

11

Perturbation Theory and Feynman Diagrams

We now turn our attention to interacting quantum field theories. All of the results that we will derive in this section apply equally to both relativistic and non-relativistic theories with only minor changes. Here we will use the path integrals approach we developed in previous chapters.

The properties of any field theory can be understood if the N -point correlation functions are known

$$G_N(x_1, \dots, x_N) = \langle 0 | T \phi(x_1) \dots \phi(x_N) | 0 \rangle \quad (11.1)$$

Much of what we will do below can be adapted to any field theory of interest. We will discuss in detail the simplest case, the relativistic self-interacting scalar field theory. It is straightforward to generalize this to other theories of interest. We will only give a summary of results for the other cases.

11.1 The generating functional in perturbation theory

The N -point function of a scalar field theory,

$$G_N(x_1, \dots, x_N) = \langle 0 | T \phi(x_1) \dots \phi(x_N) | 0 \rangle, \quad (11.2)$$

can be computed from the generating functional $Z[J]$

$$Z[J] = \langle 0 | T e^{i \int d^D x J(x) \phi(x)} | 0 \rangle \quad (11.3)$$

In $D = d + 1$ -dimensional Minkowski space-time $Z[J]$ is given by the path integral

$$Z[J] = \int \mathcal{D}\phi e^{iS[\phi] + i \int d^D x J(x) \phi(x)} \quad (11.4)$$

where the action $S[\phi]$ is the action for a relativistic scalar field. The N -point function, Eq.(11.1), is obtained by functional differentiation, i.e.

$$G_N(x_1, \dots, x_N) = (-i)^N \frac{1}{Z[J]} \frac{\delta^N Z[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0} \quad (11.5)$$

Similarly, the Feynman propagator $G_F(x_1 - x_2)$, which is essentially the 2-point function, is given by

$$G_F(x_1 - x_2) = i \langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle = -i \frac{1}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} \quad (11.6)$$

Thus, in principle, all we need to do is to compute $Z[J]$ and, once it is determined, to use it to compute the N -point functions.

We will derive an expression for $Z[J]$ in the simplest theory, the relativistic real scalar field with a ϕ^4 interaction, but the methods are very general. We will work in Euclidean space-time (i.e. in imaginary time) where the generating function takes the form

$$Z[J] = \int \mathcal{D}\phi e^{-S[\phi] + \int d^D x J(x) \phi(x)} \quad (11.7)$$

where $S[\phi]$ now is

$$S[\phi] = \int d^D x \left[\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \quad (11.8)$$

In the Euclidean theory the N -point functions are

$$G_N(x_1, \dots, x_N) = \langle \phi(x_1) \dots \phi(x_N) \rangle = \frac{1}{Z[J]} \frac{\delta^N Z[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0} \quad (11.9)$$

Let us denote by $Z_0[J]$ the generating action for the free scalar field, with action $S_0[\phi]$. In Chapter 5 we already calculated $Z_0[J]$, and obtained the result (see eq. 5.141 and eq. 5.146)

$$\begin{aligned} Z_0[J] &= \int \mathcal{D}\phi e^{-S_0[\phi] + \int d^D x J(x) \phi(x)} \\ &= [\text{Det}(-\partial^2 + m^2)]^{-1/2} e^{\frac{1}{2} \int d^D x \int d^D y J(x) G_0(x-y) J(y)} \end{aligned} \quad (11.10)$$

where ∂^2 is the Laplacian operator in D -dimensional Euclidean space, and

$G_0(x-y)$ is the free field Euclidean propagator (i.e. the two-point correlation function)

$$G_0(x-y) = \langle \phi(x)\phi(y) \rangle_0 = \langle x | \frac{1}{-\partial^2 + m^2} | y \rangle \quad (11.11)$$

where the sub-index label 0 denotes a free field expectation value.

We can write the full generating function $Z[J]$ in terms of the free field generating function $Z_0[J]$ by noting that the interaction part of the action contributes with a weight of the path-integral that, upon expanding in powers of the coupling constant λ takes the form

$$\begin{aligned} e^{-S_{\text{int}}[\phi]} &= e^{-\int d^D x \frac{\lambda}{4!} \phi^4(x)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D x_1 \dots \int d^D x_n \phi^4(x_1) \dots \phi^4(x_n) \end{aligned} \quad (11.12)$$

Hence, upon an expansion in powers of λ , the generating function $Z[J]$ is

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi e^{-S_0[\phi] + \int d^D x J(x)\phi(x) - S_{\text{int}}[\phi]} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \\ &\quad \times \int d^D x_1 \dots \int d^D x_n \int \mathcal{D}\phi \phi^4(x_1) \dots \phi^4(x_n) e^{-S_0[\phi] + \int d^D x J(x)\phi(x)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D x_1 \dots \int d^D x_n \frac{\delta^4}{\delta J(x_1)^4} \dots \frac{\delta^4}{\delta J(x_n)^4} Z_0[J] \\ &\equiv e^{-\frac{\lambda}{4!} \int d^D x \frac{\delta^4}{\delta J(x)^4}} Z_0[J] \end{aligned} \quad (11.13)$$

where the operator of the last line is defined by its power series expansion. We see that this amounts to the formal replacement

$$S_{\text{int}}[\phi] \mapsto S_{\text{int}}\left(\frac{\delta}{\delta J}\right) \quad (11.14)$$

This expression allows us to write the generating function of the full theory $Z[J]$ in terms of $Z_0[J]$, the generating function of the free field theory,

$$Z[J] = e^{-S_{\text{int}}\left(\frac{\delta}{\delta J}\right)} Z_0[J] \quad (11.15)$$

Notice that this expression holds for *any* theory, not just a ϕ^4 interaction. This result is the starting point of perturbation theory.

Before we embark on explicit calculations in perturbation theory, it is worthwhile to see what assumptions we have made along the way. We assumed, (a) that the fields obey Bose commutation relations, and (b) that the vacuum (or ground state) is non-degenerate.

The restriction to Bose statistics was made at the level of the path integral of the scalar field. This approach is, however, of general validity, and it also applies to theories with Fermi fields, which are path integrals over Grassmann fields. As we saw before, in all cases the generating functional yields vacuum (or ground state) expectation values of time ordered products of fields. The assumption of relativistic invariance is also not essential, although it simplifies the calculations significantly.

The restriction to a non-degenerate vacuum state has more subtle physical consequences. We have mentioned that, in a number of cases, the vacuum may be degenerate if a *global symmetry* is spontaneously broken. Here, the thermodynamic limit plays an essential role. For example, if the vacuum is doubly degenerate, we can do perturbation theory on one of the two vacuum states. If they are related by a global symmetry, the number of orders in perturbation theory which are necessary to have a mixing with its degenerate partner is proportional to the total number of degrees of freedom. Thus, in the thermodynamic limit, they will not mix unless the vacuum is unstable. Such an instability usually shows up in the form of infrared divergent contributions in the perturbation expansion. However, this is not a sickness of the expansion, but the consequence of having an inadequate starting point for the ground state! In general, the “safe procedure” to deal with degeneracies that are due to symmetry is to add an additional term (like a source) to the Lagrangian that breaks the symmetry and to do all the calculations in the presence of such symmetry breaking term. This term should be removed only *after* the thermodynamic limit is taken.

The case of *gauge symmetries* has other and important subtleties. In the case of Maxwell’s Electrodynamics we saw that the ground state is locally gauge invariant and that, as a result, it is unique. It turns out that this is a generic feature of theories that are locally gauge invariant for all symmetry groups and in all dimensions. There is a very powerful theorem (due to S. Elitzur, which will be discussed later) that states that in theories with local gauge invariance, not only the ground state of theories with gauge invariance is unique and gauge-invariant, but that this restriction extends to the entire spectrum of the system. Thus only gauge-invariant operators have non-zero

expectation values, and only such operators can generate the physical states. To put it differently, a local symmetry *cannot* be broken spontaneously even in the thermodynamic limit. The physical reason behind this statement is that if the symmetry is local it takes only a *finite* order in perturbation theory to mix all symmetry related states. Thus, whatever may happen at the boundaries of the system has no consequence on what happens in the interior and the thermodynamic limit does not play a role any longer. This is why we can fix the gauge and remove the enormous *redundancy* of the description of the states.

