_{\mu\nu}(x-y) = \frac{1}{\hbar}\mathcal{D}_{\mu\nu}(x-y) + \frac{e\rho}{mc^2}\delta(x-y)\tilde{g}_{\mu\nu} \tag{10.220}$$

where we introduced the diagonal tensor

$$\tilde{g}_{\mu\nu} = \begin{cases} 0 & \text{if } \mu = 0 \text{ and/or } \nu = 0 \\ -\delta_{ij} & \text{if } \mu,\nu = i,j \end{cases} \tag{10.221}$$

Since $\langle J_\mu(x)\rangle_{\rm ind}$ is gauge invariant, we can compute its form in any gauge. In the gauge $A_0 = 0$ the spatial components of $\langle J_\mu(x)\rangle_{\rm ind}$ are

$$\langle J_k(x)\rangle_{\rm ind} = -\frac{e\rho}{mc^2}A_k(x) + \int d^4x' \mathcal{D}^{\rm ret}_{k\ell}(x-x')A_\ell(x') + O(A^2) \tag{10.222}$$

where $\rho = e\langle n\rangle$ is the expectation value of the charge density and we will assume that it is uniform.

In the $A_0 = 0$ gauge, the external electric field $\boldsymbol{E}_{\rm ext}$ and magnetic field $\boldsymbol{H}$ are

$$\boldsymbol{E}_{\rm ext} = -\partial_0\boldsymbol{A}, \qquad \boldsymbol{H} = \nabla\times\boldsymbol{A} \tag{10.223}$$

Now, in Fourier space, we can write

$$\begin{aligned}\langle J_k(\boldsymbol{p},\omega)\rangle_{\text{ind}} &= -\frac{e^2\langle n\rangle}{mc^2}A_k(\boldsymbol{p},\omega) + \mathcal{D}^{\text{ret}}_{k\ell}(\boldsymbol{p},\omega)A_\ell(\boldsymbol{p},\omega)\\ &\equiv \left(\mathcal{D}^{\text{ret}}_{k\ell}(\boldsymbol{p},\omega) - \frac{e^2\langle n\rangle}{mc^2}\delta_{k\ell}\right)\frac{E^{\text{ext}}_\ell}{i\omega}(\boldsymbol{p},\omega)\end{aligned} \tag{10.224}$$

This expression is *almost* the conductivity. It is not quite that since the conductivity is a relation between the *total current* $\boldsymbol{J} = \boldsymbol{J}_{\text{ind}} + \boldsymbol{J}_{\text{ext}}$ and the *total electric field* $\boldsymbol{E}$. In order to take these electromagnetic effects into account, we must use Maxwell's equations in a medium, which involve the vector fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{H}$

$$\begin{aligned}\nabla\cdot\boldsymbol{D} &= \rho, & \nabla\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{H}}{\partial t}\\ \nabla\cdot\boldsymbol{B} &= 0, & \nabla\times\boldsymbol{H} &= \frac{\partial\boldsymbol{E}}{\partial t} + \boldsymbol{J}\end{aligned} \tag{10.225}$$

where

$$\boldsymbol{B} = \boldsymbol{H} + \boldsymbol{M} \qquad \boldsymbol{E} = \boldsymbol{E}^{\text{ext}} + \boldsymbol{E}^{\text{ind}} \tag{10.226}$$

Here $\boldsymbol{M}$ and $\boldsymbol{E}^{\text{ind}}$ are the magnetic and electric polarization vectors. In particular

$$\boldsymbol{J}^{\text{ind}} = \partial_t\boldsymbol{E}^{\text{ind}} \tag{10.227}$$

and

$$\partial_t\boldsymbol{D} = \partial_t\boldsymbol{E} + \boldsymbol{J}^{\text{ind}} \tag{10.228}$$

Linear response theory is the statement that $\boldsymbol{D}$ (not to be confused with the covariant derivative!) must be proportional to $\boldsymbol{E}$,

$$D_j = \varepsilon_{jk}E_k \tag{10.229}$$

where ε_{jk} is the dielectric tensor.

