7
Quantization of the Free Dirac Field

7.1 The Dirac equation and Quantum Field Theory

The Dirac equation is a relativistic wave equation that describes the quantum dynamics of spinors. We will see in this section that a consistent description of this theory cannot be done outside the framework of (local) relativistic Quantum Field Theory.

The Dirac Equation

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{7.1}$$

can be regarded as the equation of motion of a complex field ψ. Much as in the case of the scalar field, and also in close analogy to the theory of non-relativistic many particle systems discussed in the last chapter, we will regard the Dirac field as an operator that acts on a Fock space of states.

We have already discussed that the Dirac equation also follows from a least-action-principle. Indeed the Lagrangian

$$\mathcal{L} = \frac{i}{2}[\bar{\psi}\partial\!\!\!/\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi \equiv \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{7.2}$$

has the Dirac equation for its equation of motion. Also, the momentum $\Pi_\alpha(x)$ canonically conjugate to $\psi_\alpha(x)$ is

$$\Pi^\psi_\alpha(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi_\alpha(x)} = i\psi^\dagger_\alpha \tag{7.3}$$

Thus, they obey the equal-time Poisson Brackets

$$\{\psi_\alpha(\boldsymbol{x}), \Pi^\psi_\beta(\boldsymbol{y})\}_{PB} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{7.4}$$

Thus

$$\{\psi_\alpha(\boldsymbol{x}), \psi^\dagger_\beta(\boldsymbol{y})\}_{PB} = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{7.5}$$

In other words the field ψ_α and its adjoint $\psi^\dagger_\alpha$ are a canonical pair. This result follows from the fact that the Dirac Lagrangian is *first order* in time derivatives. Notice that the field theory of non-relativistic many-particle systems (for both fermions and bosons) also has a Lagrangian which is first order in time derivatives. We will see that, because of this property, the quantum field theory of both types of systems follows rather similar lines, at least at a formal level. As in the case of the many-particle systems, two types of statistics are available to us: Fermi-Dirac and Bose-Einstein. We will see that only the choice of *Fermi statistics* leads to a physically meaningful theory of the Dirac equation.

The Hamiltonian for the Dirac theory is

$$H = \int d^3x \, \bar{\psi}_\alpha(\boldsymbol{x}) \Big[-i\boldsymbol{\gamma} \cdot \nabla + m \Big]_{\alpha\beta} \psi_\beta(\boldsymbol{x}) \tag{7.6}$$

where the fields $\psi(x)$ and $\bar{\psi} = \psi^\dagger \gamma_0$ are operators which act on a Hilbert space to be specified below. Notice that the one-particle operator in Eq. (7.6) is just the one-particle Dirac Hamiltonian obtained if we regard the Dirac Equation as a Schrödinger Equation for spinors. We will leave the issue of their commutation relations (Fermi or Bose) open for the time being. In any event, the equations of motion are *independent* of that choice (do not depend on the statistics).

In the Heisenberg representation, we find

$$i\gamma_0 \partial_0 \psi = \Big[\gamma_0 \psi, H \Big] = (-i\boldsymbol{\gamma} \cdot \nabla + m)\psi \tag{7.7}$$

which is just the Dirac equation.

We will solve this equation by means of a Fourier expansion in modes of the form

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{\omega(p)} \left(\tilde{\psi}_+(p) e^{-ip\cdot x} + \tilde{\psi}_-(p) e^{ip\cdot x} \right) \tag{7.8}$$

where $\omega(p)$ is a quantity with units of energy, and which will turn out to be equal to $p_0 = \sqrt{\boldsymbol{p}^2 + m^2}$, and $p \cdot x = p_0 x_0 - \boldsymbol{p} \cdot \boldsymbol{x}$. In terms of $\tilde{\psi}_\pm(p)$, the Dirac equation becomes

$$(p_0 \gamma_0 - \boldsymbol{\gamma} \cdot \boldsymbol{p} \pm m) \, \tilde{\psi}_\pm(p) = 0 \tag{7.9}$$

