3

Classical Symmetries and Conservation Laws

We have used the existence of symmetries in a physical system as a guiding principle for the construction of their Lagrangians and energy functionals. We will show now that these symmetries imply the existence of conservation laws.

There are different types of symmetries which, roughly, can be classified into two classes: (a) spacetime symmetries and (b) internal symmetries. Some symmetries involve discrete operations, hence called *discrete* symmetries, while others are *continuous* symmetries. Furthermore, in some theories these are *global symmetries*, while in others they are *local symmetries*. The latter class of symmetries go under the name of *gauge symmetries*. We will see that, in the fully quantized theory, global and local symmetries play different roles.

Spacetime symmetries are the most common examples of symmetries that are encountered in Physics. They include translation invariance and rotation invariance. If the system is isolated, then time-translation is also a symmetry. A *non-relativistic* system is in general invariant under *Galilean* transformations, while *relativistic* systems, are instead *Lorentz* invariant. Other spacetime symmetries include *time-reversal* (T), *parity* (P) and *charge conjugation* (C). These symmetries are *discrete*.

In classical mechanics, the existence of symmetries has important consequences. Thus, *translation invariance*, which is a consequence of uniformity of space, implies the *conservation* of the *total momentum* $\boldsymbol{P}$ of the system. Similarly, *isotropy* implies the conservation of the *total angular momentum* $\boldsymbol{L}$ and *time translation invariance* implies the conservation of the *total energy* E.

All of these concepts have analogs in field theory. However, in field theory new symmetries will also appear which do not have an analog in the classical mechanics of particles. These are the *internal symmetries*, that will be discussed below in detail.

3.1 Continuous symmetries and Noether's theorem

We will show now that the existence of continuous symmetries has very profound implications, such as the existence of *conservation laws*. One important feature of these conservation laws is the existence of *locally conserved currents*. This is the content of the following theorem, due to Emmy Noether.

Noether's theorem: *For every continuous global symmetry there exists a global conservation law.*

Before we prove this statement, let us discuss the connection that exists between *locally conserved currents* and *constants of motion*. In particular, let us show that for every locally conserved current there exist a globally conserved quantity, i.e. a constant of motion. To this effect, let $j^\mu(x)$ be some locally conserved current, i.e. $j_\mu(x)$ satisfies the local constraint

$$\partial_\mu j^\mu(x) = 0 \tag{3.1}$$

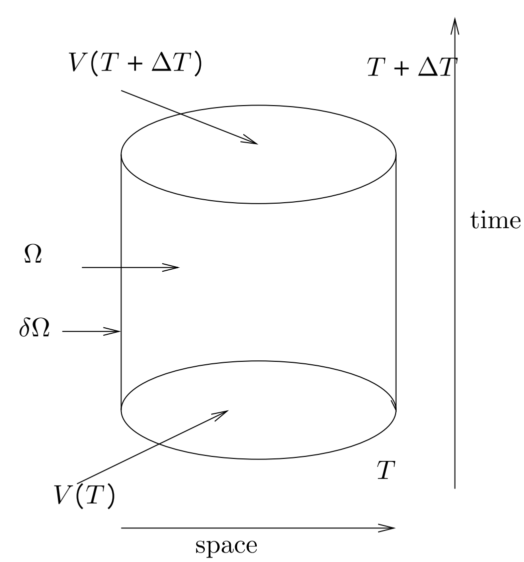

Figure 3.1 A spacetime 4-volume.

Let Ω be a bounded 4-volume of spacetime, with boundary $\partial\Omega$. Then, the

Divergence (Gauss) theorem tells us that

$$0 = \int_{\Omega} d^4x\, \partial_\mu j^\mu(x) = \oint_{\partial\Omega} dS_\mu j^\mu(x) \tag{3.2}$$

where the r.h.s. is a surface integral on the *oriented closed surface* $\partial\Omega$ (a 3-volume). Let us suppose that the 4-volume Ω extends all the way to infinity in space and has a finite extent in time ΔT.

If there are no currents at *spacial* infinity, i.e. $\lim_{|\boldsymbol{x}|\to\infty} j^\mu(\boldsymbol{x}, x_0) = 0$, then only the top (at time $T + \Delta T$) and the bottom (at time T) of the boundary $\partial\Omega$ (shown in Fig. (3.1)) will contribute to the surface (boundary) integral. Hence, the r.h.s. of Eq.(3.2) becomes

$$0 = \int_{V(T+\Delta T)} dS_0\, j^0(\boldsymbol{x}, T + \Delta T) - \int_{V(T)} dS_0\, j^0(\boldsymbol{x}, T) \tag{3.3}$$

Since $dS_0 \equiv d^3x$, the boundary contributions reduce to two oriented 3-volume integrals

$$0 = \int_{V(T+\Delta T)} d^3x\, j^0(\boldsymbol{x}, T + \Delta T) - \int_{V(T)} d^3x\, j^0(\boldsymbol{x}, T) \tag{3.4}$$

Thus, the quantity $Q(T)$

$$Q(T) \equiv \int_{V(T)} d^3x\, j^0(\boldsymbol{x}, T) \tag{3.5}$$

is a *constant of motion*, i.e.

$$Q(T + \Delta T) = Q(T) \qquad \forall\ \Delta T \tag{3.6}$$

Hence, the existence of a locally conserved current, satisfying $\partial_\mu j^\mu = 0$, implies the existence of a globally conserved *charge* (or Noether charge) $Q = \int d^3x\, j^0(\boldsymbol{x}, T)$, which is a *constant of motion*. Thus, the proof of the Noether theorem reduces to proving the existence of a locally conserved current. In the following sections we will prove Noether's theorem for internal and spacetime symmetries.

3.2 Internal symmetries

Let us begin, for simplicity, with the case of the *complex scalar field* $\phi(x) \neq \phi^*(x)$. The arguments that follow below are easily generalized to other cases. Let $\mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*)$ be the Lagrangian density. We will assume that the Lagrangian is invariant under the *continuous global* internal symmetry transformation

$$\phi(x) \mapsto \phi'(x) = e^{i\alpha}\phi(x)$$
$$\phi^*(x) \mapsto \phi'^*(x) = e^{-i\alpha}\phi^*(x) \tag{3.7}$$

where α is an arbitrary real *number* (not a function!). The system is *invariant* under the transformation of Eq.(3.7) if the Lagrangian $\mathcal{L}$ satisfies

$$\mathcal{L}(\phi', \partial_\mu\phi', \phi'^*, \partial_\mu\phi'^*) \equiv \mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*) \tag{3.8}$$

Then we say that the transformation shown in Eq.(3.7) is a global symmetry of the system.

In particular, for an infinitesimal transformation we have

$$\phi'(x) = \phi(x) + \delta\phi(x) + \ldots, \qquad \phi'^*(x) = \phi^*(x) + \delta\phi^*(x) + \ldots \tag{3.9}$$

where $\delta\phi(x) = i\alpha\phi(x)$. Since $\mathcal{L}$ is *invariant*, its variation must be identically equal to zero. The variation $\delta\mathcal{L}$ is

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\partial_\mu\phi^* \tag{3.10}$$

Using the equation of motion

$$\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\right) = 0 \tag{3.11}$$

and its complex conjugate, we can write the variation $\delta\mathcal{L}$ in the form of a total divergence

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\phi^*\right] \tag{3.12}$$

Thus, since $\delta\phi = i\alpha\phi$ and $\delta\phi^* = -i\alpha\phi^*$, we get

$$\delta\mathcal{L} = \partial_\mu\left[\; i\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\phi^*\right)\alpha\right] \tag{3.13}$$

Hence, since α is arbitrary, $\delta\mathcal{L}$ will vanish identically if and only if the 4-vector j^μ, defined by

$$j^\mu = i\Big(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\phi - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\phi^*\Big) \tag{3.14}$$

is locally conserved, i.e.

$$\delta\mathcal{L} = 0 \qquad \text{iff} \qquad \partial_\mu j^\mu = 0 \tag{3.15}$$

In particular, if $\mathcal{L}$ has the form

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - V(|\phi|^2) \tag{3.16}$$

which is manifestly invariant under the symmetry transformation of Eq.(3.7), we see that the current j^μ is given by

$$j^\mu = i\left(\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi\right) \equiv i\phi^* \overleftrightarrow{\partial}_\mu \phi \qquad (3.17)$$

Thus, the presence of a continuous internal symmetry implies the existence of a locally conserved current.

Furthermore, the conserved charge $\mathcal{Q}$ is given by

$$\mathcal{Q} = \int d^3x\, j^0(\boldsymbol{x}, x_0) = \int d^3x\, i\phi^* \overleftrightarrow{\partial}_0 \phi \qquad (3.18)$$

In terms of the canonical momentum $\Pi(x)$, the globally conserved charge $\mathcal{Q}$ of the charged scalar field is

$$\mathcal{Q} = \int d^3x\, i(\phi^* \Pi - \phi \Pi^*) \qquad (3.19)$$

3.3 Global symmetries and group representations

Let us generalize the result of the last subsection. Let us consider a scalar field ϕ^a which transforms irreducibly under a certain representation of a Lie group G. In the case considered in the previous section the group G is the group of complex numbers of unit length, the group $U(1)$. The elements of this group, $g \in U(1)$, have the form $g = e^{i\alpha}$.

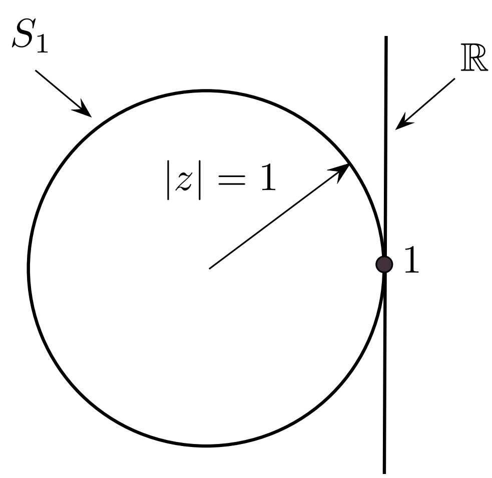

Figure 3.2 The $U(1)$ group is isomorphic to the unit circle while the real numbers $\mathbb{R}$ are isomorphic to a tangent line.

This set of complex numbers forms a group in the sense that, 1 . It is closed under complex multiplication, i.e.

$$g = e^{i\alpha} \in U(1) \quad \text{and} \quad g' = e^{i\beta} \in U(1) \Rightarrow g * g' = e^{i(\alpha+\beta)} \in U(1). \tag{3.20}$$

2 . There is an identity element, i.e. $g = 1$.

3 . For every element $g = e^{i\alpha} \in U(1)$ there is a unique inverse element $g^{-1} = e^{-i\alpha} \in U(1)$.

The elements of the group $U(1)$ are in one-to-one correspondence with the points of the unit circle S_1. Consequently, the parameter α that labels the transformation (or element of this group) is defined modulo 2π, and it should be restricted to the interval $(0, 2\pi]$. On the other hand, transformations infinitesimally close to the identity element, 1, lie essentially on the line tangent to the circle at 1 and are isomorphic to the group of real numbers $\mathbb{R}$. The group $U(1)$ is compact, in the sense that the length of the natural parametrization of its elements is 2π, which is finite. In contrast, the group $\mathbb{R}$ of real numbers is not compact (see Fig. (3.2)). In the sequel, we will almost always work with internal symmetries with a compact Lie group.