$\boldsymbol{E}$ and $\boldsymbol{E}^{\text{ext}}$ satisfy similar equations

$$\begin{aligned}-\nabla\times\nabla\times\boldsymbol{E} &= \partial_t^2\boldsymbol{E} + \partial_t\boldsymbol{J}\\ -\nabla\times\nabla\times\boldsymbol{E}^{\text{ext}} &= \partial_t^2\boldsymbol{E}^{\text{ext}} + \partial_t\boldsymbol{J}^{\text{ext}}\end{aligned} \tag{10.230}$$

Since $\nabla\times\nabla\times\boldsymbol{E} = \nabla(\nabla\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E}$, we can write, in Fourier space,

$$\begin{aligned}p_ip_jE_j(\boldsymbol{p},\omega) - \boldsymbol{p}^2E_i(\boldsymbol{p},\omega) &= -\omega^2E_j(\boldsymbol{p},\omega) - i\omega J_i(\boldsymbol{p},\omega)\\ p_ip_jE^{\text{ext}}_j(\boldsymbol{p},\omega) - \boldsymbol{p}^2E^{\text{ext}}_i(\boldsymbol{p},\omega) &= -\omega^2E^{\text{ext}}_i(\boldsymbol{p},\omega) - i\omega J^{\text{ext}}_i(\boldsymbol{p},\omega)\end{aligned} \tag{10.231}$$

Thus, we get

$$p_i p_j E_j(\boldsymbol{p},\omega) - \boldsymbol{p}^2 E_i(\boldsymbol{p},\omega) + \omega^2 E_i(\boldsymbol{p},\omega) = -i\omega J_i^{\text{ind}}(\boldsymbol{p},\omega) + p_i p_j E_j^{\text{ext}}(\boldsymbol{p},\omega) - \boldsymbol{p}^2 E_i^{\text{ext}}(\boldsymbol{p},\omega) + \omega^2 E_i^{\text{ext}}(\boldsymbol{p},\omega) \tag{10.232}$$

and

$$-i\omega J_i^{\text{ind}}(\boldsymbol{p},\omega) = \left(\delta_{ij}\frac{e^2\langle n\rangle}{mc^2} - \mathcal{D}_{ij}^R(\boldsymbol{p},\omega)\right) E_j^{\text{ext}}(\boldsymbol{p},\omega) \tag{10.233}$$

From these results we conclude that

$$(p_i p_j - \boldsymbol{p}^2\delta_{ij} + \omega^2\delta_{ij})\ E_j(\boldsymbol{p},\omega) = (\delta_{ij}\frac{e^2\langle n\rangle}{mc^2} - \mathcal{D}_{ij}^R(\boldsymbol{p},\omega) + p_i p_j - \boldsymbol{p}^2\delta_{ij} + \omega^2\delta_{ij})\ E_j^{\text{ext}}(\boldsymbol{p},\omega) \tag{10.234}$$

In matrix form, these equations have the simpler form

$$(\boldsymbol{p}\otimes\boldsymbol{p} - \boldsymbol{p}^2 I + \omega^2 I)\boldsymbol{E}(\boldsymbol{p},\omega) = \left(\frac{e^2\langle n\rangle}{mc^2} I - \mathcal{D}^R + \boldsymbol{p}\otimes\boldsymbol{p} - \boldsymbol{p}^2 I + \omega^2 I\right)\boldsymbol{E}_{\text{ext}}(\boldsymbol{p},\omega) \tag{10.235}$$

This equation allows us to write $\boldsymbol{E}_{\text{ext}}$ in terms of $\boldsymbol{E}$. We find that the induced current is

$$i\omega \boldsymbol{J}_{\text{ind}}(\boldsymbol{p},\omega) = \left(\mathcal{D}^R - \frac{e^2\langle n\rangle}{mc^2} I\right)\left[\frac{e^2\langle n\rangle}{mc^2} I - \mathcal{D}^R + \boldsymbol{p}\otimes\boldsymbol{p} - \boldsymbol{p}^2 I + \omega^2 I\right]^{-1}\left(\boldsymbol{p}\otimes\boldsymbol{p} - \boldsymbol{p}^2 I + \omega^2 I\right)\boldsymbol{E}(\boldsymbol{p},\omega) \tag{10.236}$$

and we find that the *conductivity tensor* σ_{jk} is

$$i\omega\sigma(\boldsymbol{p},\omega) = \left(\mathcal{D}^R(\boldsymbol{p},\omega) - \frac{e^2\langle n\rangle}{mc^2} I\right) + \left(\mathcal{D}^R(\boldsymbol{p},\omega) - \frac{e^2\langle n\rangle}{mc^2} I\right) \times\left[\frac{e^2\langle n\rangle}{mc^2} I - \mathcal{D}^R(\boldsymbol{p},\omega) + p\otimes p - \boldsymbol{p}^2 I + \omega^2 I\right]^{-1}\left(\mathcal{D}^R(\boldsymbol{p},\omega) - \frac{e^2\langle n\rangle}{mc^2} I\right) \tag{10.237}$$