Nevertheless we have to be very careful about two issues. Firstly, the gauge fixing procedure must select one and only one state from each gauge class. Secondly, the perturbation theory is based on the propagator of the gauge fields, $G_{\mu\nu}(x - x') = -i\langle 0|TA_\mu(x)A_\nu(x')|0\rangle$, which is *not* gauge invariant and, unless a gauge is fixed, it vanishes. If a gauge is fixed, this propagator has contributions that depend on the choice of gauge, but the *poles* of this propagator do *not* depend on the choice of gauge since they describe physical excitations, e.g. photons. Furthermore although the propagator is gauge-dependent, it will only appear in combination with *matter currents* which are *conserved*. Thus, the gauge-dependent terms of the propagator do not contribute to physical processes.

Except for these caveats, we can now proceed to do perturbation theory for all the field theories of interest.

11.2 Perturbative expansion for the two-point function

Let us discuss the perturbative computation of the two-point function in ϕ^4 field theory in D -dimensional Euclidean space-time. We recall that, under a Wick rotation, the analytic continuation to imaginary time $ix_0 \mapsto x_D$, the two-point function in D -dimensional Minkowski space-time, $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ maps onto the correlation function of two fields in D -dimensional Euclidean space-time,

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle \mapsto \langle \phi(x_1)\phi(x_2)\rangle \quad (11.16)$$

Let us formally write the two point function $G^{(2)}(x_1 - x_2)$ as a power series in the coupling constant λ ,

$$G^{(2)}(x_1 - x_2) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} G_n^{(2)}(x_1 - x_2) \quad (11.17)$$

Using the generating functional $Z[J]$ we can write

$$G^{(2)}(x_1 - x_2) = \frac{1}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} \quad (11.18)$$

where

$$Z[J] = e^{-S_{\text{int}}\left(\frac{\delta}{\delta J}\right)} Z_0[J] \quad (11.19)$$

Hence, the two-point function can be expressed as a ratio of two series expansions in powers of the coupling constant. The numerator of Eq.(11.18) is given by an expansion in powers of the coupling constant λ , which can formally be written as

$$\begin{aligned} \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!}\right)^n \\ &\times \int d^D y_1 \dots \int d^D y_n \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \frac{\delta^4}{\delta J(y_1)^4} \dots \frac{\delta^4}{\delta J(y_n)^4} Z_0[J] \Big|_{J=0} \end{aligned} \quad (11.20)$$

Likewise, the denominator of Eq.(11.18) is given by the expansion of $Z[0]$ in powers of λ , which has similar expression but without a contribution due to the external legs, corresponding to the functional derivatives with respect to the source at the external points x_1 and x_2 . The equivalent expansions in Minkowski space-time are obtained, term by term, by the replacement

$$-\lambda \mapsto i\lambda \quad (11.21)$$

at every order in the expansion.

We will now look at the form of the first few terms of the expansion of the two-point function in perturbation theory.

11.2.1 Zeroth order in λ

To zeroth order in λ , $O(\lambda^0)$, the numerator of Eq.(11.18) reduces to

$$\begin{aligned} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} Z[J] \Big|_{J=0} &= \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} Z_0[J] \Big|_{J=0} + O(\lambda) \\ &= G_0(x_1 - x_2) + O(\lambda) \end{aligned} \quad (11.22)$$

while the denominator of Eq.(11.18) is simply equal to one

$$Z[0] = Z_0[0] + O(\lambda) = 1 + O(\lambda) \quad (11.23)$$

where we dropped the contribution of the determinant in $Z_0[0]$ since it does not contribute to the expectation values of the local observables. Hence, we find that to zeroth order the two-point function is simply given by

$$G^{(2)}(x_1 - x_2) = G_0(x_1 - x_2) + O(\lambda) \quad (11.24)$$

11.2.2 First order in λ

To first order in λ , the denominator $Z[0]$ of Eq.(11.18) is given by

$$Z[0] = 1 + \frac{-1}{1!} \left(\frac{\lambda}{4!} \right) \int d^D \quad (11.25)$$

The expression inside the integrand can be calculated from the Taylor expansion of $Z_0[J]$ in powers of $J(x)$. To find a non-zero contribution we need to bring down from the exponent enough factors of J so that they can be cancelled by the functional derivatives. Since the argument of the exponential factor in $Z_0[J]$ is a *bilinear* functional of $J(x)$,

$$Z_0[J] \propto e^{\frac{1}{2} \int d^D x \int d^D y J(x) G_0(x-y) J(y)}, \quad (11.26)$$

only an even number of derivatives in $J(x)$ can be cancelled to a given order. In particular, to first order in λ , we have to cancel four derivatives. This means that we need to expand the exponential in $Z_0[J]$ to second order in its argument to obtain the only non-vanishing contribution to first order in λ to $Z[J]$ at $J = 0$,

$$\begin{aligned} Z[0] &= 1 + \frac{(-1)}{1!} \left(\frac{\lambda}{4!} \right) \\ &\times \int d^D x \frac{\delta^4}{\delta J(x)^4} \frac{1}{2!} \left(\frac{1}{2} \int d^D y_1 \int d^D y_2 J(y_1) G_0(y_1 - y_2) J(y_2) \right)^2 \Big|_{J=0} \\ &+ O(\lambda^2) \end{aligned} \quad (11.27)$$

The derivatives yield a set of delta-functions

$$\frac{\delta^4}{\delta J(x)^4} [J(y_1) \cdots J(y_4)] \Big|_{J=0} = \sum_P \prod_{j=1}^4 \delta(y_{pj} - x) \quad (11.28)$$

where P runs over the $4!$ permutations of the arguments y_1, y_2, y_3 and y_4 .

We can now write the first order correction to $Z[0]$, in the form

$$\begin{aligned}
 Z[0] &= 1 + \left(\frac{-1}{1!}\right) \left(\frac{\lambda}{4!}\right) \frac{1}{2!} \left(\frac{1}{2}\right)^2 \\
 &\quad \times \int d^D x \int d^D y_1 \dots d^D y_4 G_0(y_1 - y_2) G_0(y_3 - y_4) \sum_P \prod_{j=1}^4 \delta(y_{pj} - x) \\
 &\quad + O(\lambda^2) \\
 &= 1 + \frac{(-1)}{1!} \left(\frac{\lambda}{4!}\right) S \int d^D x G_0(x, x) G_0(x, x) + O(\lambda^2) \tag{11.29}
 \end{aligned}$$

where $S = 3$ is the multiplicity factor of this diagram, which counts the number of times this picture is obtained by contracting all the lines in a pairwise fashion without changing the topology of the graph.

It is useful to introduce a picture or *diagram* to represent this contribution. Let us mark four points y_1, \dots, y_4 and an additional point at x (which we will call a *vertex*) with four legs coming out of it. Let us join y_1 and y_2 by a line and y_3 with y_4 by another line. To each line we assign a factor of $G_0(y_1 - y_2)$ and $G_0(y_3 - y_4)$ respectively,

$$G_0(x - y) = \begin{array}{c} x \qquad \qquad \qquad y \\ \hline \end{array} \tag{11.30}$$

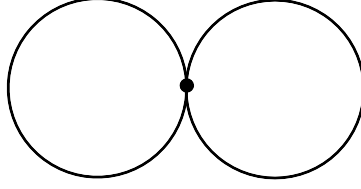
Next, because of the delta-functions, we have to identify each of the points y_1, \dots, y_4 with each one of the legs attached to y in all possible ways. The resulting expression has to be integrated over all values of the the coordinates and of x . The result is

$$Z[0] = 1 - \frac{\lambda}{8} \int d^D x (G_0(x, x))^2 + O(\lambda^2) \tag{11.31}$$

Physically, the first order contribution represents corrections to the ground state energy due to *vacuum fluctuations*. This expression can be represented more simply by the *Feynman diagram* shown in Fig.11.1.

Here, and below, we denote by a full line the bare propagator $G_0(x - y)$.

Let us now compute the first order corrections to $\left. \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0}$. We

Figure 11.1 Vacuum fluctuations: first order correction to $Z[0]$.

obtain

$$\begin{aligned}
 & \frac{(-1)}{1!} \frac{\lambda}{4!} \int d^D y \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \frac{\delta^4}{\delta J(y)^4} Z[J] \Big|_{J=0} = \\
 & = \frac{(-1)}{1!} \frac{\lambda}{4!} \int d^D y \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \frac{\delta^4}{\delta J(y)^4} \\
 & \quad \times \frac{1}{3!} \left(\frac{1}{2} \int d^D z_1 \int d^D z_2 J(z_1) G_0(z_1 - z_2) J(z_2) \right)^3 \Big|_{J=0} \quad (11.32)
 \end{aligned}$$

The non-vanishing contributions are obtained by matching the derivatives in Eq.(11.32) with an equal number of powers of J . We see that we have six factors of the source J at points z_1, \dots, z_6 and six derivatives, one at x_1 and at x_2 , and four at y . To match derivatives with powers amounts to find all possible pair-wise contractions of these two sets of points. Once the delta functions have acted we are left with just one integral over the position y of the internal vertex. Hence, the result amounts to finding all possible contractions among the external legs at x_1 and x_2 , with each other and/or with the internal vertex at y . Notice that for each contraction we get a power of the bare (unperturbed) propagator G_0 .