In other words, $\tilde{\psi}_\pm(p)$ creates one-particle states with energy $\pm p_0$. Let us make the substitution

$$\tilde{\psi}_\pm(p) = (\pm \not{p} + m) \tilde{\phi} \tag{7.10}$$

We get

$$(\not{p} \mp m)(\pm \not{p} + m)\tilde{\phi} = \pm(p^2 - m^2)\tilde{\phi} = 0 \tag{7.11}$$

This equation has non-trivial solutions only if the mass-shell condition is obeyed

$$p^2 - m^2 = 0 \tag{7.12}$$

Thus, we can identify $p_0 = \omega(\boldsymbol{p}) = \sqrt{\vec{p}^2 + m^2}$. At zero momentum these states become

$$\tilde{\psi}_\pm(p_0, \boldsymbol{p} = 0) = (\pm p_0 \gamma_0 + m)\tilde{\phi} \tag{7.13}$$

where $\tilde{\phi}$ is an arbitrary 4-spinor. Let us choose $\tilde{\phi}$ to be an eigenstate of γ_0. Recall that in the Dirac representation γ_0 is diagonal

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{7.14}$$

Thus the spinors $u^{(1)}(m, \boldsymbol{0})$ and $u^{(2)}(m, \boldsymbol{0})$

$$u^{(1)}(m, \boldsymbol{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u^{(2)}(m, \boldsymbol{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.15}$$

have γ_0-eigenvalue $+1$

$$\gamma_0 u^{(i)}(m, \boldsymbol{0}) = +u^{(\sigma)}(m, \boldsymbol{0}) \qquad \sigma = 1, 2 \tag{7.16}$$

and the spinors $v^{(\sigma)}(m, \boldsymbol{0})(\sigma = 1, 2)$

$$v^{(1)}(m, \boldsymbol{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v^{(2)}(m, \boldsymbol{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.17}$$

have γ_0-eigenvalue -1,

$$\gamma_0 v^{(\sigma)}(m, \boldsymbol{0}) = -v^{(\sigma)}(m, \boldsymbol{0}) \qquad \sigma = 1, 2 \tag{7.18}$$

Let $\varphi^{(i)}(m, 0)$ be the 2-spinors $(\sigma = 1, 2)$

$$\varphi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \varphi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.19}$$

In terms of $\varphi^{(i)}$ the solutions are

$$\tilde{\psi}_+(p) = u^{(\sigma)}(p) = \frac{(\not{p} + m)}{\sqrt{2m(p_0 + m)}} u^{(\sigma)}(m, \boldsymbol{0}) = \begin{pmatrix} \sqrt{\frac{p_0+m}{2m}} & \varphi^{(\sigma)}(m, 0) \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(p_0+m)}} & \varphi^{(\sigma)}(m, 0) \end{pmatrix} \tag{7.20}$$

and

$$\tilde{\psi}_-(p) = v^{(\sigma)}(p) = \frac{(-\not{p} + m)}{\sqrt{2m(p_0 + m)}} v^{(\sigma)}(m, \mathbf{0}) = \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(p_0+m)}} & \varphi^{(\sigma)}(m, \mathbf{0}) \\ \sqrt{\frac{p_0+m}{2m}} & \varphi^{(\sigma)}(m, \mathbf{0}) \end{pmatrix} \tag{7.21}$$

where the two solutions $\tilde{\psi}_+(p)$ have energy $+p_0 = +\sqrt{\boldsymbol{p}^2 + m^2}$ while the two solutions $\tilde{\psi}_-(p)$ have energy $-p_0 = -\sqrt{\boldsymbol{p}^2 + m^2}$.

Therefore, the one-particle states of the Dirac theory can have both *positive* and *negative* energy and, as it stands, the spectrum of the one-particle Dirac Hamiltonian, shown schematically in Fig. 7.1, is *not* positive. In addition, each Dirac state has a two-fold degeneracy due to spin.