Also, since $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, the dielectric tensor ε_{jk} and the conductivity tensor σ_{jk} are related by

$$\varepsilon = I + \frac{i}{\omega}\ \sigma \tag{10.238}$$

10.11 Correlation functions and conservation laws

In the problem discussed in the previous section, we saw that we had to consider a correlation function of currents. Since the currents are conserved

$$\partial_\mu J^\mu = 0, \tag{10.239}$$

we expect that the correlation function $\mathcal{D}_{\mu\nu}(x, x')$ should obey a similar equation. Let us compute the divergence of the retarded correlation function,

$$\partial_x^\mu \mathcal{D}^R_{\mu\nu}(x, x') = \partial_x^\mu \left(-i\Theta(x_0 - x_0') \langle \text{gnd} | [J_\mu(x), J_\nu(x')] | \text{gnd} \rangle \right) \tag{10.240}$$

Except for the contribution coming from the step function, we see that we can operate with the derivative inside the expectation value to get

$$\begin{aligned} \partial_x^\mu \mathcal{D}^R_{\mu\nu}(x, x') = & - i \left(\partial_x^\mu \Theta(x_0 - x_0') \right) \langle \text{gnd} | [J_\mu(x), J_\nu(x')] | \text{gnd} \rangle \\ & - i\Theta(x_0 - x_0') \langle \text{gnd} | [\partial_x^\mu J_\mu(x), J_\nu(x')] | \text{gnd} \rangle \end{aligned} \tag{10.241}$$

The second term vanishes since $J_\mu(x)$ is a conserved current and the first term is non zero only if $\mu = 0$. Hence we obtain the identity

$$\partial_x^\mu \mathcal{D}^R_{\mu\nu}(x, x') = -i\delta(x_0 - x_0') \langle \text{gnd} | \left[J_0(x), J_\nu(x') \right] | \text{gnd} \rangle \tag{10.242}$$

which is the v.e.v. of an equal-time commutator.

These commutators are given by(Kadanoff and Martin, 1961)

$$\begin{aligned} \langle \text{gnd} | \left[J^0(\boldsymbol{x}, x_0), J^0(\boldsymbol{x}, x_0) \right] | \text{gnd} \rangle =& 0 \\ \langle \text{gnd} | \left[J^0(\boldsymbol{x}, x_0), J^i(\boldsymbol{x}', x_0) \right] | \text{gnd} \rangle =& \frac{ie^2}{mc^2} \partial_k^x \left[\delta(\boldsymbol{x} - \boldsymbol{x}') \langle n(\boldsymbol{x}) \rangle \right] \end{aligned} \tag{10.243}$$

Identities of this type also arise in relativistic systems, e.g. the theory of Dirac fermions, where they are known as Schwinger terms. However the Schwinger terms of relativistic quantum field theory have a subtle origin (Schwinger, 1959). They are related to *anomalies* and we will discuss them in chapter 20. In the non-relativistic case they arise because in these systems the Fermi sea has a bottom, and hence there is always a high-energy regulator (or cutoff) that preserves gauge-invariance.

Hence, the divergence of $\mathcal{D}^R_{\mu\nu}$ is

$$\begin{aligned} \partial_x^\mu \mathcal{D}^R_{\mu k}(x, x') =& \frac{e^2}{mc^2} \partial_k^x \left[\delta^4(x - x') \langle n(x) \rangle \right], \qquad \partial_{x'}^\mu \mathcal{D}^R_{0\mu}(x, x') = 0 \\ \partial_{x'}^\nu \mathcal{D}^R_{k\nu}(x, x') =& -\frac{e^2}{mc^2} \partial_k^x \left[\delta^4(x - x') \langle n(x') \rangle \right], \qquad \partial_{x'}^\mu \mathcal{D}^R_{0\mu}(x, x') = 0 \end{aligned} \tag{10.244}$$

Notice that the time-ordered functions also satisfy the same identities. These identities can be used to prove that $\langle \boldsymbol{J}^{\text{ind}} \rangle$ is indeed gauge-invariant and conserved.