For infinitesimal transformations the groups $U(1)$ and $\mathbb{R}$ are essentially identical. There are, however, field configurations for which they are not. A typical case is the *vortex* configuration in two dimensions. For a vortex the phase of the field on a large circle of radius $R \to \infty$ winds by 2π. Such configurations would not exist if the symmetry group was $\mathbb{R}$ instead of $U(1)$ (note that analyticity requires that $\phi \to 0$ as $x \to 0$.)

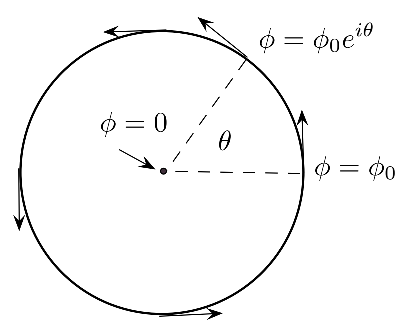

Figure 3.3 A vortex

Another example is the N-component *real* scalar field $\phi^a(x)$, with $a = 1, \ldots, N$. In this case the symmetry is the group of *rotations* in N-dimensional Euclidean space

$$\phi'^a(x) = R^{ab} \phi^b(x) \tag{3.21}$$

The field ϕ^a is said to transform like the N-dimensional (vector) representation of the Orthogonal group $O(N)$.

The elements of the orthogonal group, $R \in O(N)$, satisfy

1 . If $R_1 \in O(N)$ and $R_2 \in O(N)$, then $R_1 R_2 \in O(N)$,
2 . $\exists I \in O(N)$ such that $\forall R \in O(N)$ then $RI = IR = R$,
3 . $\forall R \in O(N)$ $\exists R^{-1} \in O(N)$ such that $R^{-1} = R^t$,

where R^t is the transpose of the matrix R.

Similarly, if the N-component vector $\phi^a(x)$ is a *complex field*, it transforms under the group of $N \times N$ *Unitary* transformations U

$$\phi'^a(x) = U^{ab} \phi^b(x) \tag{3.22}$$

The complex $N \times N$ matrices U are elements of the Unitary group $U(N)$ and satisfy

1 . $U_1 \in U(N)$ and $U_2 \in U(N)$, then $U_1 U_2 \in U(N)$,
2 .$\exists I \in U(N)$ such that $\forall U \in U(N)$, $UI = IU = U$,
3 . $\forall U \in U(N)$, $\exists U^{-1} \in U(N)$ such that $U^{-1} = U^\dagger$, where $U^\dagger = (U^t)^*$.

In the particular case discussed above ϕ^a transforms like the fundamental (spinor) representation of $U(N)$. If we impose the further restriction that $|\det U| = 1$, the group becomes $SU(N)$. For instance, if $N = 2$, the group is $SU(2)$ and

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{3.23}$$

it transforms like the spin-1/2 representation of $SU(2)$.

In general, for an arbitrary continuous Lie group G, the field transforms like

$$\phi'_a(x) = \left(\exp\left[i\lambda^k \theta^k\right]\right)_{ab} \phi_b(x) \tag{3.24}$$

where the vector $\boldsymbol{\theta}$ is arbitrary and constant (i.e. independent of x). The *matrices* λ^k are a set of $N \times N$ linearly independent matrices which span the *algebra* of the Lie group G. For a given Lie group G, the number of such matrices is $D(G)$, and it is independent of the *dimension* N of the representation that was chosen. $D(G)$ is called the *rank* of the group. The matrices λ^k_{ab} are the *generators* of the group in this representation.

In general, from a symmetry point of view, the field ϕ does not have to be a vector, as it can also be a tensor or for the matter transform under any representation of the group. For simplicity, we will only consider the case of the vector representation of $O(N)$, and the fundamental (spinor) and adjoint (vector) (see below) representations of $SU(N)$.

For an arbitrary compact Lie group G, the generators $\{\lambda^j\}$, $j = 1, \dots, D(G)$, are a set of hermitean, $\lambda_j^\dagger = \lambda_j$, traceless matrices, $\text{tr}\lambda_j = 0$, which obey the commutation relations

$$[\lambda^j, \lambda^k] = i f^{jkl} \lambda^l \tag{3.25}$$

The numerical constants f^{jkl} are known as the *structure constants* of the *Lie group* and are the same in all its representations. In addition, the generators have to be normalized. It is standard to require the normalization condition

$$\text{tr}\lambda^a \lambda^b = \frac{1}{2}\delta^{ab} \tag{3.26}$$

In the case considered above, the complex scalar field $\phi(x)$, the symmetry group is the group of unit length complex numbers of the form $e^{i\alpha}$. This group is known as the group $U(1)$. All its representations are one-dimensional, and has only one generator.

A commonly used group is $SU(2)$. This group, which is familiar from non-relativistic quantum mechanics, has three generators J_1, J_2 and J_3, that obey the angular momentum algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{3.27}$$

with

$$\text{tr}(J_i J_j) = \frac{1}{2}\delta_{ij} \quad \text{and} \quad \text{tr}J_i = 0 \tag{3.28}$$

The representations of $SU(2)$ are labelled by the angular momentum quantum number J. Each representation J is a $2J+1$-fold degenerate multiplet, i.e. the dimension of the representation is $2J+1$.

The lowest non-trivial representation of $SU(2)$, i.e. $J \neq 0$, is the spinor representation which has $J = \frac{1}{2}$ and is two-dimensional. In this representation, the field $\phi_a(x)$ is a two-component complex spinor and the generators J_1, J_2 and J_3 are given by the set of 2×2 Pauli matrices $J_j = \frac{1}{2}\sigma_j$.

The vector, or spin 1, representation is three-dimensional, and ϕ_a is a three-component vector. In this representation, the generators are very simple

$$(J_j)_{kl} = \epsilon_{jkl} \tag{3.29}$$

Notice that the dimension of this representation (3) is the same as the rank (3) of the group $SU(2)$. In this representation, known as the *adjoint* representation, the matrix elements of the generators are the structure constants. This is a general property of all Lie groups. In particular, for the group $SU(N)$, whose rank is $N^2 - 1$, it has $N^2 - 1$ infinitesimal generators, and the dimension of its adjoint (vector) representation is $N^2 - 1$. For instance, for $SU(3)$ the number of generators is eight.

Another important case is the group of rotations of N-dimensional Euclidean space, $O(N)$. In this case, the group has $N(N-1)/2$ generators which can be labelled by the matrices L^{ij} $(i, j = 1, \ldots, N)$. The fundamental (vector) representation of $O(N)$ is N-dimensional and, in this representation, the generators are

$$(L^{ij})_{kl} = -i(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}) \tag{3.30}$$

It is easy to see that the L^{ij}'s generate infinitesimal rotations in N-dimensional space.

Quite generally, in a given representation an element of a Lie group is labelled by a set of Euler angles denoted by $\boldsymbol{\theta}$. If the Euler angles $\boldsymbol{\theta}$ are infinitesimal, then the representation matrix $\exp(i\boldsymbol{\lambda}\cdot\boldsymbol{\theta})$ is close to the identity, and can be expanded in powers of $\boldsymbol{\theta}$. To leading order in $\boldsymbol{\theta}$ the change in ϕ^a is

$$\delta\phi^a(x) = i(\boldsymbol{\lambda}\cdot\boldsymbol{\theta})^{ab}\phi^b(x) + \ldots \tag{3.31}$$

If ϕ_a is real, the conserved current j^μ is

$$j_\mu^k(x) = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^a(x)}\,\lambda_{ab}^k\,\phi_b(x) \tag{3.32}$$

where $k = 1, \ldots, D(G)$. Here, the generators λ^k are real hermitean matrices. In contrast, for a complex field ϕ_a, the conserved currents are

$$j_\mu^k(x) = i\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^a(x)}\,\lambda_{ab}^k\,\phi_b(x) - \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^a(x)^*}\,\lambda_{ab}^k\,\phi_b(x)^*\right) \tag{3.33}$$

Here the generators λ^k are hermitean matrices (but are not all real).

Thus, we conclude that *the number of conserved currents is equal to the number of generators of the group.* For the particular choice

$$\mathcal{L} = (\partial_\mu\phi_a)^*(\partial^\mu\phi_a) - V(\phi_a^*\,\phi_a) \tag{3.34}$$

the conserved current is

$$j_\mu^k = i\,\lambda_{ab}^k\,\phi_a^*\overleftrightarrow{\partial_\mu}\phi_b \tag{3.35}$$

and the conserved charges are

$$Q^k = \int_V d^3x\; i\lambda_{ab}^k\,\phi_a^*\overleftrightarrow{\partial_0}\phi_b \tag{3.36}$$

where V is the volume of space.

3.4 Global and local symmetries: gauge invariance

The existence of global symmetries assumes that, at least in principle, we can measure and change all of the components of a field $\phi^a(x)$ at all points $\boldsymbol{x}$ in space at the same time. Relativistic invariance tells us that, although the theory may posses this global symmetry, in principle this experiment cannot be carried out. One is then led to consider theories which are invariant if the symmetry operations are performed *locally.* Namely, we should require that the Lagrangian be invariant under *local transformations*

$$\phi_a(x) \to \phi'_a(x) = \left(\exp\left[i\lambda^k \theta^k(x)\ \right]\right)_{ab} \phi_b(x) \tag{3.37}$$

For instance, we can demand that the theory of a complex scalar field $\phi(x)$ be invariant under local changes of phase

$$\phi(x) \to \phi'(x) = e^{i\theta(x)}\phi(x) \tag{3.38}$$

The standard local Lagrangian $\mathcal{L}$

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - V(|\phi|^2) \tag{3.39}$$

is invariant under *global* transformations with θ = constant, but it is not invariant under arbitrary smooth *local* transformations $\theta(x)$. The main problem is that since the derivative of the field does not transform like the field itself, the kinetic energy term is no longer invariant. Indeed, under a local transformation, we find

$$\partial_\mu \phi(x) \to \partial_\mu \phi'(x) = \partial_\mu \left[e^{i\theta(x)}\phi(x)\right] = e^{i\theta(x)}\left[\partial_\mu \phi + i\phi\partial_\mu \theta\right] \tag{3.40}$$

In order to make $\mathcal{L}$ *locally* invariant we must find a new *derivative operator* D_μ, the *covariant derivative*, which transforms in the same way as the field $\phi(x)$ under local phase transformations, i.e.

$$D_\mu \phi \to D'_\mu \phi' = e^{i\theta(x)} D_\mu \phi \tag{3.41}$$

From a "geometric" point of view we can picture the situation as follows. In order to define the phase of $\phi(x)$ locally, we have to define a local frame, or fiducial field, with respect to which the phase of the field is measured. Local gauge invariance is then the statement that the physical properties of the system must be independent of the particular choice of frame. From this point of view, local gauge invariance is an extension of the principle of relativity to the case of internal symmetries.