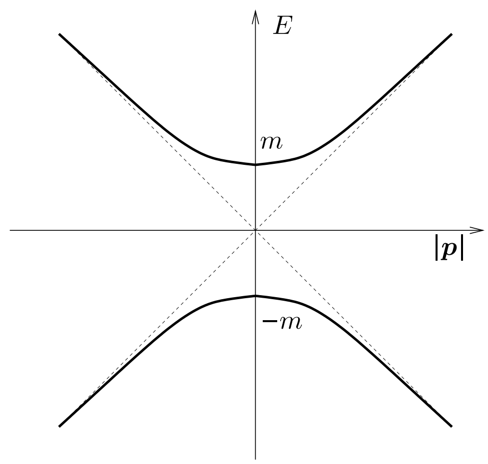

Figure 7.1 Single particle spectrum of the Dirac theory.

The spinors $u^{(i)}$ and $v^{(i)}$ obey the orthogonality conditions

$$\begin{aligned} \bar{u}^{(\sigma)}(p)u^{(\nu)}(p) =&\delta_{\sigma\nu} \\ \bar{v}^{(\sigma)}(p)v^{(\nu)}(p) =& -\delta_{\sigma\nu} \\ \bar{u}^{(\sigma)}(p)v^{(\nu)}(p) =&0 \end{aligned} \tag{7.22}$$

where $\bar{u} = u^\dagger \gamma_0$ and $\bar{v} = v^\dagger \gamma_0$. It is straightforward to check that the operators $\Lambda_\pm(p)$

$$\Lambda_\pm(p) = \frac{1}{2m}(\pm\not{p} + m) \tag{7.23}$$

are *projection operators* of the spinors onto the subspaces with positive (Λ_+)

and negative (Λ_-) energy respectively. These operators satisfy

$$\Lambda_\pm^2 = \Lambda_\pm \qquad \text{Tr}\, \Lambda_\pm = 2 \qquad \Lambda_+ + \Lambda_- = 1 \tag{7.24}$$

Hence, the four 4-spinors $u^{(\sigma)}$ and $v^{(\sigma)}$ are orthonormal and complete natural bases of the Hilbert space of single-particle states.

We can use these results to write the expansion of the field operator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \left[a_{\sigma,+}(\boldsymbol{p}) u_+^{(\sigma)}(p) e^{-ip\cdot x} + a_{\sigma,-}(\boldsymbol{p}) v_-^{(\sigma)}(p) e^{ip\cdot x} \right] \tag{7.25}$$

where the coefficients $a_{\sigma,\pm}(\boldsymbol{p})$ are operators with as yet unspecified commutation relations. The (formal) Hamiltonian for this system is

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} p_0 [a_{\sigma,+}^\dagger(\boldsymbol{p}) a_{\sigma,+}(p) - a_{\sigma,-}^\dagger(\boldsymbol{p}) a_{\sigma,-}(\boldsymbol{p})] \tag{7.26}$$

Since the single-particle spectrum does not have a lower bound, any attempt to quantize the theory with *canonical commutation relations* will have the problem that the *total energy of the system is not bounded from below.* In other words "Dirac bosons" do not have a ground state and the system is unstable since we can put as many bosons as we wish in states with arbitrarily large but negative energy.

Dirac realized that the simple and elegant way out of this problem was to *require* the electrons to obey the Pauli Exclusion Principle since, in that case, there is a natural and stable ground state. However, this assumption implies that the Dirac theory must be quantized as a theory of *fermions.* Hence we are led to quantize the theory with *canonical anticommutation relations*

$$\begin{aligned} \{a_{s,\sigma}(\boldsymbol{p}), a_{s',\sigma'}(\boldsymbol{p}')\} &= 0 \\ \{a_{s,\sigma}(\boldsymbol{p}), a_{s',\sigma'}^\dagger(\boldsymbol{p}')\} &= (2\pi)^3 \frac{p_0}{m} \delta^3(\boldsymbol{p} - \boldsymbol{p}') \delta_{ss'} \delta_{\sigma\sigma'} \end{aligned} \tag{7.27}$$

where $s = \pm$. Let us denote by $|0\rangle$ the state anihilated by the operators $a_{s,\sigma}(\boldsymbol{p})$,

$$a_{s,\sigma}(\boldsymbol{p})|0\rangle = 0 \tag{7.28}$$

We will see now that this state is not the vacuum (or ground state) of the Dirac theory.