On the other hand, by comparing Eq.(10.220), that relates the polarization tensor $\Pi_{\mu\nu}$ with the current correlation function $\mathcal{D}_{\mu\nu}$, with Eq.(10.244), we see that the presence of the Schwinger terms in the divergence of $\mathcal{D}_{\mu\nu}$ ensures that the polarization tensor $\Pi_{\mu\nu}$ is conserved and, hence, transverse, i.e.

$$\partial^{\mu}\Pi_{\mu\nu} = 0 \tag{10.245}$$

as required by gauge invariance. We will return to this question in Section 12.6 where we discuss the relation between Ward Identities and gauge invariance.

Furthermore, in momentum and frequency space, the identities become

$$\begin{aligned}
-i\omega\mathcal{D}^R_{00}(\boldsymbol{p},\omega) - ip_k\mathcal{D}^R_{k0}(\boldsymbol{p},\omega) =&0\\
-i\omega\mathcal{D}^R_{0k}(\boldsymbol{p},\omega) - ip_\ell\mathcal{D}^R_{\ell k}(\boldsymbol{p},\omega) =&-\frac{e^2\langle n\rangle}{mc^2}ip_k\\
-i\omega\mathcal{D}^R_{00}(p,\omega) - ip_k\mathcal{D}^R_{0k}(\boldsymbol{p},\omega) =&0\\
-i\omega\mathcal{D}^R_{k0}(\boldsymbol{p},\omega) - ip_\ell\mathcal{D}^R_{k\ell}(\boldsymbol{p},\omega) =&-\frac{e^2\langle n\rangle}{mc^2}ip_k
\end{aligned} \tag{10.246}$$

We can combine them to obtain the result

$$\omega^2\mathcal{D}^R_{00}(\boldsymbol{p},\omega) - p_\ell p_k\mathcal{D}^R_{\ell k}(\boldsymbol{p},\omega) = -\frac{e^2\langle n\rangle}{mc^2}\boldsymbol{p}^{\,2} \tag{10.247}$$

Hence, the density-density and the current-current correlation functions are not independent.

A number of interesting identities follow from this equation. In particular, if we take the static limit $\omega \to 0$ at fixed momentum $\boldsymbol{p}$, we get

$$\lim_{\omega\to 0} p_\ell p_k\mathcal{D}^R_{\ell k}(\boldsymbol{p},\omega) = \frac{e^2\bar{n}}{mc^2}\boldsymbol{p}^{\,2} \tag{10.248}$$

provided that $\lim_{\omega\to 0}\mathcal{D}^{\text{ret}}_{00}(\boldsymbol{p},\omega)$ is not singular for $\boldsymbol{p} \neq 0$. Also, from the equal-time commutator,

$$\langle \text{gnd}|[J_k(\boldsymbol{x},x_0), J_0(\boldsymbol{x},x_0)]|\text{gnd}\rangle = \frac{ie^2}{mc^2}\partial^x_k\left(\delta(\boldsymbol{x}-\boldsymbol{x}')\langle n(\boldsymbol{x})\rangle\right) \tag{10.249}$$

we get

$$\lim_{x_0'\to x_0}\partial^x_k\mathcal{D}^R_{k0}(x,x') = \frac{e^2}{mc^2}\,\nabla^2_{\boldsymbol{x}}\left(\delta(\boldsymbol{x}-\boldsymbol{x}')\langle n(\boldsymbol{x})\rangle\right) \tag{10.250}$$

If the system is uniform, $\langle n(\boldsymbol{x})\rangle = \bar{n}$, we can Fourier transform this last identity to obtain

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} i p_k \mathcal{D}^R_{k0}(\boldsymbol{p},\omega) = -\frac{e^2\bar{n}}{mc^2}\boldsymbol{p}^2 \tag{10.251}$$

Using the conservation laws, we find the identity

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} i\omega \mathcal{D}^R_{00}(\boldsymbol{p},\omega) = \frac{e^2\bar{n}}{mc^2}\boldsymbol{p}^2 \tag{10.252}$$

This identity is known as the *f-sum rule*.