Now, if we wish to make phase transformations that differ from point to point in spacetime, we have to specify how the phase changes as we go from one point x in spacetime to another one y. In other words, we have to define a *connection* that will tell us how to *parallel transport* the phase of ϕ from x to y as we travel along some path Γ. Let us consider the situation in which x and y are arbitrarily close to each other, i.e. $y_\mu = x_\mu + dx_\mu$ where dx_μ is an infinitesimal 4-vector. The change in ϕ is

$$\phi(x + dx) - \phi(x) = \delta\phi(x) \tag{3.42}$$

If the *transport* of ϕ along some path going from x to $x + dx$ is to *correspond* to a *phase transformation*, then $\delta\phi$ must be proportional to ϕ. So we are led to *define*

$$\delta\phi(x) = iA_\mu(x)dx^\mu\phi(x) \tag{3.43}$$

where $A_\mu(x)$ is a suitably chosen *vector field*. Clearly, this implies that the *covariant derivative* D_μ must be defined to be

$$D_\mu\phi \equiv \partial_\mu\phi(x) - ieA_\mu(x)\phi(x) \equiv \left(\partial_\mu - ieA_\mu\right)\phi \tag{3.44}$$

where e is a parameter which we will give the physical interpretation of a coupling constant.

How should $A_\mu(x)$ transform? We must choose its transformation law in such a way that $D_\mu\phi$ transforms like $\phi(x)$ itself. Thus, if $\phi \to e^{i\theta}\phi$ we have

$$D'\phi' = (\partial_\mu - ieA'_\mu)(e^{i\theta}\phi) \equiv e^{i\theta}D_\mu\phi \tag{3.45}$$

This requirement can be met if

$$i\partial_\mu\theta - ieA'_\mu = -ieA_\mu \tag{3.46}$$

Hence, A_μ should transform like

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta \tag{3.47}$$

But this is nothing but a gauge transformation! Indeed, if we define the gauge transformation $\Phi(x)$

$$\Phi(x) \equiv \frac{1}{e}\theta(x) \tag{3.48}$$

we see that the vector field A_μ transforms like the vector potential of Maxwell's electromagnetism.

We conclude that we can promote a global symmetry to a local (i.e. *gauge*) symmetry by replacing the derivative operator by the *covariant derivative*. Thus, we can make a system invariant under local gauge transformations at the price of introducing a *vector* field A_μ, the gauge field, that plays the role of a connection. From a physical point of view, this result means that the impossibility of making a comparison at a distance of the phase of the field $\phi(x)$ requires that a physical gauge field $A_\mu(x)$ must be present. This procedure, that relates the matter and gauge fields through the covariant derivative, is known as *minimal coupling*.

There is a set of configurations of $\phi(x)$ that changes only because of the presence of the gauge field. These are the *geodesic* configurations $\phi_c(x)$. They satisfy the equation

$$D_\mu \phi_c = (\partial_\mu - ieA_\mu)\phi_c \equiv 0 \tag{3.49}$$

which is equivalent to the linear equation (see Eq. (3.43))

$$\partial_\mu \phi_c = ieA_\mu \phi_c \tag{3.50}$$

Let us consider, for example, two points x and y in spacetime at the ends of a path $\Gamma(x, y)$. For a *given* path $\Gamma(x, y)$, the solution of Eq. (3.49) is the path-ordered exponential of a line integral

$$\phi_c(x) = e^{-ie\int_{\Gamma(x,y)} dz_\mu A^\mu(z)}\ \phi_c(y) \tag{3.51}$$

Indeed, under a gauge transformation, the line integral transforms like

$$\begin{aligned} e\int_{\Gamma(x,y)} dz_\mu A^\mu \mapsto e\int_{\Gamma(x,y)} dz_\mu A^\mu + e\int_{\Gamma(x,y)} dz_\mu \frac{1}{e}\partial^\mu \theta \\ = e\int_{\Gamma(x,y)} dz_\mu A^\mu(z) + \theta(y) - \theta(x) \end{aligned} \tag{3.52}$$

Hence, we get

$$\begin{aligned} \phi_c(y)\ e^{-ie\int_\Gamma dz_\mu A^\mu} &\mapsto \phi_c(y)\ e^{-ie\int_\Gamma dz_\mu A^\mu} e^{-i\theta(y)} e^{i\theta(x)} \\ &\equiv e^{i\theta(x)} \phi_c(x) \end{aligned} \tag{3.53}$$

as it should be.

However, we may now want to ask how the change of phase of ϕ_c depends on the choice of the path Γ. Thus, let $\phi_c^{\Gamma_1}(y)$ and $\phi_c^{\Gamma_2}(y)$ be solutions of the geodesic equations for two different paths Γ_1 and Γ_2 with the same end points, x and y. Clearly, we have that the change of phase $\Delta\gamma$ given by

$$\Delta\gamma = -e\int_{\Gamma_1} dz_\mu A^\mu + e\int_{\Gamma_2} dz_\mu A^\mu \equiv -e\oint_{\Gamma^+} dz_\mu A^\mu \tag{3.54}$$

Here Γ^+ is the *closed oriented path*

$$\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^- \tag{3.55}$$

and

$$\int_{\Gamma_2^-} dz_\mu A_\mu = -\int_{\Gamma_2^+} dz_\mu A_\mu \tag{3.56}$$

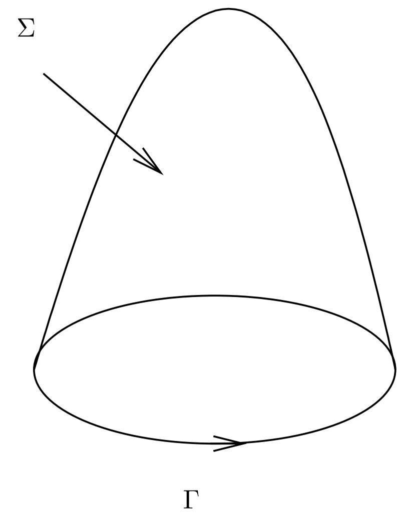

Figure 3.4 A closed path Γ is the boundary of the open surface Σ.

Using Stokes theorem we see that, if Σ^+ is an *oriented surface* whose *boundary* is the oriented *closed* path Γ^+, $\partial\Sigma^+ \equiv \Gamma^+$ (see Fig.(3.4), then $\Delta\gamma$ is given by the flux $\Phi(\Sigma)$ of the curl of the vector field A_μ through the surface Σ^+, i.e.

$$\Delta\gamma = -\frac{e}{2}\int_{\Sigma^+} dS_{\mu\nu}F^{\mu\nu} = -e\ \Phi(\Sigma) \tag{3.57}$$

where $F^{\mu\nu}$ is the field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.58}$$

$dS_{\mu\nu}$ is the *oriented* area element, and $\Phi(\Sigma)$ is the flux through the surface Σ. Both $F^{\mu\nu}$ and $dS_{\mu\nu}$ are antisymmetric in their spacetime indices. In particular, $F^{\mu\nu}$ can also be written as a commutator of two covariant derivatives

$$F^{\mu\nu} = \frac{i}{e}[D^\mu, D^\mu] \tag{3.59}$$

Thus, $F^{\mu\nu}$ measures the (infinitesimal) incompatibility of displacements along two independent directions. In other words, $F^{\mu\nu}$ is a *curvature* tensor. These results show very clearly that if $F^{\mu\nu}$ is non-zero in some region of spacetime, then the *phase* of ϕ cannot be uniquely determined: the *phase* of ϕ_c depends on the *path* Γ along which it is measured.

3.5 The Aharonov-Bohm effect

The path dependence of the phase of ϕ_c is closely related to *Aharonov-Bohm Effect*. This is a subtle effect, which was first discovered in the context of elementary quantum mechanics, and plays a fundamental role in (quantum) field theory as well.

Consider a quantum mechanical particle of charge e and mass m moving on a plane. The particle is coupled to an external electromagnetic field A_μ (here $\mu = 0, 1, 2$ only, since there is no motion out of the plane). Let us consider the geometry shown in Fig.(3.5) in which an infinitesimally thin solenoid pierces the plane in the vicinity of some point $\boldsymbol{r} = 0$. The

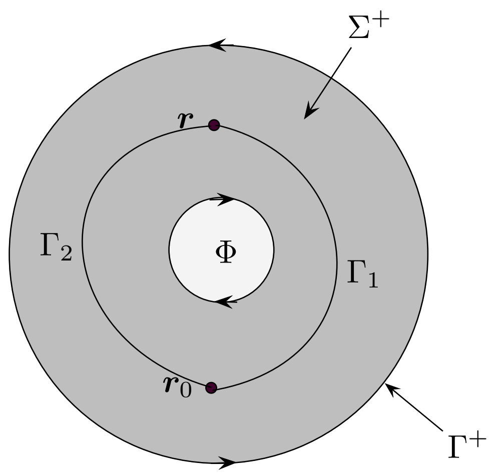

Figure 3.5 Geometric setup of the Aharonov-Bohm effect: a magnetic flux $\Phi \neq 0$ is thread through the small hole in the punctured plane Σ^+. Here Γ^+ are the oriented outer and inner edges of Σ^+; Γ_1 and Γ_2 are two inequivalent paths from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ described in the text.

Schrödinger Equation for this problem is

$$H\Psi = i\hbar \frac{\partial \Psi}{\partial t} \tag{3.60}$$

where

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\boldsymbol{A}\right)^2 \tag{3.61}$$

is the Hamiltonian. The magnetic field $\boldsymbol{B} = B\hat{z}$ vanishes everywhere except at $\boldsymbol{r} = 0$,

$$B = \Phi_0\, \delta(\boldsymbol{r}) \tag{3.62}$$

Using Stokes theorem we see that the flux of $\boldsymbol{B}$ through an arbitrary region Σ^+ with boundary Γ^+ is

$$\Phi = \int_{\Sigma^+} d\boldsymbol{S} \cdot \boldsymbol{B} = \oint_{\Gamma^+} d\boldsymbol{\ell} \cdot \boldsymbol{A} \tag{3.63}$$

Hence, $\Phi = \Phi_0$ *for all* surfaces Σ^+ that enclose the point $\boldsymbol{r} = 0$, and it is equal to zero otherwise. Hence, although the magnetic field is zero for $\boldsymbol{r} \neq 0$, the vector potential does not (and cannot) vanish.