The only non-vanishing terms resulting from this process are represented by the Feynman diagrams of Fig.11.2.

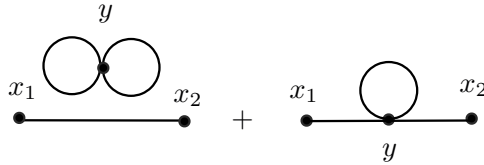


Figure 11.2 First order contribution to the two-point function.

The first contribution, Fig.11.2(a), is the product of the bare propagator $G_0(x_1 - x_2)$ between the external points and the first order correction of the

vacuum diagrams:

$$(a) = G_0(x_1 - x_2) \times \left[-\left(\frac{\lambda}{8}\right) \int d^D y \left(G_0(y, y)\right)^2 \right] \quad (11.33)$$

The second term, the “tadpole” diagram of Fig.11.2(b), is given by the expression

$$(b) = -\left(\frac{\lambda}{4!}\right) S \int d^D y G_0(x_1, y)G_0(y, y)G_0(y, x_2) \quad (11.34)$$

where the multiplicity factor is $S = 4 \times 3$. It counts the number of ways (12) of contracting the external points to the internal vertex: there are four different ways of contracting one external point to one of the four lines attached to the internal vertex at y , and three different ways of contracting the remaining external point to the one of the three remaining lines of the internal vertex. There is only one way to contract the two leftover internal lines attached to the vertex at y .

By collecting terms we get the result shown in Fig. 11.3.

$$G^{(2)} = \frac{\text{---} + \text{---}\bigcirc\bigcirc\text{---} + \text{---}\bigcirc\text{---}}{1 + \bigcirc\bigcirc}$$

Figure 11.3 Diagrams for the two-point function to first order in λ .

To first order in λ , the expansion of the two-point function can also be written in the form shown in Fig. 11.4.

$$G^{(2)} = \frac{(1 + \bigcirc\bigcirc + O(\lambda^2)) \left(\text{---} + \frac{\bigcirc}{\text{---}} + O(\lambda^2) \right)}{(1 + \bigcirc\bigcirc + O(\lambda^2))}$$

Figure 11.4 Factorization of Feynman diagrams for the two-point function to first order in λ .

At least at this order in perturbation theory, a number of diagrams that contribute to the expansion of the numerator get exactly cancelled by the expansion of the denominator, the vacuum diagrams. The diagrams that get

cancelled are unlinked in the sense that one can split a diagram in two by drawing a line that does not cut any of the propagator lines. These diagrams contain a factor consisting of terms of the vacuum diagrams. We will see shortly that this is a feature of this expansion to all orders in λ .

Thus, to first order in λ , the two-point function is given by (see figure)

$$G^{(2)}(x_1, x_2) = G_0^{(2)}(x_1, x_2) - \frac{\lambda}{2} \int d^D y G_0^{(2)}(x_1, y) G_0^{(2)}(y, y) G_0^{(2)}(y, x_2) + O(\lambda^2) \quad (11.35)$$

as shown in Fig.11.5.

$$G^{(2)} = \text{---} + \text{---} \circ \text{---} + \dots$$

Figure 11.5 The two-point function to first order in λ .

11.3 Cancellation of the vacuum diagrams

The cancellation of the vacuum diagrams is a general feature of perturbation theory. Let us reexamine this issue in more general terms. We will give the arguments for the case of the two-point function, but they are trivial to generalize to any N -point function. This feature also holds in all theories, relativistic or not, bosonic or fermionic, provided the fields satisfy local canonical commutation (or anti-commutation) relations.

The expansion of the two point function has the form

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{1}{Z[0]} \sum_{n=0}^{\infty} \int d^D y_1 \dots d^D y_n \frac{(-1)^n}{n!} \langle \phi(x_1) \phi(x_2) \prod_{j=1}^n \mathcal{L}_{\text{int}}(\phi(y_j)) \rangle_0 \quad (11.36)$$

The denominator factor $Z[0]$ has a similar expansion

$$Z[0] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^D y_1 \dots d^D y_n \langle \prod_{j=1}^n \mathcal{L}_{\text{int}}(\phi(y_j)) \rangle_0 \quad (11.37)$$

where $\langle \mathcal{O}[\phi] \rangle_0$ denotes the expectation value of the operator $\mathcal{O}[\phi]$ in free field theory.

Let us consider first the numerator. Each expectation value involves a sum of products of pairwise contractions. If we assign a Feynman diagram to each contribution, it is clear that we can classify these terms into two classes: (a) linked and (b) unlinked diagrams. A diagram is said to be *unlinked* if it

contains a *sub-diagram* in which a set of internal vertices are linked with each other but not to an external vertex. The *linked* diagrams satisfy the opposite property. Since the vacuum diagrams by definition do not contain any external vertices, they are unlinked.

All the expectation values that appear in the numerator can be written as a sum of terms, each of the form of a linked diagram times a vacuum graph, i.e.

$$\begin{aligned} \langle \phi(x_1)\phi(x_2)\mathcal{L}_{\text{int}}(\phi(y_1))\dots\mathcal{L}_{\text{int}}(\phi(y_n)) \rangle_0 &= \\ &= \sum_{k=0}^n \binom{n}{k} \langle \phi(x_1)\phi(x_2) \prod_{j=1}^k \mathcal{L}_{\text{int}}(\phi(y_j)) \rangle_0^\ell \langle \prod_{j=k+1}^n \mathcal{L}_{\text{int}}(\phi(y_j)) \rangle_0 \end{aligned} \quad (11.38)$$

where the super-index ℓ denotes a *linked* factor, i.e. a factor that does not contain any vacuum sub-diagram. Thus, the numerator has the form

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^n}{n!} \binom{n}{k} \langle \phi(x_1)\phi(x_2) \prod_{j=1}^k \mathcal{L}_{\text{int}}(y_k) \rangle_0^\ell \langle \prod_{j=k+1}^n \mathcal{L}_{\text{int}}(\phi(y_j)) \rangle_0 \quad (11.39)$$

which factorizes into

$$\begin{aligned} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int d^D y_1 \dots d^D y_k \langle \phi(x_1)\phi(x_2) \prod_{j=1}^k \mathcal{L}_{\text{int}}(\phi(y_j)) \rangle_0^\ell \right) \\ \times \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^D y_1 \dots d^D y_n \langle \prod_{j=1}^n \mathcal{L}_{\text{int}}(\phi(y_j)) \rangle_0 \right) \end{aligned} \quad (11.40)$$

We can clearly recognize that the second factor is *exactly* equal to the denominator $Z[0]$.

Hence we find that we can write the two-point function as a sum of linked Feynman diagrams:

$$\langle \phi(x_1)\phi(x_2) \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^D y_1 \dots d^D y_n \langle \phi(x_1)\phi(x_2) \prod_{j=1}^n \mathcal{L}_{\text{int}}(\phi(y_n)) \rangle_0^\ell \quad (11.41)$$

This result is known as the *linked-cluster theorem*. This theorem, which proves that the vacuum diagrams cancel exactly out to all orders in perturbation theory, is valid for all the N -point functions (not just for two-point function) and for any local theory. It also holds in Minkowski space-time upon the replacement $(-1)^n \leftrightarrow i^n$. It holds for all theories with a local canonical structure, relativistic or not.