If the system is isotropic, these relations can be used to yield a simpler form for the conductivity tensor. Indeed, for an isotropic system $\mathcal{D}^R_{k\ell}(\boldsymbol{p},\omega)$ can only have the form of a sum of a longitudinal part $\mathcal{D}^R_{\|}$ and a transverse part $\mathcal{D}^R_{\perp}$

$$\mathcal{D}^R_{\ell k}(\boldsymbol{p},\omega) = \mathcal{D}^R_{\|}(\boldsymbol{p},\omega)\frac{p_\ell p_k}{\boldsymbol{p}^2} + \mathcal{D}^R_{\perp}(\boldsymbol{p},\omega)\left(\frac{p_\ell p_k}{\boldsymbol{p}^2} - \delta_{\ell k}\right) \tag{10.253}$$

Thus, we get a relation between $\mathcal{D}^R_{00}$ and $\mathcal{D}^R_{\|}$

$$\omega^2\mathcal{D}^R_{00}(p,\omega) - \boldsymbol{p}^2\mathcal{D}^R_{\|}(\boldsymbol{p},\omega) = -\frac{e^2\bar{n}}{mc^2}\boldsymbol{p}^2 \tag{10.254}$$

Hence

$$\mathcal{D}^R_{00}(\boldsymbol{p},\omega) = \frac{\boldsymbol{p}^2}{\omega^2}\left(\mathcal{D}^R_{\|}(\boldsymbol{p},\omega) - \frac{e^2\bar{n}}{mc^2}\right) \tag{10.255}$$

and

$$\lim_{\omega\to 0}\mathcal{D}^R_{\|}(\boldsymbol{p},\omega) = \frac{e^2\bar{n}}{mc^2} \tag{10.256}$$

for all $\boldsymbol{p}$.

The conductivity tensor can also be separated into longitudinal $\sigma_{\|}$ and transverse $\sigma_\perp$ pieces

$$\sigma_{ij} = \sigma_{\|}\frac{p_i p_j}{\boldsymbol{p}^2} + \sigma_\perp\left(\frac{p_i p_j}{\boldsymbol{p}^2} - \delta_{ij}\right) \tag{10.257}$$

we find

$$\sigma_{\|} = i\omega\left[\frac{\mathcal{D}^R_{\|} - \frac{e^2\bar{n}}{mc^2}}{-\mathcal{D}^R_{\|} + \frac{e^2\bar{n}}{mc^2} + \omega^2}\right] \tag{10.258}$$

and

$$\sigma_\perp = \frac{1}{i\omega}\left(\mathcal{D}_\perp^R - \frac{e^2\bar{n}}{mc^2}\right)\left[1 + \frac{\mathcal{D}_\perp^R - \frac{e^2\bar{n}}{mc^2}}{\frac{e^2\bar{n}}{mc^2} - \mathcal{D}_\perp^R + \omega^2 - \boldsymbol{p}^2}\right] \tag{10.259}$$

These relations tell us that the real part of $\sigma_\parallel$ is determined by the imaginary part of $\mathcal{D}_\parallel^R$. Thus, the *resistive part* (the real part) of the longitudinal conductivity $\sigma_\parallel$, which reflects the dissipation in the system, is determined by the imaginary part of a response function. This general result is known as the fluctuation-dissipation theorem.

10.12 The Dirac propagator in a background electromagnetic field

Let us consider briefly the Dirac propagator in a background electromagnetic field and use it to compute the S-matrix for Coulomb scattering. By a background field we mean a classical (fixed but possibly time-dependent) electromagnetic field $A_\mu(x)$. We will denote by $S_F(x,x'|A)$ the propagator for a free Dirac field in a background gauge field A_μ.

$S_F(x,x'|A)$ obeys the Green function equation

$$\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right) S_F(x,x'|A) = \delta^4(x-x') \tag{10.260}$$

In the absence of a background field the Dirac propagator $S_F(x,x')$ obeys instead

$$\left(i\partial\!\!\!/ - m\right) S_F(x,x') = \delta^4(x-x') \tag{10.261}$$

Thus, we can also write Eq.(10.260) as

$$\left(S_F^{-1} - eA\!\!\!/\right) S_F(A) = 1 \tag{10.262}$$

Hence

$$S_F(x,x'|A) = S_F(x-x') + e\int d^4y\; S_F(x-y)\; A\!\!\!/(y)\; S_F(y,x'|A) \tag{10.263}$$

or, in components,

$$S_F^{\alpha\beta}(x,x'|A) = S_F^{\alpha\beta}(x-x') + e\int d^4y\; S_F^{\alpha\lambda}(x-y)\; \left[A_\mu(y)\gamma^\mu\right]^{\lambda\sigma}\; S_F^{\sigma\beta}(y,x'|A) \tag{10.264}$$

As an explicit application we will consider the case of Coulomb scattering of (free) Dirac electrons from a fixed nucleus with positive electric charge Ze. We will now compute the S-matrix for this problem in the Born approximation. As in non-relativistic Quantum Mechanics, in this approximation we replace the propagator in the integrand of Eqs. (10.263) and (10.264) by the free Dirac propagator, $S_F(x - x')$.