The wave function $\Psi(\boldsymbol{r})$ can be calculated in a very simple way. Let us define

$$\Psi(\boldsymbol{r}) = e^{i\theta(\boldsymbol{r})} \Psi_0(\boldsymbol{r}) \tag{3.64}$$

where $\Psi_0(\boldsymbol{r})$ satisfies the Schrödinger Equation *in the absence* of the field, i.e.

$$H_0 \Psi_0 = i\hbar \frac{\partial \Psi_0}{\partial t} \tag{3.65}$$

with

$$H_0 = -\frac{\hbar^2}{2m} \nabla^2 \tag{3.66}$$

Since the wave function Ψ has to be differentiable and Ψ_0 is single valued, we must also demand the boundary condition that

$$\lim_{\boldsymbol{r} \to 0} \Psi_0(\boldsymbol{r}, t) = 0 \tag{3.67}$$

The wave function $\Psi = \Psi_0 \, e^{i\theta}$ looks like a gauge transformation. But we will discover that there is a subtlety here. Indeed, θ can be determined as follows. By direct substitution we get

$$\left(\frac{\hbar}{i} \nabla + \frac{e}{c} \boldsymbol{A} \right) (e^{i\theta} \Psi_0) = e^{i\theta} \left(\hbar \, \nabla \theta + \frac{e}{c} \boldsymbol{A} + \frac{\hbar}{i} \nabla \right) \Psi_0 \tag{3.68}$$

Thus, in order to succeed in our task, we only have to require that $\boldsymbol{A}$ and θ must obey the relation

$$\hbar \nabla \theta + \frac{e}{c} \boldsymbol{A} \equiv 0 \tag{3.69}$$

Or, equivalently,

$$\nabla \theta(\boldsymbol{r}) = -\frac{e}{\hbar c} \boldsymbol{A}(\boldsymbol{r}) \tag{3.70}$$

However, if this relation holds, θ cannot be a smooth function of $\boldsymbol{r}$. In fact, the line integral of $\nabla \theta$ on an arbitrary *closed* path Γ^+ is given by

$$\int_{\Gamma^+} d\boldsymbol{\ell} \cdot \nabla \theta = \Delta \theta \tag{3.71}$$

where $\Delta\theta$ is the total change of θ in one full counterclockwise turn around the path Γ. It is the immediate to see that $\Delta\theta$ is given by

$$\Delta\theta = -\frac{e}{\hbar c}\oint_{\Gamma^+} d\boldsymbol{\ell}\cdot\boldsymbol{A} \tag{3.72}$$

We must conclude that, in general, $\theta(\boldsymbol{r})$ is a *multivalued* function of $\boldsymbol{r}$ which has a branch cut going from $\boldsymbol{r} = 0$ out to some arbitrary point at infinity. The actual position and shape of the branch cut is irrelevant, but the *discontinuity* $\Delta\theta$ of θ across the cut is not irrelevant.

Hence, Ψ_0 is chosen to be a smooth, single valued, solution of the Schrödinger equation in the *absence* of the solenoid, satisfying the boundary condition of Eq. (3.67). Such wave functions are (almost) plane waves.

Since the function $\theta(\boldsymbol{r})$ is multivalued and, hence, path-dependent, the wave function Ψ is also multivalued and path-dependent. In particular, let $\boldsymbol{r}_0$ be some arbitrary point on the plane and $\Gamma(\boldsymbol{r}_0,\boldsymbol{r})$ is a path that begins in $\boldsymbol{r}_0$ and ends at $\boldsymbol{r}$. The phase $\theta(\boldsymbol{r})$ is, for that choice of path, given by

$$\theta(\boldsymbol{r}) = \theta(\boldsymbol{r}_0) - \frac{e}{\hbar c}\int_{\Gamma(\boldsymbol{r}_0,\boldsymbol{r})} d\boldsymbol{x}\cdot\boldsymbol{A}(\boldsymbol{x}) \tag{3.73}$$

The overlap of two wave functions that are defined by two different paths $\Gamma_1(\boldsymbol{r}_0,\boldsymbol{r})$ and $\Gamma_2(\boldsymbol{r}_0,\boldsymbol{r})$ is (with $\boldsymbol{r}_0$ fixed)

$$\begin{aligned}\langle\Gamma_1|\Gamma_2\rangle &= \int d^2\boldsymbol{r}\ \Psi^*_{\Gamma_1}(\boldsymbol{r})\ \Psi_{\Gamma_2}(\boldsymbol{r})\\ &\equiv \int d^2\boldsymbol{r}\ |\Psi_0(\boldsymbol{r})|^2 \exp\left\{+\frac{ie}{\hbar c}\left(\int_{\Gamma_1(\boldsymbol{r}_0,\boldsymbol{r})} d\boldsymbol{\ell}\cdot\boldsymbol{A} - \int_{\Gamma_2(\boldsymbol{r}_0,\boldsymbol{r})} d\boldsymbol{\ell}\cdot\boldsymbol{A}\right)\right\}\end{aligned} \tag{3.74}$$

If Γ_1 and Γ_2 are chosen in such a way that the origin (where the solenoid is piercing the plane) is always to the left of Γ_1 but it is also always to the right of Γ_2, the difference of the two line integrals is the circulation of $\boldsymbol{A}$

$$\int_{\Gamma_1(\boldsymbol{r}_0,\boldsymbol{r})} d\boldsymbol{\ell}\cdot\boldsymbol{A} - \int_{\Gamma_2(\boldsymbol{r}_0,\boldsymbol{r})} d\boldsymbol{\ell}\cdot\boldsymbol{A} \equiv \oint_{\Gamma^+_{(\boldsymbol{r}_0)}} d\boldsymbol{\ell}\cdot\boldsymbol{A} \tag{3.75}$$

on the closed, positively oriented, contour $\Gamma^+ = \Gamma_1(\boldsymbol{r}_0,\boldsymbol{r}) \cup \Gamma_2(\boldsymbol{r},\boldsymbol{r}_0)$. Since this circulation is constant, and equal to the flux Φ, we find that the overlap $\langle\Gamma_1|\Gamma_2\rangle$ is

$$\langle\Gamma_1|\Gamma_2\rangle = \exp\left\{\frac{ie}{\hbar c}\Phi\right\} \tag{3.76}$$

where we have taken Ψ_0 to be normalized to unity. The result of Eq.(3.76) is known as the *Aharonov-Bohm Effect.*

We find that the overlap is a pure phase factor which, in general, is different from one. Notice that, although the wave function is always defined up to a *constant* arbitrary phase factor, *phase changes* are physical effects. In addition, for some special choices of Φ the wave function becomes single valued. These values correspond to the choice

$$\frac{e}{\hbar c}\Phi = 2\pi n \tag{3.77}$$

where n is an arbitrary integer. This requirement amounts to a quantization condition for the magnetic flux Φ, i.e.

$$\Phi = n\left(\frac{hc}{e}\right) \equiv n\Phi_0 \tag{3.78}$$

where Φ_0 is the *flux quantum*, $\Phi_0 = \frac{hc}{e}$.

In 1931 Dirac considered the effects of a monopole configuration of magnetic fields on the quantum mechanical wave functions of charged particles (Dirac, 1931). In Dirac's construction, a magnetic monopole is represented as a long thin solenoid in three-dimensional space. The magnetic field near the end of the solenoid is the same as that of a magnetic charge m equal to the magnetic flux going through the solenoid. Dirac argued that for the solenoid (the "Dirac string") to be unobservable, the wave function must be single-valued. This requirement leads to the Dirac quantization condition for the smallest magnetic charge,

$$me = 2\pi\hbar c \tag{3.79}$$

which we recognize is the same as the flux quantization condition of Eq.(3.78).

3.6 Non-abelian gauge invariance

Let us now consider systems with a non-abelian global symmetry. This means that the field ϕ transforms like some representation of a Lie group G,

$$\phi'_a(x) = U_{ab}\phi_b(x) \tag{3.80}$$

where U is a matrix that represents the action of a group element. The local Lagrangian density

$$\mathcal{L} = \partial_\mu \phi_a^* \partial^\mu \phi^a - V(|\phi|^2) \tag{3.81}$$

is invariant under *global* transformations.

Suppose now that we want to promote this *global* symmetry to a *local* one. However, it is also correct for this general case as well that while the potential term $V(|\phi|^2)$ is invariant even under local transformations $U(x)$, the first term of the Lagrangian of Eq.(3.81) is not. Indeed, the gradient of ϕ does not transform properly (i.e. covariantly) under the action of the Lie group G,

$$\begin{aligned}\partial_\mu \phi'(x) =&\partial_\mu[U(x)\phi(x)]\\ =&\left(\partial_\mu U(x)\right)\phi(x) + U(x)\partial_\mu\phi(x)\\ =&U(x)[\partial_\mu\phi(x) + U^{-1}(x)\partial_\mu U(x)\phi(x)]\end{aligned} \tag{3.82}$$

Hence $\partial_\mu\phi$ does not transform in the same way as the ϕ field.

We can now follow the same approach that we used in the abelian case and *define* a *covariant derivative* operator D_μ which should have the property that $D_\mu\phi$ should obey the same transformation law as the field ϕ, i.e.

$$\left(D_\mu\phi(x)\right)' = U(x)\left(D_\mu\phi(x)\right) \tag{3.83}$$

It is clear that D_μ is now both a differential operator as well as a matrix acting on the field ϕ. Thus, D_μ depends on the representation that was chosen for the field ϕ. We can now proceed in analogy with what we did in the case of electrodynamics, and guess that the covariant derivative D_μ should be of the form

$$D_\mu = I\,\partial_\mu - igA_\mu(x) \tag{3.84}$$

where g is a coupling constant, I is the $N\times N$ identity matrix, and A_μ is a matrix-valued vector field. If ϕ has N components, the vector field $A_\mu(x)$ is an $N\times N$ hermitian matrix which can be expanded in the basis of the group generators λ^k_{ab} (with $k = 1,\ldots,D(G)$, and $a,b = 1,\ldots,N$) which span the *algebra* of the Lie group G,

$$\left(A_\mu(x)\right)_{ab} = A^k_\mu(x)\lambda^k_{ab} \tag{3.85}$$

Thus, the vector field $A_\mu(x)$ is parametrized by the $D(G)$-component 4-vectors $A^k_\mu(x)$. We will choose the transformation properties of $A_\mu(x)$ in such a way that $D_\mu\phi$ transforms covariantly under gauge transformations,

$$\begin{aligned}D'_\mu\phi(x)' \equiv& D'_\mu\left(U(x)\phi(x)\right) = \left(\partial_\mu - igA'_\mu(x)\right)\left(U\phi(x)\right)\\ =&U(x)[\partial_\mu\phi(x) + U^{-1}(x)\partial_\mu U(x)\phi(x) - igU^{-1}(x)A'_\mu(x)U(x)\phi(x)]\\ \equiv&U(x)\;D_\mu\phi(x)\end{aligned} \tag{3.86}$$

This condition is met if we require that

$$U^{-1}(x)igA'_\mu(x)U(x) = igA_\mu(x) + U^{-1}(x)\partial_\mu U(x) \tag{3.87}$$

or, equivalently, that A_μ obeys the transformation law

$$A'_\mu(x) = U(x)A_\mu(x)U^{-1}(x) - \frac{i}{g}\left(\partial_\mu U(x)\right)U^{-1}(x) \tag{3.88}$$

Since the matrices $U(x)$ are unitary and invertible, we have

$$U^{-1}(x)U(x) = I \tag{3.89}$$

we can equivalently write the transformed vector field $A'_\mu(x)$ in the form

$$A'_\mu(x) = U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g}U(x)\left(\partial_\mu U^{-1}(x)\right) \tag{3.90}$$

This is the general form of a gauge transformation for a non-abelian Lie group G.

In the case of an abelian symmetry group, such as the group $U(1)$, the matrix reduces a simple phase factor, $U(x) = e^{i\theta(x)}$, and the field $A_\mu(x)$ is a real number-valued vector field. It is easy to check that, in this case, A_μ transforms as follows

$$\begin{aligned} A'_\mu(x) =& e^{i\theta(x)}A_\mu(x)e^{-i\theta(x)} + \frac{i}{g}e^{i\theta(x)}\partial_\mu(e^{-i\theta(x)}) \\ \equiv& A_\mu(x) + \frac{1}{g}\partial_\mu\theta(x) \end{aligned} \tag{3.91}$$

which recovers the correct form for an abelian gauge transformation.