11.4 Summary of Feynman rules for ϕ^4 theory

11.4.1 Position space

The general rules to construct the diagrams for the N -point function $\langle 0|T\phi(x_1)\dots\phi(x_N)|0\rangle$ in ϕ^4 theory in Minkowski space and $\langle\phi(x_1)\dots\phi(x_N)\rangle$ in Euclidean space, in *position space* are

- 1 A general graph for the N -point function has N external points and n internal interaction vertices, where n is the order in perturbation theory. Each vertex is a point with a coordinate label and 4 lines (for a ϕ^4 theory) coming out of it.
- 2 Draw all topologically distinct graphs obtained by connecting the external points and the internal vertices in all possible ways. Discard all graphs that contain sub-diagrams not linked to at least one external point.
- 3 The following weight is assigned to each graph:
 - 1 For every vertex a factor of $-i\frac{\lambda}{4!}$ in Minkowski space and $-\frac{\lambda}{4!}$ in Euclidean space .
 - 2 For every line connecting a pair of points z_1 and z_2 , a propagator factor of $\langle 0|T\phi(z_1)\phi(z_2)|0\rangle_0 = -iG_0^{(2)}(z_1, z_2)$ in Minkowski space, or $\langle\phi(z_1)\phi(z_2)\rangle_0 = G_0(z_1 - z_2)$ in Euclidean space.
 - 3 An overall factor of $\frac{1}{n!}$.
 - 4 A multiplicity factor that counts the number of ways in which the lines can be joined without changing the topology of the graph.
 - 5 Integrate over all internal coordinates $\{z_i\}$.

For example, the 4-point function

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \quad (11.42)$$

has the two contributions at order λ^2 shown in the diagram of figure 11.6. These two diagrams have exactly the same weight (if $G_0^{(2)}(1, 2) = G_0^{(2)}(2, 1)$),

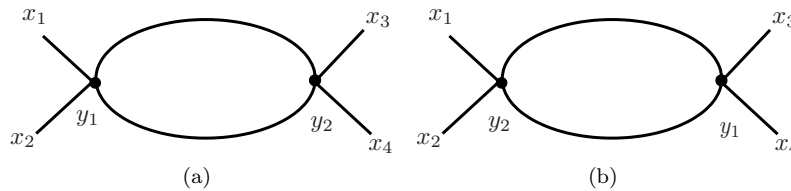


Figure 11.6 Two topologically distinct contributions to the four-point function to order λ^2 that have the same analytic expression.

and their total contribution to the 4-point function is

$$\frac{1}{2!} \left(\frac{-\lambda}{4!} \right)^2 S \int d^D y_1 \int d^D y_2 G_0^{(2)}(x_1, y_1) G_0^{(2)}(x_2, y_1) \\ \times \left[G_0^{(2)}(y_1, y_2) \right]^2 G_0^{(2)}(x_3, y_2) G_0^{(2)}(x_4, y_2) \quad (11.43)$$

but are *topologically distinct* and, thus, count as separate contributions. The multiplicity factor is $S = (4 \times 3)^2 \times 2 \times 2$.

11.4.2 Momentum space

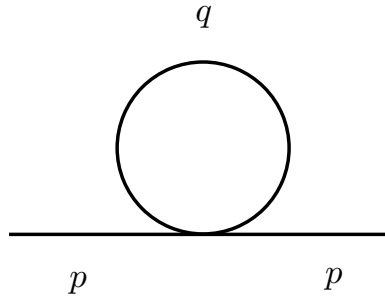


Figure 11.7 First order contribution to the two-point function in momentum space.

The N -point functions can also be computed in momentum space. The rules for constructing Feynman diagrams in momentum space are:

- 1 A graph has N external legs, labelled by a set of external momenta k_1, \dots, k_N , flowing into the diagram and n internal vertices (for order n in perturbation theory). Each vertex has 4 lines (for ϕ^4 theory) each carrying a momentum q_1, \dots, q_4 (flowing out of the vertex). All lines must be connected in pairs. All vacuum terms have to be discarded.
- 2 Draw all the topologically different graphs.
- 3 Weight of each diagram:
 - for each vertex, a factor of $(\frac{-\lambda}{4!})(2\pi)^D \delta^D(\sum_{i=1}^4 q_i)$.
 - each line carries a momentum p_μ and contributes to the weight with a factor $G_0^{(2)}(p) = \frac{1}{p^2 + m^2}$, in Euclidean space. In Minkowski space it becomes $-iG_0^{(2)} = \frac{-i}{p^2 - m^2 + i\epsilon}$.
 - all the numerical factors are the same as in position space.
 - we must integrate over all the internal momenta.

For example, the first order contribution to the two-point function is the tadpole diagram shown in Fig.11.7. It has the algebraic weight

$$\left(\frac{-\lambda}{4!}\right) \frac{1}{1!} (4 \times 3) \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \left(\frac{1}{p^2 + m^2}\right)^2 \quad (11.44)$$

11.5 Feynman rules for theories with fermions and gauge fields

The Feynman rules that we derived for ϕ^4 theory can be easily extended to other theories, relativistic or not. An important difference occurs in theories with fermions in which the rules need to be modified to account to the Fermi-Dirac statistics. As an example we will consider the case of quantum electrodynamics (QED), the theory of electrons and photons. Similar rules are formulated in quantum chromodynamics (QCD) that we will discuss in a later chapter. Feynman rules for self-interacting fermions are also devised in the case of the Gross-Neveu model (or the Nambu-Jona-Lasinio model) and in the case of the non-relativistic electron gas.

The Lagrangian density of QED, in the Feynman-'t Hooft gauges, is

$$\mathcal{L}_{\text{QED}} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \quad (11.45)$$

The partition function in the presence of separate sources $\bar{\eta}(x)$ for the Dirac (spinor) field $\psi(x)$ (and $\eta(x)$ for the adjoint Dirac field $\bar{\psi}(x)$), and $J_\mu(x)$ for the electromagnetic gauge field $A_\mu(x)$, is

$$Z[\eta, \bar{\eta}, J_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{i \int d^4 x [\mathcal{L}_{\text{QED}} + J_\mu(x) A^\mu(x) + \bar{\eta}(x) \psi(x) + \bar{\psi}(x) \eta(x)]} \quad (11.46)$$

where we dropped the Faddeev-Popov determinant which is trivial in this abelian gauge theory (see section 9.5). In what follows we drop the spinor labels to simplify the notation.

As in the case of ϕ^4 theory, the partition function $Z[\eta, \bar{\eta}, J_\mu]$ of Eq.(11.46) can be expressed in terms of the partition function of a free field theory with sources. In this case we have two partition functions of free field theories: the partition function for free Dirac fields with sources,

$$Z_{\text{Dirac}}[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int d^4 x (\bar{\psi} (i \not{D} - m) \psi + \bar{\eta} \psi + \bar{\psi} \eta) \right] \quad (11.47)$$

and the partition function of the free Maxwell field, in the Feynman-'t Hooft

gauges, with sources J_μ given by

$$Z_{\text{Maxwell}}[J_\mu] = \int \mathcal{D}A_\mu \exp \left[i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right) + J_\mu A^\mu \right] \quad (11.48)$$

where, here too, we dropped the Faddeev-Popov determinant.

In QED the interaction term of the Lagrangian is

$$\mathcal{L}_{\text{int}}[\psi, \bar{\psi}, A_\mu] = e A_\mu \bar{\psi} \gamma^\mu \psi \quad (11.49)$$

Then, the full generating function of eq.(11.46) is expressed as

$$Z[\eta, \bar{\eta}, J_\mu] = \exp \left(i \int d^4x \mathcal{L}_{\text{int}} \left[-i \frac{\delta}{\delta \bar{\eta}}, -i \frac{\delta}{\delta \eta}, -i \frac{\delta}{\delta J_\mu} \right] \right) Z_{\text{Dirac}}[\eta, \bar{\eta}] Z_{\text{Maxwell}}[J_\mu] \quad (11.50)$$

where we made the replacements

$$\psi(x) \rightarrow -i \frac{\delta}{\delta \bar{\eta}(x)}, \quad \bar{\psi}(x) \rightarrow -i \frac{\delta}{\delta \eta(x)}, \quad A_\mu(x) \rightarrow -i \frac{\delta}{\delta J^\mu(x)} \quad (11.51)$$

inside the interaction term of the lagrangian.

The Feynman rules then follow by the same line of argument that we used in ϕ^4 theory. The main differences are that

1. Now we have two propagators: the propagator of the free massive Dirac field, $S_F(x-y) = -i \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle$, which was given explicitly in section 7.2, and the propagator of the Maxwell field (in the Feynman-'t Hooft gauges) $G_{\mu\nu}(x-y, \alpha) = i \langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle$, given explicitly in section 9.6.
2. The interaction vertex involves two Dirac fermions and a gauge field, shown in Fig. 11.8. Each bare vertex has a weight in a Feynman diagram of $e\gamma_\mu$.
3. Each closed Dirac fermion loop contributes with a minus sign, and each diagram now has a weight $(-1)^F$, where F is the number of fermion loops. This rule follows from the form of the path integral for fermions (see, chapter 7). It applies to all theories of fermions, relativistic or not.