Consider now an incoming state, a spinor the we will denote by $\Psi_i(x)$, with a particle with positive energy (an electron) and spin up (say in the z direction), and momentum $\boldsymbol{p}_i$. This incoming (initial) state is (for $x_0 \to -\infty$)

$$\Psi_i(x) = \frac{1}{\sqrt{V}}\, u^{(\alpha)}(p_i)\, \sqrt{\frac{m}{E_i}}\, e^{-ip_i \cdot x} \tag{10.265}$$

The outgoing (final) state $\Psi_f(x)$ is a spinor representing also a particle with positive energy (an electron) with spin up (also in the z direction) and momentum $\boldsymbol{p}_f$, and it is given by

$$\Psi_f(y) = \frac{1}{\sqrt{V}}\, u^{(\beta)}(p_f)\, \sqrt{\frac{m}{E_f}}\, e^{-ip_f \cdot y} \tag{10.266}$$

The S-matrix is

$$S_{fi} = i \lim_{x_0 \to -\infty} \lim_{y_0 \to +\infty} \int d^3x \int d^3y\, \overline{\Psi}_f(\boldsymbol{y}, y_0)\, S_F(y, x|A)\, \Psi_i(\boldsymbol{x}, x_0) \tag{10.267}$$

At the level of the Born approximation we can write

$$\begin{aligned} S_{fi} = i &\lim_{x_0 \to -\infty} \lim_{y_0 \to +\infty} \int d^3x \int d^3y\, \overline{\Psi}_f(\boldsymbol{y}, y_0)\, S_F(y, x)\, \Psi_i(\boldsymbol{x}, x_0) + \\ i &\lim_{x_0 \to -\infty} \lim_{y_0 \to +\infty} \int d^3x \int d^3y \int d^4z\, \overline{\Psi}_f(\boldsymbol{y}, y_0)\, S_F(y, z)\, \not{A}(z)\, S_F(z, x)\, \Psi_i(\boldsymbol{x}, x_0) \\ &+ \ldots \end{aligned} \tag{10.268}$$

We now recall the expression for the free Dirac propagator

$$\begin{aligned} S_F(x - x') &= -\, i\langle 0|T\psi_\alpha(x)\bar{\psi}_{\alpha'}(x')|0\rangle \\ &= -\, i \int \frac{d^3p}{(2\pi)^3} \left(\frac{m}{E(p)}\right) \Big(\Theta(x_0' - x_0) e^{-ip \cdot (x' - x)} \Lambda_+(p) \\ &\qquad + \Theta(x_0 - x_0') e^{-ip \cdot (x - x'} \Lambda_-(p)\Big) \end{aligned} \tag{10.269}$$

where $\Lambda_\pm(p)$ are projection operators onto positive (particle) and negative (anti-particle) energy states:

$$\Lambda_\pm(p) = \frac{1}{2m}\left(\pm\not{p} + m\right) \tag{10.270}$$

Alternatively, we can express the propagator in terms of the basis spinors $u^\sigma(p)$ (which span the positive energy states), and $v^\sigma(p)$ (which span the negative energy states), as

$$S_F(x'-x) = -\,i\Theta(x_0'-x_0)\int d^3p \sum_{\sigma=1,2} u_p^{(\sigma)}(x')\bar{u}_p^{(\sigma)}(x) + i\Theta(x_0-x_0')\int d^3p \sum_{\sigma=1,2} v_p^{(\sigma)}(x')\bar{v}_p^{(\sigma)}(x) \tag{10.271}$$

where we have used the notation

$$u_p^{(\sigma)}(x) \equiv u^{(\sigma)}(p)\, e^{-ip\cdot x}, \quad v_p^{(\sigma)}(x) = v^{(\sigma)}(p)\, e^{-ip\cdot x} \tag{10.272}$$