7.1.1 Ground state and normal ordering

We will show now that the *ground state* or *vacuum* $|\text{vac}\rangle$ is the state in which *all the negative energy states are filled* (as shown in Fig.7.2)

$$|\text{vac}\rangle = \prod_{\sigma,\boldsymbol{p}} a^\dagger_{-,\sigma}(\boldsymbol{p})|0\rangle \tag{7.29}$$

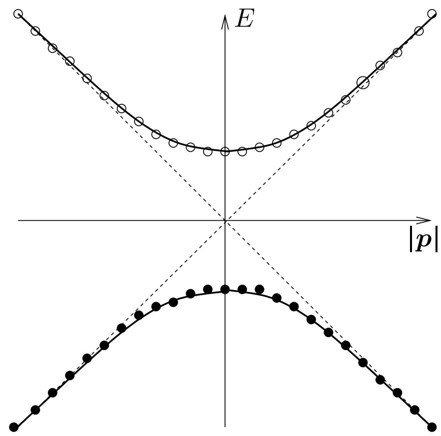

Figure 7.2 Ground State of the Dirac theory.

We will normal-order all the operators relative to the vacuum state $|\text{vac}\rangle$. This amounts to a particle-hole transformation for the negative energy states. Thus, we define the fermion creation and annihilation operators $b_\sigma(\boldsymbol{p}), b^\dagger_\sigma(\boldsymbol{p})$ and $d_\sigma(\boldsymbol{p}), d^\dagger_\sigma(\boldsymbol{p})$ to be

$$\begin{aligned} b_\sigma(\boldsymbol{p}) &= a_{\sigma,+}(\boldsymbol{p}) \\ d_\sigma(\boldsymbol{p}) &= a^\dagger_{\sigma,-}(\boldsymbol{p}) \end{aligned} \tag{7.30}$$

which obey

$$b_\sigma(\boldsymbol{p})|\text{vac}\rangle = d_\sigma(\boldsymbol{p})|\text{vac}\rangle = 0 \tag{7.31}$$

The Hamiltonian now reads

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}p_0 \sum_{\sigma=1,2}[b^\dagger_\sigma(\boldsymbol{p})b_\sigma(\boldsymbol{p}) - d_\sigma(\boldsymbol{p})d^\dagger_\sigma(\boldsymbol{p})] \tag{7.32}$$

We now normal order $\hat{H}$ relative to the vacuum state

$$H =: H : +E_0 \tag{7.33}$$

with a normal-ordered Hamiltonian

$$: H : = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} p_0 \Big[b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) + d_\sigma^\dagger(\boldsymbol{p}) d_\sigma(\boldsymbol{p}) \Big] \tag{7.34}$$

The constant E_0 is the (negative and divergent) ground state energy

$$E_0 = -2V \int d^3p \sqrt{\boldsymbol{p}^2 + m^2} \tag{7.35}$$

similar to the expression we already countered in the Klein-Gordon theory, but with opposite sign. The factor of 2 is due to spin.