Aside from these changes the Feynman rules are the same.

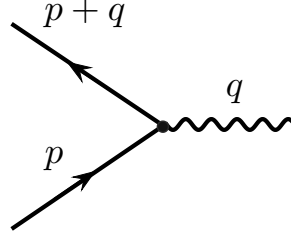


Figure 11.8 The bare interaction vertex of QED: the full lines are Dirac fermion propagators and the wavy line is a photon propagator. The bare vertex has a weight $e\gamma_\mu$.

11.6 The two-point function and the self-energy in ϕ^4 theory

We now return to ϕ^4 theory. To first order in λ , and in momentum space, the two point function in Euclidean space is

$$G^{(2)}(p) = \frac{1}{p^2 + m^2} + \frac{-\lambda}{4!}(4 \times 3) \left(\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \right) \left(\frac{1}{p^2 + m^2} \right)^2 + O(\lambda^2) \quad (11.52)$$

Let us define by μ^2 the *effective* or *renormalized mass* (squared) such that

$$\frac{1}{p^2 + \mu^2} = \frac{1}{p^2 + m^2} \left\{ 1 - \frac{\lambda}{2} \left(\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \right) \left(\frac{1}{p^2 + m^2} \right) + O(\lambda^2) \right\} \quad (11.53)$$

Again, to first order in λ , we can write the equivalent expression

$$G^{(2)}(p) = \frac{1}{p^2 + m^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2}} + O(\lambda^2) \quad (11.54)$$

This expression leads us to define μ^2 to be

$$\mu^2 = m^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} + \dots \quad (11.55)$$

This equation is equivalent to a sum of a large number of diagrams with higher order in λ . How do we know that it is consistent? Let us first note that we have summed diagrams of the form shown in figure 11.9. These diagrams have the very special feature that it is possible to split the diagram into two sub-diagrams by cutting only a *single* internal line. Momentum conservation requires that the momentum of that line be *equal* to the momentum on the incoming external leg. Thus, once again we have two types of diagrams: (a)

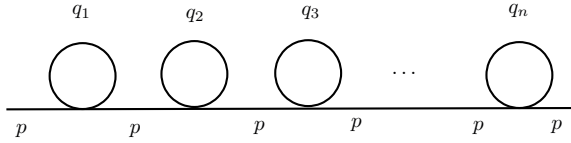


Figure 11.9 The set of all one-particle reducible diagrams of the two-point function to leading order in λ .

one-particle *reducible* diagrams (which satisfy the property defined above) and (b) the one-particle *irreducible* graphs that do not. Hence, the total contribution to the two-point function is the solution of the equation shown in Fig 11.10.

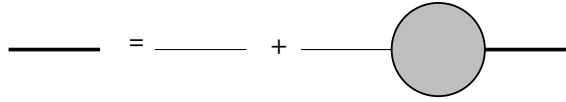


Figure 11.10 The Dyson equation for the two-point function.

Here the thick lines are the *full* propagator, the thin line is the *bare* propagator, and the *shaded blob* represents *all* the irreducible diagrams, i.e. diagrams with amputated external legs. We represent the blob by the *self-energy* operator $\Sigma(p)$, shown in Fig.11.11



Figure 11.11 Feynman diagrams summed by the Dyson equation.

Thus, the total sum satisfies the Dyson equation

$$G^{(2)}(p) = G_0^{(2)}(p) + G_0^{(2)}(p) \Sigma(p) G^{(2)}(p) \quad (11.56)$$

The inverse of $G^{(2)}(p)$, $\Gamma^{(2)}(p)$, satisfies

$$\Gamma^{(2)}(p) = G_0^{(2)}(p)^{-1} - \Sigma(p) = p^2 + m^2 - \Sigma(p) \quad (11.57)$$

To first order in λ , $\Sigma(p)$ is just the tadpole term

$$\Sigma(p) = -\frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} + O(\lambda^2) \quad (11.58)$$

which, to this order in λ , happens to be independent of the external momentum p_μ . Of course, the higher order terms in general will be functions of p_μ .

In terms of the renormalized mass μ^2 , to order one-loop (i.e. $O(\lambda)$) we get

$$\mu^2 = m^2 - \Sigma(p) = m^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} + O(\lambda^2) \quad (11.59)$$

Thus, we conclude that, to $O(\lambda)$, vacuum fluctuations renormalize the mass. However, a quick look at Eq.(11.59) reveals that this is very large renormalization. Indeed, fluctuations *of all momenta*, ranging from long wave-lengths (and low energies) with $q \sim 0$, to short wave-lengths (or high energies) contribute to the mass renormalization. In fact, the high-energy fluctuations, with $q^2 \gg m^2$, yield the *largest* contributions to Eq.(11.59), since the mass effectively cuts-off the contributions in the infrared, IR, $q \rightarrow 0$. Moreover, for all dimensions $D \geq 2$ the high-energy (or ultraviolet, UV, $q \rightarrow \infty$) contribution is *divergent*. If we were to cutoff the integral at a high-momentum scale Λ , in general space-time dimension D the contribution from this diagram will diverge as Λ^{D-2} . In particular, the tadpole contribution to the mass renormalization is *logarithmically divergent* for $D = 2$ (i.e. $1 + 1$) dimensions, and *quadratically divergent* for $D = 4$ dimensions.

Thus, although it is consistent (and quite physical) to regard the leading effect of fluctuations as a mass renormalization, they amount to a divergent change. The reason for this divergence is that all wavelengths contribute, from the IR to the UV. This happens because space-time is continuous, and we assumed that there is no intrinsic short-distance scale below which local field theory should not be valid.

There is a way to think about this problem. The problem of how to understand the physics of these singular contributions, indeed of how the *continuum limit* (a theory without cutoff) of quantum field theory is defined is the central purpose of the Renormalization Group (RG). We will study this approach in detail in a later chapter. Here we will discuss some qualitative features. From the point of view of the RG the problem is that the continuum theory (i.e. defining a theory without a UV cutoff) cannot be defined naively. We will see that for such a procedure to work it is necessary to be able to define the theory in a regime in which there is no scale, i.e. in a scale-invariant regime. This requirement means that one should look at a regime in which the renormalized mass becomes arbitrarily small, $\mu^2 \rightarrow 0$. As we will see below, this requires to fine tune the *bare coupling constant* and the *bare mass* to some determined *critical values*. It turns out that,

near such a *critical point* a continuum field theory (without a UV cutoff) can be defined. The RG point of view relates the problem of the *definition* of a Quantum Field Theory to that of finding a *continuous phase transition*, which a central problem in Statistical Physics.

However, there are alternative descriptions, such as String Theory, that postulate that local field theory is not the correct description at short distances, typically near the *Planck scale*, $\ell_{\text{Planck}} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}$ cm (!), where G is Newton's gravitational constant. From this view point, these singular contributions at high energies signal a breakdown of the theory at those scales.

Before we try to compute $\Sigma(p)$, it is worth to mention the *Hartree Approximation*. It consists in summing up all tadpole diagrams (and only the tadpole diagrams) to all orders in λ . A typical graph is shown in figure 11.12.

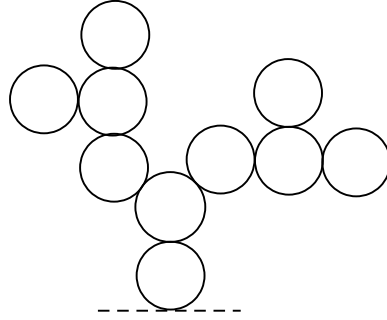


Figure 11.12 A typical tree diagram.