Returning now to the non-abelian case, we see that under an infinitesimal transformation $U(x)$

$$(U(x))_{ab} = \left[\exp\left(i\lambda^k\theta^k(x)\right)\right]_{ab} \cong \delta_{ab} + i\lambda^k_{ab}\,\theta^k(x) + \dots \tag{3.92}$$

the scalar field $\phi(x)$ transforms as

$$\delta\phi_a(x) \cong i\lambda^k_{ab}\phi_b(x)\,\theta^k(x) + \dots \tag{3.93}$$

while the vector field A^k_μ now transforms as

$$\delta A^k_\mu(x) \cong if^{ksj}A^j_\mu(x)\,\theta^s(x) + \frac{1}{g}\partial_\mu\theta^k(x) + \dots \tag{3.94}$$

Thus, $A^k_\mu(x)$ transforms as a vector in the *adjoint representation* of the Lie group G since, in that representation, the matrix elements of the generators are the group structure constants f^{ksj}. Notice that A^k_μ is *always* in the adjoint representation of the group G, regardless of the representation in which $\phi(x)$ happens to be in.

From the discussion given above, it is clear that the field $A_\mu(x)$ can be interpreted as a generalization of the vector potential of electromagnetism. Furthermore, A_μ provides for a natural *connection* which tell us how the "internal coordinate system," in reference to which the field $\phi(x)$ is defined, changes from one point x_μ of spacetime to a neighboring point $x_\mu + dx_\mu$. In particular, the configurations $\phi^a(x)$ which are solutions of the geodesic equation

$$D_\mu^{ab}\phi_b(x) = 0 \tag{3.95}$$

correspond to the *parallel transport* of ϕ from some point x to some point y. This equation can be written in the equivalent form

$$\partial_\mu \phi_a(x) = igA_\mu^k(x)\,\lambda_{ab}^k\,\phi_b(x) \tag{3.96}$$

This linear partial differential equation can be solved as follows. Let x_μ and y_μ be two arbitrary points in spacetime and $\Gamma(x,y)$ a fixed path with endpoints at x and y. This path is parametrized by a mapping z_μ from the real interval $[0,1]$ to Minkowski space $\mathcal{M}$ (or any other space), $z_\mu : [0,1] \mapsto \mathcal{M}$, of the form

$$z_\mu = z_\mu(t), \qquad t \in [0,1] \tag{3.97}$$

with the boundary conditions

$$z_\mu(0) = x_\mu \quad \text{and} \quad z_\mu(1) = y_\mu \tag{3.98}$$

By integrating the geodesic equation Eq.(3.95) along the path Γ we obtain

$$\int_{\Gamma(x,y)} dz_\mu\, \frac{\partial \phi_a(z)}{\partial z_\mu} = ig \int_{\Gamma(x,y)} dz_\mu\, A_{ab}^\mu(z)\,\phi_b(z) \tag{3.99}$$

Hence, we find that $\phi(x)$ must be the solution of the integral equation

$$\phi(y) = \phi(x) + ig \int_{\Gamma(x,y)} dz_\mu\, A^\mu(z)\phi(z) \tag{3.100}$$

where we have omitted all the indices to simplify the notation. In terms of the parametrization $z_\mu(t)$ of the path $\Gamma(x,y)$ we can write

$$\phi(y) = \phi(x) + ig\int_0^1 dt\,\frac{dz_\mu}{dt}\,A^\mu\left(z(t)\right)\,\phi\left(z(t)\right) \tag{3.101}$$

We will solve this equation by means of an iteration procedure, similar to what it is used for the evolution operator in quantum theory. By substituting repeatedly the l.h.s. of this equation into its r. h. s. we get the series

$$\begin{aligned}\phi(y) &= \phi(x) + ig \int_0^1 dt\, \frac{dz_\mu(t)}{dt}\, A^\mu\left(z(t)\right)\phi(x)\\ &+ (ig)^2 \int_0^1 dt_1 \int_0^{t_1} dt_2 \frac{dz_{\mu_1}(t_1)}{dt_1}\, \frac{dz_{\mu_2}(t_2)}{dt_2} A^{\mu_1}\left(z(t_1)\right) A^{\mu_2}\left(z(t_2)\right)\phi(x)\\ &+ \ldots + (ig)^n \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \prod_{j=1}^{n}\left(\frac{dz_{\mu_j}(t_j)}{dt_j} A^{\mu_j}\left(z(t_j)\right)\right)\phi(x)\\ &+ \ldots \end{aligned} \tag{3.102}$$

Here we need to keep in mind that the A^μ's are matrix-valued fields which are ordered from left to right!.

The nested integrals of Eq.(3.102) can be written in the form

$$\begin{aligned} I_n &= (ig)^n \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, F(t_1)\cdots F(t_n)\\ &\equiv \frac{(ig)^n}{n!} \widehat{P}\left[\left(\int_0^1 dt\, F(t)\right)^n\right] \end{aligned} \tag{3.103}$$

where the F's are matrices and the operator $\widehat{P}$ means the path-ordered product of the objects sitting to its right. If we *formally* define the exponential of an operator to be equal to its power series expansion,

$$e^A \equiv \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{3.104}$$

where A is some arbitrary matrix, we see that the geodesic equation has the formal solution

$$\phi(y) = \widehat{P}\left[\exp\left(+ig \int_0^1 dt\, \frac{dz_\mu}{dt}\, A^\mu(z(t))\right)\right]\, \phi(x) \tag{3.105}$$

or, what is the same

$$\phi(y) = \widehat{P}\, \left[\exp\left(ig \int_{\Gamma(x,y)} dz_\mu\, A^\mu(z)\right)\right]\, \phi(x) \tag{3.106}$$

Thus, $\phi(y)$ is given by an operator, the *path-ordered* exponential of the line integral of the vector potential A^μ, acting on $\phi(x)$.

By expanding the exponential in a power series, it is easy to check that, under an arbitrary local gauge transformation $U(z)$, the path ordered exponential transforms as follows

$$\widehat{P}\left[\exp\Big(ig\int_{\Gamma(x,y)} dz^\mu\, A'_\mu(z(t))\Big)\right]$$
$$\equiv U(y)\,\hat{P}\left[\exp\left(ig\int_{\Gamma(x,y)} dz^\mu\, A_\mu(z)\right)\right]U^{-1}(x) \tag{3.107}$$

In particular we can consider the case of a *closed path* $\Gamma(x,x)$, where x is an *arbitrary* point on Γ. The path-ordered exponential $\widehat{W}_{\Gamma(x,x)}$

$$\widehat{W}_{\Gamma(x,x)} = \widehat{P}\left[\exp\left(ig\int_{\Gamma(x,x)} dz^\mu\, A_\mu(z)\right)\right] \tag{3.108}$$

is *not* gauge invariant since, under a gauge transformation it transforms as

$$\widehat{W}'_{\Gamma(x,x)} = \hat{P}\left[\exp\left(ig\int_{\Gamma(x,x)} dz^\mu\, A'_\mu(z)\right)\right]$$
$$= U(x)\,\hat{P}\left[\exp\left(ig\int_{\Gamma(x,x)} dz^\mu\, A_\mu(t)\right)\right]U^{-1}(x) \tag{3.109}$$

Therefore $\widehat{W}_{\Gamma(x,x)}$ transforms like a *group element*,

$$\widehat{W}_{\Gamma(x,x)} = U(x)\,\widehat{W}_{\Gamma(x,x)}\,U^{-1}(x) \tag{3.110}$$

However, the *trace* of $\widehat{W}_\Gamma(x,x)$, which we denote by

$$W_\Gamma = \text{tr}\,\widehat{W}_{\Gamma(x,x)} \equiv \text{tr}\,\widehat{P}\left[\exp\left(ig\int_{\Gamma(x,x)} dz^\mu\, A_\mu(z)\right)\right] \tag{3.111}$$

not only *is gauge-invariant* but it is also independent of the choice of the point x. However, it is a functional of the path Γ. The quantity W_Γ, which is known as the *Wilson loop*, plays a crucial role in gauge theories. In the quantum theory this object will become the Wilson loop operator.

Let us now consider the case of a *small* closed path Γ. If Γ is small, then the minimal area $a(\Gamma)$ enclosed by Γ and its length $\ell(\Gamma)$ are both infinitesimal. In this case, we can expand the exponential in powers and retain only the leading terms. We get

$$\widehat{W}_\Gamma \approx I + ig\widehat{P}\oint_\Gamma dz^\mu A_\mu(z) + \frac{(ig)^2}{2!}\widehat{P}\left(\oint_\Gamma dz^\mu A_\mu(z)\right)^2 + \cdots \tag{3.112}$$

Stokes theorem says that the first integral, the circulation of the vector field A_μ on the closed path Γ, is given by

$$\oint_\Gamma dz_\mu A^\mu(z) = \int\int_\Sigma dx^\mu \wedge dx^\nu\, \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{3.113}$$

where $\partial\Sigma = \Gamma$ is the infinitesimal area element bounded by Γ, and $dx^\mu \wedge dx^\nu$ is the oriented infinitesimal area element. Furthermore, the quadratic term in Eq.(3.112) can be expressed as follows

$$\frac{1}{2!}\widehat{P}\left(\oint_\Gamma dz^\mu A_\mu(z)\right)^2 \equiv \frac{1}{2}\int\int_\Sigma dx^\mu \wedge dx^\nu \left(-[A_\mu, A_\nu]\right) + \dots \tag{3.114}$$

Therefore, for an infinitesimally small loop, we get

$$\widehat{W}_{\Gamma(x,x')} \approx I + \frac{ig}{2}\int\int_\Sigma dx^\mu \wedge dx^\nu \, F_{\mu\nu} + O(a(\Sigma)^2) \tag{3.115}$$

where $F_{\mu\nu}$ is the field tensor, defined by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = i\left[D_\mu, D_\nu\right] \tag{3.116}$$

Keep in mind that since the fields A_μ are matrices, the field tensor $F_{\mu\nu}$ is also a matrix.

Notice also that now $F_{\mu\nu}$ *is not* gauge invariant. Indeed, under a local gauge transformation $U(x), F_{\mu\nu}$ transforms as a similarity transformation

$$F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^{-1}(x) \tag{3.117}$$

This property follows from the transformation properties of A_μ. However, although $F_{\mu\nu}$ itself is not gauge invariant, other quantities such as $\text{tr}(F_{\mu\nu}F^{\mu\nu})$ are gauge invariant.

Let us finally note the form of $F_{\mu\nu}$ in components. By expanding $F_{\mu\nu}$ in the basis of the group generators λ^k (hence, in the algebra of the gauge group),

$$F_{\mu\nu} = F^k_{\mu\nu}\lambda^k \tag{3.118}$$

we find that the components, $F^k_{\mu\nu}$ are

$$F^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + gf^{k\ell m}A^\ell_\mu A^m_\nu \tag{3.119}$$

The natural local, gauge-invariant, theory for a non-abelian gauge group is the Yang-Mills Lagrangian

$$\mathcal{L} = -\frac{1}{4}\text{tr}F_{\mu\nu}F^{\mu\nu} \tag{3.120}$$

for a general compact Lie group G, that we will call the *gauge group*. Notice that the apparent similarity with the Maxwell Lagrangian is only superficial, since in this theory it is not quadratic in the vector potentials. We will see in later chapters that other Lagrangians are possible in other dimensions if some symmetry, e.g. time-reversal, is violated.