Let us begin by computing the first term in Eq.(10.268), the projection of the free propagator onto the initial and final states. By expanding the propagator we find,

$$\begin{aligned}\int d^3x d^3y\, \bar{\psi}_f(y) S_F(y-x)\psi_i(x) &= \\ = -\,i\Theta(y_0-x_0)\int d^3p &\sum_{\sigma=1,2}\int d^3x \int d^3y\, \bar{\psi}_f(y) u_p^{(\sigma)}(y)\bar{u}_p^{(\sigma)}(x)\psi_i(x) \\ + i\Theta(x_0-y_0)\int d^3p &\sum_{\sigma=1,2}\int d^3x d^3y\, \bar{\psi}_f(y) v_p^{(\sigma)}(y)\bar{v}_p^{(\sigma)}(x)\psi_i(x)\end{aligned} \tag{10.273}$$

We now use the orthogonality relations of the Dirac basis spinors to find

$$\begin{aligned}\int d^3y\, \bar{\psi}_f^{(\beta)}(y) u_p^{(\sigma)}(y) &= \delta^{\beta\sigma}\delta^3(\boldsymbol{p}-\boldsymbol{p}_f) \\ \int d^3x\, \bar{u}_p^{(\sigma)}(x)\psi_i^{(\alpha)}(x) &= \delta^{\sigma\alpha}\delta^3(\boldsymbol{p}-\boldsymbol{p}_i)\end{aligned} \tag{10.274}$$

Hence, to leading order the matrix element S_{fi} of the S-matrix is

$$S_{fi} = \delta^3(\boldsymbol{p}_f-\boldsymbol{p}_i)\delta^{\alpha\beta} + \text{Born term} \tag{10.275}$$

Let us now compute the Born term (the first Born approximation). We will need to compute first an expression for

$$\int d^3y\, \bar{\psi}_f(y) S_F(y,z) \tag{10.276}$$

and for

$$\int d^3x\, S_F(z,x)\psi_i(x) \tag{10.277}$$

Using once again the expansion of the propagator we find that Eq.(10.276) is

$$\begin{aligned}
&\int d^3y\, \bar{\psi}_f(y) S_F(y,z) = \\
&= \int d^3y\, \bar{\psi}_f^{(\beta)}(y)(-i)\Theta(y_0 - z_0) \sum_{\sigma=1,2} \int d^3p\, u_p^{(\sigma)}(y)\bar{u}_p^{(\sigma)}(z) \\
&\quad + \int d^3y\, \bar{\psi}_f^{(\beta)}(i)\Theta(z_0 - y_0) \sum_{\sigma=1,2} v_p^{(\sigma)}(y)\bar{v}_p^{(\sigma)}(z) \\
&= -i\Theta(y_0 - z_0) \sum_{\sigma=1,2} \int d^3p \left(\int d^3y\, \bar{\psi}_f^{(\beta)}(y) u_p^{\sigma)}(y) \right) \bar{u}_p^{(\sigma)}(z) \\
&\quad + i\Theta(z_0 - y_0) \sum_{\sigma=1,2} \int d^3p \left(\int d^3y\, \bar{\psi}_f^{(\beta)}(y) v_p^{\sigma)}(y) \right) \bar{v}_p^{(\sigma)}(z) \\
&= \begin{cases} -i\Theta(y_0 - z_0)\, \bar{\psi}_f^{(\beta)}(z), & \text{if the final state is a particle} \\ +i\Theta(z_0 - y_0)\, \bar{\psi}_f^{(\beta)}(z), & \text{if the final state is an antiparticle} \end{cases}
\end{aligned} \tag{10.278}$$

The other expression, Eq.(10.277), can be computed similarly. Putting it all together we find that the Born term is

$$\text{Born term} = -ie \int d^4z\, \bar{\psi}_f^{(\beta)}(z)\, \not{A} \psi_i^{(\alpha)}(z)\, \Theta(y_0 - z_0)\, \Theta(z_0 - x_0) \tag{10.279}$$

corresponding to an electron propagating forward in time.

Let us evaluate this expression for the case of a Coulomb potential,

$$A^\mu = (A_0, 0), \quad A_0 = \frac{-Ze}{4\pi r} \tag{10.280}$$

with $r = |\boldsymbol{z}|$. The Born term now becomes

$$\begin{aligned}
\text{Born term} &= -ie \int_{-\infty}^{\infty} dz_0 \int d^3z\, \bar{\psi}_f^{(\beta)}(z)\gamma_0 \psi_i^{(\alpha)}(z)\, \Theta(y_0 - z_0) \left(\frac{-Ze}{4\pi r} \right) \\
&= \frac{ie}{V} \frac{m}{\sqrt{E_i E_f}} \frac{Ze}{4\pi} \int_{-\infty}^{\infty} dz_0 \int d^3z\, e^{i(p_f - p_i)\cdot z}\, \frac{1}{r}\, \bar{u}^{(\beta)}(p_f)\gamma_0 u^{(\alpha)}(p_i)
\end{aligned} \tag{10.281}$$