The sum of all the tadpole diagrams can be done by means of a very simple trick. Let us modify the expression for the self-energy to make it self consistent, i.e.

$$\Sigma_0(p) = -\frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2 - \Sigma_0(q)} \tag{11.60}$$

This formula is equivalent to a Dyson equation in which the internal propagator is replaced by the full propagator, as in figure 11.15. This approximation becomes exact for a theory of an N -component real scalar field $\phi_a(x)$ ($a = 1, \dots, N$), with $O(N)$ symmetry, and interaction (shown in Fig.11.13)

$$\mathcal{L}_{\text{int}}[\phi] = \frac{\lambda}{4!} ((\phi)^2)^2 = \frac{\lambda}{4!} \left(\sum_{a=1}^N \phi_a(x) \phi_a(x) \right)^2 \tag{11.61}$$

in the large N limit, $N \rightarrow \infty$. Otherwise, the solution of this integral equa-

tion just yields the leading correction. The reason why this works can be seen

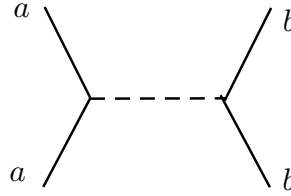


Figure 11.13 The vertex for the real scalar field with $O(N)$ symmetry; here $a, b = 1, \dots, N$.

by a direct counting of the powers of N in the expansion of the two-point function to order one-loop, as shown in Fig.11.14.

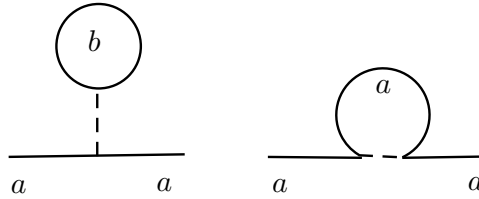


Figure 11.14 The 2-point function to one-loop order in the $O(N)$ theory; here $a, b = 1, \dots, N$. The diagram on the left is of order N while the diagram on the right is of order 1.

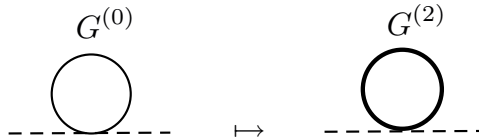


Figure 11.15 The self-energy in the one-loop (Hartree) approximation.

Eq.(11.60) is an integral non-linear equation for $\Sigma_0(p)$. Equations of this type are common in many-body physics. For example, the gap equation of the BCS theory of superconductivity has a similar form (although in BCS theory one sums ladder diagrams, instead of bubble diagrams.)

Let us now evaluate the integral in the equation for $\Sigma_0(p)$, Eq.(11.60). Clearly $\Sigma_0(p)$ is a correction due to *virtual* fluctuations with momenta q_μ ranging from zero to infinity. These “off-shell” fluctuations do *not* obey the mass shell condition $p^2 = m^2$. Notice that, at this level of approximation, $\Sigma(p)$ is *independent of the momentum*. This is only correct to order one-loop.

Before computing the integral, let us rewrite Eq.(11.60) in terms of the effective or renormalized mass μ^2 ,

$$\mu^2 = m^2 - \Sigma(p) = m^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \mu^2} \quad (11.62)$$

Let us denote by m_c^2 the value of the bare mass such that the renormalized mass vanishes, $\mu^2 = 0$:

$$0 = m_c^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2} \quad (11.63)$$

Clearly, m_c^2 is IR divergent for $D \leq 2$ and UV divergent for $D \geq 2$. Let us now express the renormalized mass μ^2 in terms of m_c^2 , and define $\delta m^2 = m^2 - m_c^2$. We find,

$$\begin{aligned} \mu^2 &= \delta m^2 + m_c^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \mu^2} \\ &= \delta m^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \left(\frac{1}{q^2 + \mu^2} - \frac{1}{q^2} \right) \\ &= \delta m^2 - \mu^2 \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q^2 + \mu^2)} \end{aligned} \quad (11.64)$$

In $D = 4$ dimensions the integral contributes with a logarithmic divergence (*c.f.* Eq.(11.99) of section 11.8):

$$\mu^2 = \delta m^2 - \frac{\lambda}{2} \mu^2 \left(\frac{1}{8\pi^2} \ln \left(\frac{\Lambda}{\mu} \right) - \frac{\gamma}{16\pi^2} \right) \quad (11.65)$$

Thus, although the stronger, quadratic divergence, was absorbed by a renormalization of the mass, a weaker logarithmic singularity remains. It turns out, as we will discuss in a later chapter, that this remaining singular contribution can be absorbed only by a renormalization of the coupling constant λ . In any case, what is clear is that, already at the lowest order in perturbation theory, the leading corrections can (and do) yield a larger contribution to the behavior of physical quantities than the bare, unperturbed, values, and that these corrections are not small.

We will end with a discussion of what these large perturbative corrections mean in the context of the theory of phase transitions since, after all, it is also described by a theory with the same form. We saw before that in the Landau theory of phase transitions the mass (squared) parameter, the coefficient of the ϕ^2 term in the action, is related to the difference $m^2 \equiv T - T_0$, i.e. the

temperature T measured from the mean field critical temperature, T_0 . We can think of $m_c^2 \equiv T_c - T_0$, defining a corrected critical temperature, T_c ,

$$T_c = T_0 - \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2} \quad (11.66)$$

It shows that fluctuations suppress T_c *downwards* from T_0 due to (precisely) the large contributions from fluctuations at short distances. Thus, the UV singular effects can be absorbed in a new (and lower) T_c . The *subtracted* mass, $\delta m^2 \equiv T - T_c$, is now the new control parameter, as shown in the last line of Eq.(11.64).

Returning to the mass renormalization of Eq.(11.64), we can use the integral of Eq.(11.98) (of section 11.8), to find the result

$$\mu^2 = \delta m^2 - \lambda \frac{(\mu^2)^{\frac{D}{2}-1} \Gamma\left(2 - \frac{D}{2}\right)}{(4\pi)^{D/2} (D-2)} \quad (11.67)$$

Since $\delta m^2 = T - T_c$, we can write this result in the suggestive form

$$\delta m^2 = T - T_c = \mu^2 + \left[\frac{\lambda}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{D-2} \right] (\mu^2)^{\frac{D}{2}-1} \quad (11.68)$$

We will now look at the IR behavior, where the renormalized mass $\mu^2 \rightarrow 0$. We see that if $D > 4$, the exponent of μ^2 in the second term of the r.h.s. of Eq.(11.68) is always larger than 1 and hence this contribution is small compared to the first term. However, for $D < 4$ the reverse is true and for μ^2 small enough the fluctuations dominate. hence, perturbation theory breaks down for some small value of μ^2 where the two contributions are comparable. Hence perturbation theory is reliable (in the IR) for

$$\delta T \gtrsim \mu^2 \approx \left(\frac{\lambda}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{D-2} \right)^{2/(4-D)} \quad (11.69)$$

and breaks down for δT (or μ^2) smaller than that value. This is known as the Ginzburg Criterion which gives an estimate of the width of the critical region: the range temperatures (or renormalized mass) for which the perturbation theory breaks down in the IR and does not describe the true behavior.

Let us recall that the relation between the susceptibility and the effective (or renormalized) mass: $\mu^2 = \chi^{-1}$. Hence, at one-loop order, we would predict

that

$$\chi(T) \propto \begin{cases} (T - T_c)^{-2/(D-2)} & : D < 4 \\ (T - T_c)^{-1} & : D > 4 \\ (T - T_c)^{-1} \times \text{small logarithmic corrections} & : D = 4 \end{cases} \quad (11.70)$$

We see that one effect of these fluctuations can be to change the dependence of a physical quantity, such as the susceptibility, on the control parameter, $T - T_c$, which sets how close the theory is the *massless* or *critical* regime. A key purpose of the renormalization group program is the prediction of *critical exponents* such as the one we found in Eq.(11.70). We will see that this result is actually the beginning of a set of controlled approximations to the exact values. Just as important will be the fact that the renormalization group will give a deeper interpretation of the meaning of renormalization.

11.7 The four point function and the effective coupling constant

We will now discuss briefly the perturbative contributions to the four-point function,

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \quad (11.71)$$

which is also known as the “two-particle” correlation function. We will discuss its connection with the effective (or renormalized) coupling constant.

To zeroth order in perturbation theory, $O(\lambda^0)$, the four point function factorizes into a product of all (three) possible two point functions obtained by pair-wise contractions of the four field operators (shown in Fig.11.16)

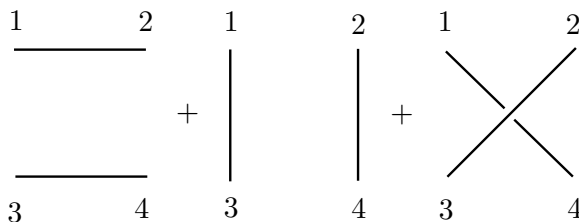


Figure 11.16 The four point function $G^{(4)}(1, 2, 3, 4)$ to order $O(\lambda^0)$.