3.7 Gauge invariance and minimal coupling

We are now in a position to give a general prescription for the coupling of matter and gauge fields. Since the issue here is local gauge invariance, this prescription is valid for both relativistic and non-relativistic theories.

So far, we have considered two cases: (a) fields that describe the dynamics of matter and (b) gauge fields that describe electromagnetism and chromodynamics. In our description of Maxwell's electrodynamics we saw that, if the Lagrangian is required to respect local gauge invariance, then only conserved currents can couple to the gauge field. However, we have also seen that the presence of a global symmetry is a sufficient condition for the existence of a locally conserved current. This is not only a necessary condition since a local symmetry also requires the existence of a conserved current.

We will now consider more general Lagrangians that will include both matter and gauge fields. In the last sections we saw that if a system with Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ has a global symmetry $\phi \to U\phi$, then by replacing all derivatives by covariant derivatives we promote a global symmetry into a local (or gauge) symmetry. We will proceed with our general philosophy and write down gauge-invariant Lagrangians for systems which contain both matter and gauge fields. I will give a few explicit examples

3.7.1 Quantum electrodynamics

Quantum Electrodynamics (QED) is a theory of electrons and photons. The electrons are described by *Dirac spinor fields* $\psi_\alpha(x)$. The reason for this choice will become clear when we discuss the quantum theory and the Spin-Statistics theorem. Photons are described by a $U(1)$ gauge field A_μ. The Lagrangian for *free electrons* is just the *free Dirac Lagrangian*, $\mathcal{L}_{\text{Dirac}}$

$$\mathcal{L}_{\text{Dirac}}(\psi, \bar{\psi}) = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{3.121}$$

The Lagrangian for the gauge field is the free Maxwell Lagrangian $\mathcal{L}_{\text{gauge}}(A)$

$$\mathcal{L}_{\text{gauge}}(A) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv -\frac{1}{4}F^2 \tag{3.122}$$

The prescription that we will adopt, known as *minimal coupling*, consists in requiring that the *total* Lagrangian be invariant under *local* gauge transformations.

The free Dirac Lagrangian is invariant under the *global* phase transformation (i.e. with the same phase factor for *all* the Dirac components)

$$\psi_\alpha(x) \to \psi'_\alpha(x) = e^{i\theta}\psi_\alpha(x) \tag{3.123}$$

if θ is a constant, arbitrary phase, but it is not invariant not under the *local phase* transformation

$$\psi_\alpha(x) \to \psi'_\alpha(x) = e^{i\theta(x)}\psi_\alpha(x) \tag{3.124}$$

As we saw before, the matter part of the Lagrangian can be made invariant under the local transformations

$$\begin{aligned} \psi_\alpha(x) &\to \psi'_\alpha(x) = e^{i\theta(x)}\psi_\alpha(x) \\ A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x) \end{aligned} \tag{3.125}$$

if the derivative $\partial_\mu\psi$ is replaced by the *covariant* derivative D_μ

$$D_\mu = \partial_\mu - ieA_\mu(x) \tag{3.126}$$

The total Lagrangian is now given by the sum of two terms

$$\mathcal{L} = \mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) + \mathcal{L}_{\text{gauge}}(A) \tag{3.127}$$

where $\mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A)$ is the gauge-invariant extension of the Dirac Lagrangian, i.e.

$$\begin{aligned} \mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) &= \bar{\psi}(i\not{D} - m)\psi \\ &= \bar{\psi}\left(i\not{\partial} - m\right)\psi + e\bar{\psi}\gamma_\mu\psi A^\mu \end{aligned} \tag{3.128}$$

$\mathcal{L}_{\text{gauge}}(A)$ is the usual Maxwell term and $\not{D}$ is a shorthand for $D_\mu\gamma^\mu$. Thus, the total Lagrangian for QED is

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^2 \tag{3.129}$$

Notice that now both matter and gauge fields are dynamical degrees of freedom.

The QED Lagrangian has a local gauge invariance. Hence, it also has a locally conserved current. In fact the argument that we used above to show that there are conserved (Noether) currents if there is a continuous global symmetry, is also applicable to gauge invariant Lagrangians. As a matter of fact, under an arbitrary infinitesimal gauge transformation

$$\delta\psi = i\theta\psi \quad \delta\bar{\psi} = -i\theta\bar{\psi} \quad \delta A_\mu = \tfrac{1}{e}\partial_\mu\theta \tag{3.130}$$

the QED Lagrangian remains invariant, i.e. $\delta\mathcal{L} = 0$. An arbitrary variation of $\mathcal{L}$ is

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta\partial_\mu\psi + (\psi \leftrightarrow \bar{\psi}) + \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta\partial_\mu A_\nu + \frac{\delta\mathcal{L}}{\delta A_\mu}\delta A_\mu \tag{3.131}$$

After using the equations of motion and the form of the gauge transformation, $\delta\mathcal{L}$ can be written in the form

$$\delta\mathcal{L} = \partial_\mu[j^\mu(x)\theta(x)] - \frac{1}{e}F^{\mu\nu}(x)\partial_\mu\partial_\nu\theta(x) + \frac{\delta\mathcal{L}}{\delta A_\mu}\frac{1}{e}\partial_\mu\theta(x) \tag{3.132}$$

where $j^\mu(x)$ is the *electron number current*

$$j^\mu = i\left(\frac{\partial\mathcal{L}}{\delta\partial_\mu\psi}\psi - \bar{\psi}\frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\psi}}\right) \tag{3.133}$$

For smooth gauge transformations $\theta(x)$, the term $F^{\mu\nu}\partial_\mu\partial_\nu\theta$ vanishes because of the antisymmetry of the field tensor $F_{\mu\nu}$. Hence we can write

$$\delta\mathcal{L} = \theta(x)\partial_\mu j^\mu(x) + \partial_\mu\theta(x)\left[j^\mu(x) + \frac{1}{e}\frac{\delta\mathcal{L}}{\delta A_\mu(x)}\right] \tag{3.134}$$

The first term tells us that since the infinitesimal gauge transformation $\theta(x)$ is arbitrary, the *Dirac current* $j^\mu(x)$ *locally is conserved*, i.e. $\partial_\mu j^\mu = 0$.

Let us define the *charge* (or *gauge*) current $J^\mu(x)$ by the relation

$$J^\mu(x) \equiv \frac{\delta\mathcal{L}}{\delta A_\mu(x)} \tag{3.135}$$

which is the current that enters in the Equation of Motion for the gauge field A_μ, i.e. the Maxwell equations. The vanishing of the second term of Eq. (3.134), required since the changes of the infinitesimal gauge transformations are also arbitrary, tells us that the charge current and the number current are related by

$$J_\mu(x) = -ej_\mu(x) = -e\,\bar{\psi}\gamma_\mu\psi \tag{3.136}$$

This relation tells us that since $j^\mu(x)$ is locally conserved, then the global conservation of Q_0

$$Q_0 = \int d^3x j_0(x) \equiv \int d^3x\, \psi^\dagger(x)\psi(x) \tag{3.137}$$

implies the global conservation of the electric charge Q

$$Q \equiv -eQ_0 = -e\int d^3x\, \psi^\dagger(x)\psi(x) \tag{3.138}$$

This property justifies the interpretation of the *coupling constant* e as the *electric charge*. In particular the gauge transformation of Eq.(3.125), tells us that the matter field $\psi(x)$ represents excitations that carry the unit of charge, $\pm e$. From this point of view, the electric charge can be regarded as a *quantum number*. This point of view becomes very useful in the quantum theory in the strong coupling limit. In this case, under special circumstances, the excitations may acquire unusual quantum numbers. This is *not* the case of Quantum Electrodynamics, but it is the case of a number of theories in one and two space dimensions, with applications in Condensed Matter systems such as polyacetylene, or the two-dimensional electron gas in high magnetic fields, i.e. the fractional quantum Hall effect, or in gauge theories with magnetic monopoles).

3.7.2 Quantum chromodynamics

Quantum Chromodynamics (QCD) is the gauge field theory of strong interactions in hadron physics. In this theory the elementary constituents of hadrons, the *quarks*, are represented by the Dirac spinor field $\psi^i_\alpha(x)$. The theory also contains a set of gauge fields $A^a_\mu(x)$ that represent the *gluons*. The quark fields have both *Dirac* indices $\alpha = 1, \ldots, 4$ and *color* indices $i = 1, \ldots, N_c$, where N_c is the number of colors. In the Standard Model of weak, strong interactions, and electromagnetic in particle physics, and in QCD, there are in addition $N_f = 6$ flavors of quarks, grouped into three generations, each labeled by a flavor index, and six flavors of leptons, also grouped into three generations. The flavor symmetry is a global symmetry of the theory.

Quarks are assumed to transform under the *fundamental representation* of the gauge (or color) group G, say $SU(N_c)$. The theory is invariant under the group of gauge transformations. In QCD, the color group is $SU(3)$ and so $N_c = 3$. The color symmetry is a non-abelian gauge symmetry. The gauge field A_μ is needed in order to enforce local gauge invariance. In components, we get $A_\mu = A^a_\mu \lambda^a$, where λ^a are the generators of $SU(N_c)$. Thus, $a = 1, \ldots, D(SU(N_c))$, and $D(SU(N_c) = N_c^2 - 1$. Thus, A_μ is an $N_c^2 - 1$ dimensional vector in the adjoint representation of G. For $SU(3), N_c^2 - 1 = 8$ and there are eight generators.

The gauge-invariant matter term of the Lagrangian, $\mathcal{L}_{\text{matter}}$ is

$$\mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) = \bar{\psi}(i \not{D} - m)\psi \tag{3.139}$$

where $\not{D} = \not{\partial} - ig\not{A} \equiv \not{\partial} - ig\not{A}^a \lambda^a$ is the covariant derivative. The gauge field term of the Lagrangian $\mathcal{L}_{\text{gauge}}$

$$\mathcal{L}_{\text{gauge}}(A) = -\frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} \equiv -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \tag{3.140}$$

is the Yang-Mills Lagrangian. The total Lagrangian for QCD is $\mathcal{L}_{\text{QCD}}$

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{matter}}(\psi, \bar{\psi}, A) + \mathcal{L}_{\text{gauge}}(A) \tag{3.141}$$

Can we define a *color charge*? Since the color group is non-abelian it has more than one generator. We showed before that there are as many conserved currents as generators are in the group. Now, in general, the group generators do not commute with each other. For instance, in $SU(2)$ there is only *one* diagonal generator, J_3, while in $SU(3)$ there are only *two* diagonal generators, etc. Can all the global *charges* Q^a

$$Q^a \equiv \int d^3x\, \psi^\dagger(x)\lambda^a\psi(x) \tag{3.142}$$

be defined simultaneously? It is straightforward to show that the Poisson Brackets of any pair of charges are, in general, different from zero. We will see below, when we *quantize* the theory, that the charges Q^a obey the same commutation relations as the group generators themselves do. So, in the quantum theory, the only charges that can be assigned to *states* are precisely the same as the quantum numbers that label the representations. Thus, if the group is $SU(2)$, we can only assign to the states the values of the quadratic Casimir operator $\boldsymbol{J}^2$ and of the projection J_3. Similar restrictions apply to the case of $SU(3)$ and to other Lie groups.