where V is the volume. Using now that

$$\int_{-\infty}^{\infty} dz_0\, e^{i(E_f - E_i)z_0} = 2\pi\delta(E_f - E_i) \tag{10.282}$$

we can write the Born term as

$$\text{Born term} = \frac{iZ\alpha}{V} \frac{m}{\sqrt{E_i E_f}\, 2\pi\delta(E_i - E_f} \int d^3r\, \frac{e^{-i\boldsymbol{q}\cdot r}}{r}\, \bar{u}^{(\beta)}(p_f)\gamma_0 u^{(\alpha)}(p_i) \tag{10.283}$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant, $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$ is the momentum transfer, and

$$\int d^3r \, \frac{e^{-i\boldsymbol{q}\cdot r}}{r} = \frac{4\pi}{|\boldsymbol{q}|} \tag{10.284}$$

The matrix element of the S-matrix, in the Born approximation, is then equal to

$$S_{fi} = \delta^{\alpha\beta}\delta^3(\boldsymbol{p}_f - \boldsymbol{p}_i) + i\frac{Z\alpha}{V}\,\frac{M}{\sqrt{E_iE_f}}\,2\pi\delta(E_i - E_f)\,\frac{4\pi}{q^2}\,\bar{u}^{(\beta)}(p_f)\gamma_0 u^{(\alpha)}(p_i) \tag{10.285}$$

Since

$$\#\text{ states with } \boldsymbol{p}_f \text{ within } d^3p_f = V\frac{d^3p_f}{(2\pi)^3} \tag{10.286}$$

we can write the transition probability per particle into these final states as

$$|S_{fi}|^2\, V\, \frac{d^3p_f}{(2\pi^3} = Z^2\frac{(4\pi\alpha)^2}{E_iV}\, m^2\, \frac{|\bar{u}^{(\beta)}(p_f)\gamma_0 u^{(\alpha)}(p_i)|^2}{|\boldsymbol{q}|^4}\, \frac{d^3p_f}{(2\pi)^3E_f}\, 2\pi\delta(E_f - E_i)\, T \tag{10.287}$$

where T is the time of measurement (this is Fermi's Golden rule).

Thus, the number of transitions per particle and unit time is

$$\frac{dP_{fi}}{dt} = \int \left|\frac{iZ\alpha}{V}\,\frac{m}{\sqrt{E_fE_i}}\,\frac{4\pi}{|\boldsymbol{q}|^2}\,\bar{u}^{(\beta)}(p_f)\gamma_0 u^{(\alpha)}(p_i)\right|^2 2\pi\delta(E_f - E_i)V\frac{d^3p_f}{(2\pi)^3} \tag{10.288}$$

Dividing out this expression by the incoming flux, $\frac{1}{V}\frac{|\boldsymbol{p}_i|}{E_i}$, we obtain an expression for the differential cross section

$$d\sigma_{fi} = \left(\int dp_f p_f^2\, \frac{4Z^2\alpha^2m^2}{|\boldsymbol{p}_i|E_f|\boldsymbol{q}|^4}|\bar{u}^{(\beta)}(p_f)\gamma_0 u^{(\alpha)}(p_i)|^2\delta(E_f - E_i)\right)^2 d\Omega_f \tag{10.289}$$

For elastic scattering, $|\boldsymbol{p}_i| = |\boldsymbol{p}_f| = p_f$ and $EdE = p_f dp_f$, we obtain that the differential cross section is

$$d\sigma_{fi} = \frac{4Z^2\alpha^2m^2}{|\boldsymbol{q}|^4}|\bar{u}^{(\beta)}(p_f)\gamma_0 u^{(\alpha)}(p_i)|^2 d\Omega_f \tag{10.290}$$

For an unpolarized beam we get

$$\left.\frac{d\sigma_{fi}}{d\Omega}\right|_{\text{unpolarized}} = \frac{Z^2\alpha^2}{4|\boldsymbol{p}|^2\beta^2\sin^4(\Theta/2)}\left(1 - \beta^2\sin^2\frac{\Theta}{2}\right) \tag{10.291}$$

where $\beta = v/c$.