In terms of the operators b_σ and d_σ the Dirac field has the mode expansion

$$\psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \Big[b_\sigma(\boldsymbol{p}) u_\alpha^{(\sigma)}(\boldsymbol{p}) e^{-ip\cdot x} + d_\sigma^\dagger(\boldsymbol{p}) v_\alpha^{(\sigma)}(\boldsymbol{p}) e^{ip\cdot x} \Big] \tag{7.36}$$

which satisfy equal-time canonical anticommutation relations

$$\begin{aligned} \{\psi_\alpha(\boldsymbol{x}), \psi_\beta^\dagger(\boldsymbol{x}')\} &= \delta_{\alpha\beta}\delta^3(\boldsymbol{x} - \boldsymbol{x}') \\ \{\psi_\alpha(\boldsymbol{x}), \psi_\beta(\boldsymbol{x}')\} &= \{\psi_\alpha^\dagger(\boldsymbol{x}), \psi_\beta^\dagger(\boldsymbol{x}')\} = 0 \end{aligned} \tag{7.37}$$

7.1.2 One-particle states

The excitations of this theory can be constructed by using the same methods employed for non-relativistic many-particle systems. Let us first construct the total four-momentum operator P^μ

$$P^\mu = \int d^3x \, T^{0\mu} = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} p^\mu \sum_{\sigma=1,2} : b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) - d_\sigma(\boldsymbol{p}) d_\sigma^\dagger(\boldsymbol{p}) : \tag{7.38}$$

Hence

$$: P^\mu := \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} p^\mu \sum_{\sigma=1,2} \Big[b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) + d_\sigma^\dagger(\boldsymbol{p}) d_\sigma(\boldsymbol{p}) \Big] \tag{7.39}$$

The states $b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$ and $d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$ have energy $p_0 = \sqrt{\boldsymbol{p}^2 + m^2}$ and momentum $\boldsymbol{p}$,

$$\begin{aligned} : H : b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle &= p_0 b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle \\ : H : d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle &= p_0 d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle \\ : P^i : b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle &= p^i b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle \\ : P^i : d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle &= p^i d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle \end{aligned} \tag{7.40}$$

We see that there are four different states which have the same energy and momentum. Let us find quantum numbers to classify these states.

7.1.3 Spin

The angular momentum tensor $\mathcal{M}_{\mu\nu\lambda}$ for the Dirac theory is

$$\mathcal{M}_{\mu\nu\lambda} = \int d^3x\, \bar{\psi}(x)\, \gamma^\mu \left[i\left(x^\nu\partial^\lambda - x^\lambda\partial^\nu\right) + \frac{1}{2}\sigma^{\nu\lambda}\right] \psi(x) \tag{7.41}$$

where $\sigma^{\nu\lambda}$ is the matrix

$$\sigma^{\nu\lambda} = \frac{i}{2}\left[\gamma^\nu, \gamma^\lambda\right] \tag{7.42}$$

The conserved angular momentum $J^{\nu\lambda}$ is

$$J^{\nu\lambda} = \mathcal{M}^{0\nu\lambda} = \int d^3x\, \psi^\dagger(x)\left[i(x^\nu\partial^\lambda - x^\lambda\partial^\nu) + \frac{1}{2}\sigma^{\nu\lambda}\right] \psi(x) \tag{7.43}$$

In particular, out of its space components J^{ij}, we can construct the total angular momentum three-vector $\boldsymbol{J}$

$$J^i = \frac{1}{2}\epsilon^{ijk}J^{jk} = \int d^3x\, \psi^\dagger \left(i\epsilon^{ijk}x^j\partial^k + \frac{1}{2}\epsilon^{ijk}\sigma^{jk}\right)\psi \tag{7.44}$$

It is easy to check that, in the Dirac representation, the last term represents the spin.

In the quantized theory, the angular momentum operator is

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{7.45}$$

where $\boldsymbol{L}$ is the *orbital angular momentum*

$$\boldsymbol{L} = \int d^3x\, \psi^\dagger(\boldsymbol{x})\boldsymbol{x} \times i\boldsymbol{\partial}\psi(\boldsymbol{x}) \tag{7.46}$$

while $\boldsymbol{S}$ is the *spin*

$$\boldsymbol{S} = \int d^3x\, \psi^\dagger(\boldsymbol{x})\, \boldsymbol{\Sigma}\, \psi(\boldsymbol{x}) \tag{7.47}$$

where $\boldsymbol{\Sigma}$ is the 4×4 matrix

$$\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \equiv \frac{1}{2}\boldsymbol{\sigma} \tag{7.48}$$