3.8 Spacetime symmetries and the energy-momentum tensor

Until now we have considered only the role of internal symmetries. We now turn to spacetime symmetries, and consider the role of coordinate transformations. In this more general setting we will have to require the invariance of the *action* rather than only of the *Lagrangian*, as we did for internal symmetries.

There are three continuous spacetime symmetries that will be important to us: a) translation invariance, b) rotation invariance and c) homogeneity of time. While rotations are a subgroup of Lorentz transformation, space and time translations are examples of inhomogeneous Lorentz transformations (in the relativistic case) and of Galilean transformations (in the non-relativistic case). Inhomogeneous Lorentz transformations also form a group, known as the Poincaré group. Note that the transformations discussed above are particular cases of more general coordinate transformations. However, it is important to keep in mind that, in most cases, general coordinate transformations are not symmetries of an arbitrary system. They are the symmetries of General Relativity.

In what follows we are going to consider the response of a system to infinitesimal coordinate transformations of the form

$$x'_\mu = x_\mu + \delta x_\mu \tag{3.143}$$

where δx_μ may be a function of the spacetime point x_μ. Under a coordinate transformation the fields change as

$$\phi(x) \to \phi'(x') = \phi(x) + \delta\phi(x) + \partial_\mu\phi\, \delta x^\mu \tag{3.144}$$

where $\delta\phi$ is the variation of ϕ in the *absence* of a change of coordinates , i.e. a functional change. In this notation, a uniform infinitesimal translation by a constant vector a_μ has $\delta x_\mu = a_\mu$ and an infinitesimal rotation of the space axes is $\delta x_0 = 0$ and $\delta x_i = \epsilon_{ijk}\theta_j x_k$.

In general, the action of the system is not invariant under arbitrary changes in both coordinates and fields. Indeed, under an arbitrary change of coordinates $x_\mu \to x'_\mu(x_\mu)$, the volume element d^4x is not invariant and changes by a multiplicative factor of the form

$$d^4x' = d^4x\; J \tag{3.145}$$

where J is the Jacobian of the coordinate transformation

$$J = \frac{\partial x'_1 \cdots x'_4}{\partial x_1 \cdots x_4} \equiv \left|\det\left(\frac{\partial x'_\mu}{\partial x_j}\right)\right| \tag{3.146}$$

For an infinitesimal transformation, $x'_\mu = x_\mu + \delta x_\mu(x)$, we get

$$\frac{\partial x'_\mu}{\partial x_\nu} = g_\mu^\nu + \partial^\nu \delta x_\mu \tag{3.147}$$

Since δx_μ is small, the Jacobian can be approximated by

$$J = \left|\det\left(\frac{\partial x'_\mu}{\partial x_\nu}\right)\right| = \left|\det\left(g_\mu^\nu + \partial^\nu \delta x_\mu\right)\right| \approx 1 + \mathrm{tr}(\partial^\nu \delta x_\mu) + O(\delta x^2) \tag{3.148}$$

Thus

$$J \approx 1 + \partial^\mu \delta x_\mu + O(\delta x^2) \tag{3.149}$$

The Lagrangian itself is in general not invariant. For instance, even though we will always be interested in systems whose Lagrangians are not an explicit function of x, still they are not in general invariant under the given transformation of coordinates. Also, under a coordinate change the fields may also transform. Thus, in general, $\delta\mathcal{L}$ does not vanish.

The most general variation of $\mathcal{L}$ is

$$\delta\mathcal{L} = \partial_\mu\mathcal{L}\, \delta x^\mu + \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi \tag{3.150}$$

If ϕ obeys the equations of motion,

$$\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \tag{3.151}$$

then, the general change $\delta\mathcal{L}$ obeyed by the solutions of the equations of motion is

$$\delta \mathcal{L} = \delta x^\mu \; \partial_\mu \mathcal{L} + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \phi \right) \tag{3.152}$$

The total change in the action is a sum of two terms

$$\delta S = \delta \int d^4x \; \mathcal{L} = \int \delta d^4x \; \mathcal{L} + \int d^4x \; \delta \mathcal{L} \tag{3.153}$$

where the change in the integration measure is due to the Jacobian factor,

$$\delta d^4 x = d^4x \; \partial_\mu \delta x^\mu \tag{3.154}$$

Hence, δS is given by

$$\delta S = \int d^4x \left[(\partial_\mu \delta x^\mu) \; \mathcal{L} + \delta x^\mu \; \partial_\mu \mathcal{L} + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \delta \phi \right) \right] \tag{3.155}$$

Since the *total* variation of ϕ, $\delta_T \phi$, is the sum of the functional change of the fields plus the changes in the fields caused by the coordinate transformation,

$$\delta_T \phi \equiv \delta \phi + \partial_\mu \phi \delta x^\mu \tag{3.156}$$

we can write δS as a sum of two contributions: one due to change of coordinates and another due to functional changes of the fields:

$$\delta S = \int d^4x \; \left[(\partial_\mu \delta x^\mu) \; \mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \; (\delta_T \phi - \partial_\nu \phi \delta x^\nu) \right) \right] \tag{3.157}$$

Therefore, for the change of the action we get

$$\delta S = \int d^4x \; \left\{ \partial_\mu \left[\left(g^\mu_\nu \mathcal{L} - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial_\nu \phi \right) \; \delta x^\nu \right] + \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \; \delta_T \phi \right] \right\} \tag{3.158}$$

We have already encountered the second term when we discussed the case of internal symmetries. The first term represents the change of the action S as a result of a change of coordinates.

We will now consider a few explicit examples. To simplify matters we will consider the effects of coordinate transformations alone. For simplicity, here we will restrict our discussion to the case of the scalar field.

3.8.1 Spacetime translations

Under a uniform infinitesimal translation $\delta x_\mu = a_\mu$, the field ϕ does not change

$$\delta_T \phi = 0 \tag{3.159}$$

The change of the action now is

$$\delta S = \int d^4x \, \partial_\mu \left(g^{\mu\nu} \mathcal{L} - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial^\nu \phi \right) a^\nu \tag{3.160}$$

For a system which is *isolated* and *translationally invariant* the action must not change under a redefinition of the origin of the coordinate system. Thus, $\delta S = 0$. Since a_μ is *arbitrary*, it follows that the tensor $T^{\mu\nu}$

$$T^{\mu\nu} \equiv -g^{\mu\nu} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial^\nu \phi \tag{3.161}$$

is conserved,

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.162}$$

The tensor $T^{\mu\nu}$ is known as the *energy-momentum tensor*. The reason for this name is the following. Given that $T^{\mu\nu}$ is locally conserved, by Noether's theorem we can define the 4-vector P^ν

$$P^\nu = \int d^3x \, T^{0\nu}(\boldsymbol{x}, x_0) \tag{3.163}$$

which is a *constant of motion*. In particular, P^0 is given by

$$P^0 = \int d^3x \, T^{00}(\boldsymbol{x}, x_0) \equiv \int d^3x \left[-\mathcal{L} + \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} \partial^0 \phi \right] \tag{3.164}$$

But $\frac{\delta \mathcal{L}}{\delta \partial_0 \phi}$ is just the canonical momentum $\Pi(x)$,

$$\Pi(x) = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} \tag{3.165}$$

Then we can easily recognize that the quantity in brackets in Eq.(3.164) in the definition of P^0 is just the Hamiltonian density $\mathcal{H}$

$$\mathcal{H} = \Pi \, \partial^0 \phi - \mathcal{L} \tag{3.166}$$

Therefore, the time component of P^ν is just the *total energy* of the system

$$P^0 = \int d^3x \, \mathcal{H} \tag{3.167}$$

The space components P^j are

$$P^j = \int d^3x \, T^{0j} = \int d^3x \left[-g^{0j} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \partial_j \phi} \partial^j \phi \right] \tag{3.168}$$

Thus, since $g^{0j} = 0$, we get

$$\boldsymbol{P} = \int d^3x \, \Pi(x) \, \boldsymbol{\partial} \phi(x) \tag{3.169}$$

The vector $\boldsymbol{P}$ is identified with the *total linear momentum* since (a) it is a constant of motion, and (b) it is the generator of infinitesimal space translations. For the same reasons we will denote the component $T^{0j}(x)$ with the *linear momentum density* $\mathcal{P}^j(x)$. It is important to stress that the *canonical momentum* $\Pi(x)$ and the *total linear momentum density* $\mathcal{P}^j$ are obviously completely different physical quantities. While the canonical momentum is a field which is canonically conjugate to the field ϕ, the total momentum is the linear momentum stored in the field, i.e. the linear momentum of the center of mass.

3.8.2 Rotations

If the action is invariant under global infinitesimal Lorentz transformations, of which spacial rotations are a particular case,

$$\delta x_\mu = \omega_\mu^\nu \, x_\nu \tag{3.170}$$

where $\omega^{\mu\nu}$ is infinitesimal, and antisymmetric, then the variation of the action is zero, $\delta S = 0$. If ϕ is a *scalar field*, then $\delta_T \phi$ is also zero. This is not the case for spinor or vector fields which transforms under Lorentz transformations. Because of their transformation properties of these fields, the angular momentum tensor that we define below will be missing the contribution representing the spin of the field (which vanishes for a scalar field). Here we will consider only the case of scalars.

Then, since for a scalar field $\delta_T \phi = 0$, we find

$$\delta S = 0 = \int d^4x \, \partial_\mu \left[\left(g^{\mu\nu} \mathcal{L} - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \partial^\nu \phi \right) \omega^{\nu\rho} \, x_\rho \right] \tag{3.171}$$

Since $\omega^{\nu\rho}$ is constant and arbitrary, the magnitude in brackets must also define a conserved current.

Let $M^{\mu\nu\rho}$ be the tensor defined by

$$M^{\mu\nu\rho} \equiv T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu \tag{3.172}$$

in terms of which, the quantity in brackets in Eq.(3.171) becomes $\frac{1}{2}\omega_{\nu\rho}M^{\mu\nu\rho}$. Thus, for arbitrary constant ω, we find that the tensor $M^{\mu\nu\rho}$ is locally conserved,

$$\partial_\mu M^{\mu\nu\rho} = 0 \tag{3.173}$$

In particular, the transformation

$$\delta x_0 = 0, \qquad \delta x_j = \omega_{jk} x_k \tag{3.174}$$

represents an infinitesimal rotation of the spatial axes with

$$\omega_{jk} = \epsilon_{jkl}\theta_l \tag{3.175}$$

where θ_l are three infinitesimal Euler angles. Thus, we suspect that $M^{\mu\nu\rho}$ must be related with the total angular momentum. Indeed, the local conservation of the current $M^{\mu\nu\rho}$ leads to the global conservation of the tensorial quantity $L^{\nu\rho}$

$$L^{\nu\rho} \equiv \int d^3x \; M^{0\nu\rho}(\boldsymbol{x}, x_0) \tag{3.176}$$

In particular, the space components of $L_{\nu\rho}$ are

$$\begin{aligned} L_{jk} &= \int d^3x \; (T_{0j}(x)\; x_k - T_{0k}(x)\; x_j) \\ &= \int d^3x \; (\mathcal{P}_j(x)\; x_j - \mathcal{P}_k(x)\; x_j) \end{aligned} \tag{3.177}$$

If we denote by L_j the (pseudo) vector

$$L_j \equiv \frac{1}{2}\,\epsilon_{jkl}\; L_{kl} \tag{3.178}$$

we get

$$L_j \equiv \int d^3x \; \epsilon_{jkl}\; x_k\; \mathcal{P}_l(x) \equiv \int d^3x \; \ell_j(x) \tag{3.179}$$

The vector L_j is the generator of infinitesimal rotations and is thus identified with the total *angular momentum*, whereas $\ell_j(x)$ is the corresponding (spacial) angular momentum density. Notice that, since we are dealing with a *scalar* field, there is no spin contribution to the angular momentum density.