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) = \\ = G_0(x_1, x_3)G_0(x_2, x_4) + G_0(x_1, x_2)G_0(x_3, x_4) + G_0(x_1, x_4)G_0(x_2, x_3) + O(\lambda) \end{aligned} \quad (11.72)$$

As it is apparent, to zeroth order in λ , the four point function reduces to just products of bare two-point functions and hence nothing new is learned from it. We will show in the next chapter, Eq.(12.40), that, to all orders on perturbation theory, the four-point function has the following structure:

$$\begin{aligned}
 G^{(4)}(x_1, x_2, x_3, x_4) = & G^{(2)}(x_1, x_3)G^{(2)}(x_2, x_4) \\
 & + G^{(2)}(x_1, x_2)G^{(2)}(x_3, x_4) + G^{(2)}(x_1, x_4)G^{(2)}(x_2, x_3) \\
 - \int d^D y_1 \dots \int d^D y_4 & G^{(2)}(x_1, y_1)G^{(2)}(x_2, y_2)G^{(2)}(x_3, y_3)G^{(2)}(x_4, y_4) \\
 & \times \Gamma^{(4)}(y_1, y_2, y_3, y_4)
 \end{aligned} \tag{11.73}$$

where the factors of $G^{(2)}(x, x')$ represent the *exact two-point function*, and the new four-point function, $\Gamma^{(4)}(y_1, y_2, y_3, y_4)$, is known as the *four-point vertex function*. The vertex function is defined as the set of one-particle irreducible (1PI) Feynman diagrams, i.e. diagrams that cannot be split in two by cutting a single propagator line, with the external lines “amputated” (they are already accounted for in the propagator factors).

In momentum space, due to momentum conservation at the vertex, $\Gamma^{(4)}$ has the form

$$\Gamma^{(4)}(p_1, \dots, p_4) = (2\pi)^D \delta^D \left(\sum_{i=1}^4 p_i \right) \overline{\Gamma^{(4)}}(p_1, \dots, p_4) \tag{11.74}$$

The *lowest order* contribution to $\Gamma^{(4)}(y_1, y_2, y_3, y_4)$ appears at order λ

$$\Gamma^{(4)}(y_1, y_2, y_3, y_4) = \lambda + O(\lambda^2) \tag{11.75}$$

depicted by the *tree level* diagrams of Fig.11.17. In momentum space the

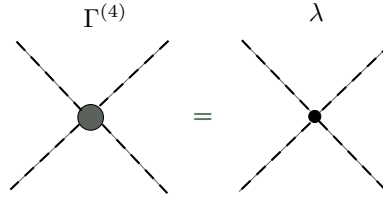


Figure 11.17 The one-particle irreducible four-point (vertex) function at the tree level.

one-particle irreducible four-point function is

$$\overline{\Gamma^{(4)}}(p_1, \dots, p_4) = \lambda + O(\lambda^2) \tag{11.76}$$

To one-loop order, $O(\lambda^2)$, the four-point vertex function is a sum of (three) Feynman diagrams of the form of Fig.11.18. The total contribution to the

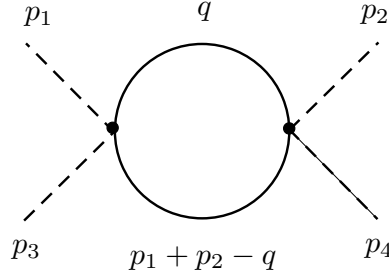


Figure 11.18 The one-loop contribution to the one-particle irreducible four-point (vertex) function.

vertex function $\overline{\Gamma^{(4)}}$, to order one-loop, is

$$\begin{aligned} \overline{\Gamma^{(4)}}(p_1, \dots, p_4) = \\ \lambda - \frac{\lambda^2}{2} \left\{ \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)((p_1 + p_2 - q)^2 + m^2)} + \text{two permutations} \right\} \\ + O(\lambda^3) \end{aligned} \quad (11.77)$$

This expression has a logarithmic UV divergence in $D = 4$, and more severe divergencies for $D > 4$. To address this problem let us proceed by analogy with the mass renormalization and define the physical or *renormalized* coupling constant g by the value of $\overline{\Gamma^{(4)}}(p_1, \dots, p_4)$ at zero external momenta, $p_1 = \dots = p_4 = 0$. (It is up to us to define it at any momentum scale we wish.)

$$g \equiv \lim_{p_i \rightarrow 0} \overline{\Gamma^{(4)}}(p_1, \dots, p_4) = \overline{\Gamma^{(4)}}(0, \dots, 0) \quad (11.78)$$

This definition is convenient and simple but it is problematic if the renormalized mass μ^2 vanishes (i.e. in the massless or critical theory). To order one-loop, the renormalized coupling constant g is

$$g = \lambda - 3 \frac{\lambda^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)^2} + O(\lambda^3) \quad (11.79)$$

Using the same line of argument we used to define the self energy, Σ , we will now sum all the one loop diagrams, as shown in Fig.11.19. This “bubble” sum is a geometric series, and it is equivalent to the replacement

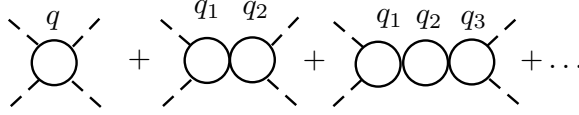


Figure 11.19 The sum of bubble diagrams.

of Eq.(11.79) by

$$g = \lambda - 3\frac{g^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)^2} + O(\lambda^3) \quad (11.80)$$

or, alternatively, to write the bare coupling constant λ in terms of the renormalized coupling g as

$$\lambda = g + \frac{3}{2} g^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + \mu^2)^2} + O(g^3) \quad (11.81)$$

where we have replaced the bare mass m^2 with the renormalized mass μ^2 . This amounts to add the insertions of tadpole diagrams in the internal propagators. This is consistent at this order in perturbation theory.

Written in terms of the renormalized coupling constant g and of the renormalized mass μ^2 , the vertex function becomes

$$\begin{aligned} \overline{\Gamma^{(4)}}(p_1, \dots, p_4) &= \\ &= g - \frac{g^2}{2} \int \frac{d^D q}{(2\pi)^D} \left[\frac{1}{(q^2 + \mu^2)((p_1 + p_2 - q)^2 + \mu^2)} - \frac{1}{(q^2 + \mu^2)^2} \right] \\ &\quad + \text{two permutations} + O(g^3) \end{aligned} \quad (11.82)$$

which is UV finite for $D < 6$. Thus, the renormalization of the coupling constant leads to a subtraction of the singular expression for the vertex function.

After the renormalization of the coupling constant, the singular behavior of the integral now appears only in the relation between the bare coupling constant λ and the renormalized coupling constant g , given in Eq.(11.81). Clearly there are a number of ways to interpret the meaning of this relation. One interpretation is to say that at a fixed value of the bare coupling constant λ , Eq.(11.81) relates the regulator Λ (which is a momentum scale) and the renormalized coupling constant. In other terms, the effective of renormalized coupling constant g has become a function of a momentum (or energy) scale!. Conversely, we can fix the renormalized coupling and ask how do we

have to change the bare coupling constant λ as we send the regulator to infinity.

It will be useful to work with dimensionless quantities. Since the bare and the renormalized coupling constants have units of Λ^{4-D} , we define the *dimensionless* bare coupling constant u by

$$\lambda = \Lambda^{4-D} u \quad (11.83)$$

Then

$$\begin{aligned} u &= \Lambda^{D-4} \left(g + \frac{3}{2} g^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2)^2} + O(g^3) \right) \\ &= \Lambda^{D-4} \left(g + \frac{1}{(4\pi)^{D/2}} \frac{6}{(D-2)(D-4)} \Lambda^{D-4} g^2 + O(g^3) \right) \end{aligned} \quad (11.84)$$

We will vary the momentum scale Λ and the dimensionless bare coupling constant u at fixed g . The differential change of the dimensionless coupling constant is known as the renormalization group *beta-function*,

$$\beta(u) = -\Lambda \left. \frac{\partial u}{\partial \Lambda} \right|_g \quad (11.85)$$

where I have used the sign convention standard in Statistical Physics (opposite to that in High Energy Physics!). Hence (for $D \rightarrow 4$)

$$\beta(u) = (4-D)u - \frac{3}{16\pi^2} u^2 + O(u^3) \quad (11.86)$$

The behavior of the Renormalization group beta-function (or Gell-Mann-Low) for $D < 4$, $D = 4$ and $D > 4$ is shown in Fig.11.20. The infrared (IR) RG *flows*, i.e. the flow of the dimensionless coupling constant u as the momentum scale λ is *decreased*, is also shown in Fig.11.20 (bottom). The UV flows, i.e. the flow of u as the momentum scale is *increased*, is obtained by reversing the direction of the RG flows.

Clearly for $D \geq 4$ the dimensionless coupling constant flows to 0 in the infrared (at low energies and long distances). That means that in the IR regime, $\Lambda \rightarrow 0$, the theory becomes *weakly coupled*, and perturbation theory becomes reliable in that regime. However, at short distances (or at high energies), $\Lambda \rightarrow \infty$, the opposite happens: the dimensionless coupling becomes large and perturbation theory breaks down. Four dimensions is special in that the approach and departure from the decoupled limit, $u = 0$, is very slow, and leads to logarithmic corrections to the free field values.

However, for $D < 4$ something new happens: there is a *non-trivial fixed point* at u^* where $\beta(u^*) = 0$. *At the fixed point*, the coupling constant does

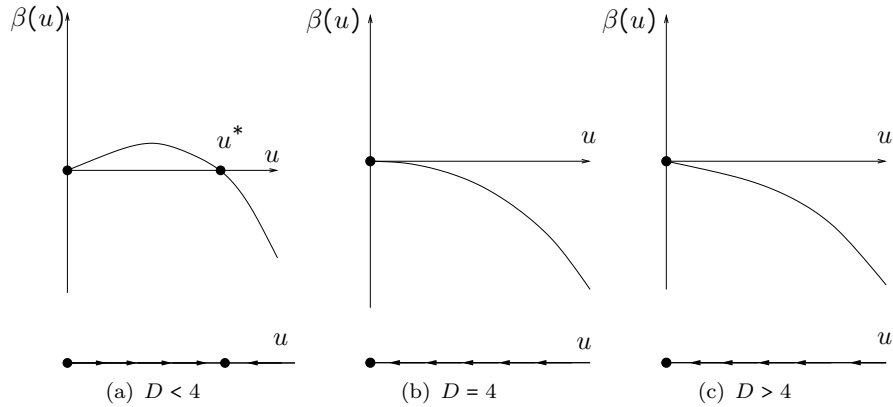


Figure 11.20 The beta-function (top) and the IR RG flows (bottom) for $D < 4$, $D = 4$ and $D > 4$. The UV RG flows are the reverse.

not flow as the momentum scale changes. Hence, *at a fixed point* it is possible to send the momentum scale $\Lambda \rightarrow \infty$ and effectively have a theory without a cutoff. Notice that even infinitesimally away from the fixed point the UV flows are unstable. Also, and for the same reason, at the fixed point it is also possible to go into the deep infrared regime and have a theory with a finite coupling constant u^* . It turns out that this behavior is central for the theory of phase transitions. We will come back to the problem of the renormalization group in another chapter where we will develop it in detail, and discuss its application to different theories.

11.8 One-loop Integrals

Here we will sketch the computation of one-loop integrals (in Euclidean space-time). We introduce a momentum cutoff Λ and to suppress the contributions at large momenta, $q \gg \Lambda$ of integrals of the form

$$I_D \left(\frac{\mu^2}{\Lambda^2} \right) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \mu^2} e^{-\frac{q^2}{\Lambda^2}} \quad (11.87)$$

where we used a Gaussian cutoff function (or *regulator*). We will only be interested in the regime $\mu^2 \ll \Lambda^2$. Using a Feynman-Schwinger parametriza-

tion we can write

$$\begin{aligned}
 I_D\left(\frac{\mu^2}{\Lambda^2}\right) &= \int_0^\infty d\alpha \int \frac{d^D q}{(2\pi)^D} e^{-\frac{q^2}{\Lambda^2} - \alpha(q^2 + \mu^2)} \\
 &= \int_0^\infty d\alpha e^{-\alpha\mu^2} \int \frac{d^D q}{(2\pi)^D} e^{-\left(\frac{1}{\Lambda^2} + \alpha\right)q^2} \\
 &= e^{\frac{\mu^2}{\Lambda^2}} \frac{(\mu^2)^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \int_{\mu^2/\Lambda^2}^\infty dt t^{-\frac{D}{2}} e^{-t}
 \end{aligned} \tag{11.88}$$

Hence

$$I_D\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{(\mu^2)^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}, \frac{\mu^2}{\Lambda^2}\right) e^{\frac{\mu^2}{\Lambda^2}} \tag{11.89}$$

where $\Gamma(\nu, z)$ is the incomplete gamma function, with $z = \frac{\mu^2}{\Lambda^2}$ and $\nu = 1 - \frac{D}{2}$,

$$\Gamma(\nu, z) = \int_z^\infty dt t^{\nu-1} e^{-t} \tag{11.90}$$

and $\Gamma(\nu, 0) = \Gamma(\nu)$ is the Gamma function

$$\Gamma(\nu) = \int_0^\infty dt t^{\nu-1} e^{-t} \tag{11.91}$$

If the regulator Λ is removed (i.e. if we take the limit $\Lambda \rightarrow \infty$), $I_D(\mu^2)$ formally becomes:

$$I_D(\mu^2) = \frac{(\mu^2)^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \tag{11.92}$$

For general D , $\Gamma(1 - D/2)$ is a meromorphic function of the complex variable D , and has simple poles for $\nu = 0$ or any *negative* integer, $\Gamma(1 - D/2)$ has poles for $D = 2, 4, 6, \dots$

In $D = 4$ dimensions, $\nu = -1$ where $\Gamma(\nu)$ has a pole, the incomplete Gamma function at $\nu = -1$ is (as $z \rightarrow 0$)

$$\Gamma(-1, z) = \left(\frac{1}{z} - \ln \frac{1}{z}\right) e^{-z} + \gamma \tag{11.93}$$

where γ is the Euler-Mascheroni constant

$$\gamma = - \int_0^\infty dt e^{-t} \ln t = 0.5772\dots \tag{11.94}$$

Hence, for $\mu^2 \ll \Lambda^2$, I_4 is

$$I_4\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{\Lambda^2}{16\pi^2} - \frac{\mu^2}{8\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) + \frac{\gamma}{16\pi^2} \mu^2 \quad (11.95)$$

Here we see that the leading singularity is quadratic in the regulator Λ , with a sub-leading logarithmic piece.

In two dimensions I_2 has instead a logarithmic singularity for $\mu^2 \ll \Lambda^2$

$$\begin{aligned} I_2\left(\frac{\mu^2}{\Lambda^2}\right) &= \frac{1}{4\pi} \Gamma\left(0, \frac{\mu^2}{\Lambda^2}\right) \\ &= \frac{1}{2\pi} \ln\left(\frac{\Lambda}{\mu}\right) - \frac{\gamma}{4\pi} \end{aligned} \quad (11.96)$$

A second integral of interest is

$$\begin{aligned} J_D\left(\frac{\mu^2}{\Lambda^2}\right) &= \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q^2 + \mu^2)} \\ &= \frac{1}{\mu^2} \left(I_D(0) - I_D\left(\frac{\mu^2}{\Lambda^2}\right) \right) \end{aligned} \quad (11.97)$$

$J_D(\mu^2/\Lambda^2)$ is UV finite if $D < 4$, where it is given by

$$J_D(\mu^2) = \frac{(\mu^2)^{\frac{D}{2}-2}}{(4\pi)^{D/2}} \frac{\Gamma\left(\frac{D}{2} - 2\right)}{\frac{D}{2} - 1} \quad (11.98)$$

In four dimensions, J_4 , has a logarithmic divergence

$$J_4\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{1}{8\pi^2} \ln\left(\frac{\Lambda}{\mu}\right) - \frac{\gamma}{16\pi^2} \quad (11.99)$$

A third useful integral is

$$I'_D\left(\frac{\mu^2}{\Lambda^2}\right) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + \mu^2)^2} e^{-\frac{q^2}{\Lambda^2}} = -\frac{\partial I_D}{\partial \mu^2} \quad (11.100)$$

In the massless limit it becomes

$$I'_D(0) = \frac{1}{(4\pi)^{D/2}} \frac{4}{(D-2)(D-4)} \Lambda^{D-4} \quad (11.101)$$

In four dimensions it becomes

$$I_4' \left(\frac{\mu^2}{\Lambda^2} \right) = \frac{1}{8\pi^2} \ln \left(\frac{\Lambda}{\mu} \right) - \left(\frac{\gamma + 1}{16\pi^2} \right) \quad (11.102)$$