The generalized angular momentum tensor $L^{\nu\rho}$ of Eq.(3.176) is not translationally invariant since under a displacement of the origin of the coordinate system by a^μ, $L^{\nu\rho}$ changes by an amount $a^\nu P^\rho - a^\rho P^\nu$. A truly intrinsic angular momentum is given by the Pauli-Lubanski vector W^μ,

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\frac{L^{\nu\lambda}P^\rho}{\sqrt{P^2}} \tag{3.180}$$

which, in the rest frame $\boldsymbol{P} = 0$, reduces to the angular momentum.

Finally, we find that if the angular momentum tensor $M^{\mu\nu\lambda}$ has the form

$$M^{\mu\nu\lambda} = T^{\mu\nu}x^{\lambda} - T^{\mu\lambda}x^{\nu} \tag{3.181}$$

Then, the conservation of the energy-momentum tensor $T^{\mu\nu}$ and of the angular momentum tensor $M^{\mu\nu\lambda}$ together lead to the condition that the energy-momentum tensor should be a symmetric second rank tensor,

$$T^{\mu\nu} = T^{\mu\nu} \tag{3.182}$$

Thus, we conclude that the conservation of angular momentum requires that the energy momentum tensor $T^{\mu\nu}$ for a *scalar field* be symmetric.

The expression for $T^{\mu\nu}$ that we derived in Eq.(3.161) is not manifestly symmetric. However, if $T^{\mu\nu}$ is conserved, then the "improved" tensor $\tilde{T}^{\mu\nu}$

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_{\lambda}K^{\mu\nu\lambda} \tag{3.183}$$

is also conserved, provided the tensor $K^{\mu\nu\lambda}$ is antisymmetric in (μ, λ) and (ν, λ). It is always possible to find such a tensor $K^{\mu\nu\lambda}$ to make $\tilde{T}^{\mu\nu}$ symmetric. The improved, symmetric, energy-momentum tensor is known as the Belinfante energy-momentum tensor.

In particular, for the scalar field $\phi(x)$ whose Lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)^2 - V(\phi) \tag{3.184}$$

the locally conserved energy-momentum tensor $T^{\mu\nu}$ is

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi}\partial^{\nu}\phi \equiv -g^{\mu\nu}\mathcal{L} + \partial^{\mu}\phi\partial^{\nu}\phi \tag{3.185}$$

which is symmetric. The conserved energy-momentum 4-vector is

$$P^{\mu} = \int d^3x \, (-g^{0\mu}\mathcal{L} + \partial^0\phi\partial^{\mu}\phi) \tag{3.186}$$

Thus, we find that

$$P^0 = \int d^3x \, (\Pi \, \partial_0\phi - \mathcal{L}) = \int d^3x \, [\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)] \tag{3.187}$$

is the total energy of the field, and

$$\boldsymbol{P} = \int d^3x \, \Pi(x) \, \nabla\phi(x) \tag{3.188}$$

is the linear momentum $\boldsymbol{P}$ of the field. Both are constants of motion.

3.9 The energy-momentum tensor for the electromagnetic field

For the case of the Maxwell field A_μ, a straightforward application of these methods yields an energy-momentum tensor $T^{\mu\nu}$ of the form

$$\begin{aligned} T^{\mu\nu} &= -g^{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta\partial_\nu A_\lambda}\partial^\mu A_\lambda \\ &= \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\nu\lambda}\partial^\mu A_\lambda \end{aligned} \tag{3.189}$$

It obeys $\partial_\mu T^{\mu\nu} = 0$ and, hence, is locally conserved. However, this tensor is not gauge invariant. We can construct a gauge-invariant and conserved energy momentum tensor by exploiting the ambiguity in the definition of $T^{\mu\nu}$. Thus, if we choose $K_{\mu\nu\lambda} = F_{\nu\lambda}A_\mu$, which is anti-symmetric in the indices ν and λ, we can construct the required gauge-invariant and conserved energy momentum-tensor

$$\tilde{T}^{\mu\nu} = \frac{1}{4}g^{\mu\nu}F^2 - F^\nu_\lambda F^{\mu\lambda} \tag{3.190}$$

where we used the equation of motion of the free electromagnetic field, $\partial^\lambda F_{\nu\lambda} = 0$. Notice that this "improved" energy-momentum tensor is both gauge-invariant and symmetric.

From here we find that the 4-vector

$$P^\mu = \int_{x_0 \text{ fixed}} d^3x\, \tilde{T}^{\mu 0} \tag{3.191}$$

is a constant of motion. Thus, we identify

$$P^0 = \int_{x_0 \text{ fixed}} d^3x\, \tilde{T}^{00} = \int_{x_0 \text{ fixed}} d^3x\, \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \tag{3.192}$$

with the Hamiltonian, and

$$P^i = \int_{x_0 \text{ fixed}} d^3x\, \tilde{T}^{i0} = \int_{x_0 \text{ fixed}} d^3x\, (\boldsymbol{E}\times\boldsymbol{B})_i \tag{3.193}$$

with the linear momentum (or Poynting vector) of the electromagnetic field.

3.10 The energy-momentum tensor and changes in the geometry

The energy momentum tensor $T^{\mu\nu}$ appears in classical field theory as a result of the translation invariance, in both space and time, of the physical system. We have seen in the previous section that for a scalar field $T^{\mu\nu}$ is a symmetric tensor as a consequence of the conservation of angular momentum. The definition that we found does not require that $T^{\mu\nu}$ should have any definite symmetry. However we found that it is always possible to modify $T^{\mu\nu}$ by adding a suitably chosen antisymmetric conserved tensor to find a symmetric version of $T^{\mu\nu}$. Given this fact, it is natural to ask if there is a way to define the energy-momentum tensor in a such a way that it is always symmetric. This issue becomes important if we want to consider theories for systems which contain fields which are not scalars.

It turns out that it is possible to regard $T^{\mu\nu}$ as the change in the *action* due to a change of the *geometry* in which the system lives. From classical physics, we are familiar with the fact that when a body is distorted in some manner, in general its energy increases since we have to perform some work against the body in order to deform it. A deformation of a body is a change of the *geometry* in which its component parts evolve. Examples of such changes of geometry are shear distortions, dilatations, bendings, and twists. On the other hand, there are changes that do not cost any energy since they are symmetry operations. Examples of symmetry operations are translations and rotations. These symmetry operations can be viewed as simple changes in the *coordinates* of the parts of the body which do not change its geometrical properties, i.e. the distances and angles of different points. Thus, coordinate transformations do not alter the energy of the system. The same type of arguments apply to *any* dynamical system. In the most general case, we have to consider transformations which leave the *action* invariant. This leads us to consider how changes in the geometry of the spacetime affect the action of a dynamical system.

The information about the geometry in which a system evolves is encoded in the *metric tensor* of the space (and spacetime). The metric tensor is a *symmetric* tensor that specifies how to measure the distance $|ds|$ between a pair of nearby points x and $x + dx$, namely

$$ds^2 = g^{\mu\nu}(x)\, dx_\mu\, dx_\nu \tag{3.194}$$

Under an arbitrary local *change of coordinates* $x_\mu \to x_\mu + \delta x_\mu$, the metric tensor changes as follows

$$g'_{\mu\nu}(x') = g_{\lambda\rho}(x)\frac{\partial x^\lambda}{\partial x'^\mu}\,\frac{\partial x^\rho}{\partial x'^\nu} \tag{3.195}$$

For an infinitesimal change, this means that the functional change in the metric tensor $\delta g_{\mu\nu}$ is

$$\delta g_{\mu\nu} = -\frac{1}{2}\left(g_{\mu\lambda}\partial^\nu \delta x_\lambda + g_{\lambda\nu}\partial^\mu \delta x_\lambda + \partial^\lambda g_{\mu\nu}\delta x_\lambda\right) \tag{3.196}$$

The volume element, invariant under coordinate transformations, is $d^4x\,\sqrt{g}$, where g is the determinant of the metric tensor.

Coordinate transformations change the metric of spacetime but do not change the action. *Physical* or geometric changes, are changes in the metric tensor which are not due to coordinate transformations. For a system in a space with metric tensor $g^{\mu\nu}(x)$, not necessarily the Minkowski (or Euclidean) metric, the *change* of the action is a linear function of the infinitesimal change in the metric $\delta g^{\mu\nu}(x)$ (i.e. "Hooke's Law"). Thus, we can write the change of the action due to an arbitrary infinitesimal change of the metric in the form

$$\delta S = \int d^4x \, \sqrt{g} \; T^{\mu\nu}(x) \, \delta g_{\mu\nu}(x) \tag{3.197}$$

Below we will identify the proportionality constant, the tensor $T^{\mu\nu}$, with the conserved energy-momentum tensor. This definition implies that $T^{\mu\nu}$ can be regarded as the derivative of the action with respect to the metric

$$T^{\mu\nu}(x) \equiv \frac{\delta S}{\delta g_{\mu\nu}(x)} \tag{3.198}$$

Since the metric tensor is symmetric, this definition always yields a symmetric energy momentum tensor.

In order to prove that this definition of the energy-momentum tensor agrees with the one we obtained before (which was not unique!) we have to prove that this form of $T^{\mu\nu}$ that we have just defined is a *conserved* current for coordinate transformations. Under an arbitrary local change of coordinates, which leave the distance ds unchanged, the metric tensor changes by the $\delta g_{\mu\nu}$ given above. The change of the action δS must be zero for this case. We see that if we substitute the expression for $\delta g_{\mu\nu}$ in δS, then an integration by parts will yield a conservation law. Indeed, for the particular case of a flat metric, such as the Minkowski or Euclidean metrics, the change δS is

$$\delta S = -\frac{1}{2} \int d^4x \, T^{\mu\nu}(x) \, \left(g_{\mu\lambda} \partial_\nu \delta x^\lambda + g_{\lambda\nu} \partial_\mu \delta x^\lambda \right) \tag{3.199}$$

since for a global coordinate transformation the Jacobian factor $\sqrt{g}$ and the measure of spacetime d^4x are constant. Thus, if δS is to vanish for an arbitrary change δx, the tensor $T^{\mu\nu}(x)$ has to be a locally conserved current, i.e. $\partial_\mu T^{\mu\nu}(x) = 0$. This definition can be extended to the case of more general spaces. We should note, however, that the energy-momentum tensor can be made symmetric only if the space does not have a property known as torsion.

Finally, let us remark that this definition of the energy-momentum tensor also allows us to identify the spacial components of $T^{\mu\nu}$ with the stress-energy tensor of the system. Indeed, for a change in geometry which is does not vary with time, the change of the action reduces to a change of the total energy of the system. Hence, the space components of the energy momentum tensor tell us how much does the total energy change for a specific deformation of the geometry. But this is precisely what the stress energy tensor is!