In order to measure the spin polarization of a state we first go to the rest frame in which $\boldsymbol{p} = 0$. In this frame we can consider the four-vector W^μ

$$W^\mu = (0, m\boldsymbol{\Sigma}) \tag{7.49}$$

Let n^μ be the space-like 4-vector $n^\mu = (0, \boldsymbol{n})$, where $\boldsymbol{n}$ has unit length. Thus, $n^\mu n_\mu = -1$. We will use n^μ to fix the direction of polarization in the rest frame.

The scalar product $W_\mu n^\mu$ is a Lorentz invariant scalar and, hence, its value is independent of the choice of frame. In the rest frame we have

$$W_\mu n^\mu = -m\boldsymbol{n}\cdot\boldsymbol{\Sigma} \equiv -\frac{m}{2}\vec{n}\cdot\boldsymbol{\sigma} = -\frac{m}{2}\begin{pmatrix}\boldsymbol{n}\cdot\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{n}\cdot\boldsymbol{\sigma}\end{pmatrix} \tag{7.50}$$

In particular, if $\boldsymbol{n} = e_z$ then $W_\mu n^\mu$ is

$$W_\mu n^\mu = -\frac{m}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} \tag{7.51}$$

which is diagonal. The operator $-\frac{1}{m}W\cdot n$ is a Lorentz scalar which measures the spin polarization:

$$\begin{aligned}
-\frac{1}{m}W\cdot n u^{(1)}(p) &= +\frac{1}{2}u^{(1)}(p)\\
-\frac{1}{m}W\cdot n u^{(2)}(p) &= -\frac{1}{2}u^{(2)}(p)\\
-\frac{1}{m}W\cdot n v^{(1)}(p) &= +\frac{1}{2}v^{(1)}(p)\\
-\frac{1}{m}W\cdot n v^{(2)}(p) &= -\frac{1}{2}v^{(2)}(p)
\end{aligned} \tag{7.52}$$

It is straightforward to check that $-\frac{1}{m}W\cdot n$ is the Lorentz scalar

$$-\frac{1}{m}W\cdot n = \frac{1}{4m}\epsilon_{\mu\nu\lambda\rho}n^\mu p^\nu \sigma^{\lambda\rho} = \frac{1}{2m}\gamma_5 \not{n}\not{p} \tag{7.53}$$

which enables us to write the spin projection operator $P(n)$

$$P(n) = \frac{1}{2}(I + \gamma_5 \not{n}) \tag{7.54}$$

where we used that

$$\begin{aligned}
\frac{1}{2m}\gamma_5 \not{n}\not{p}\, u^{(\sigma)}(p) &= \frac{1}{2}\gamma_5 \not{n}\, u^{(\sigma)}(p) = (-1)^\sigma \frac{1}{2}u^{(\sigma)}(p)\\
\frac{1}{2m}\gamma_5 \not{n}\not{p}\, v^{(\sigma)}(p) &= -\frac{1}{2}\gamma_5 \not{n}\, v^{(\sigma)}(p) = (-1)^\sigma \frac{1}{2}v^{(\sigma)}(p)
\end{aligned} \tag{7.55}$$

7.1.4 Charge

The Dirac Lagrangian is invariant under the global (phase) transformation

$$\begin{aligned} \psi &\to \psi' = e^{i\alpha}\psi \\ \bar{\psi} &\to \bar{\psi}' = e^{-i\alpha}\bar{\psi} \end{aligned} \tag{7.56}$$

Consequently, it has a locally conserved current j^μ

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{7.57}$$

which is also locally gauge invariant. As a result it has a conserved total charge $Q = -e\int d^3x j^0(x)$. The corresponding operator in the quantized theory Q is

$$Q = -e\int d^3x\, j^0(x) = -e\int d^3x\, \psi^\dagger(x)\psi(x) \tag{7.58}$$

The total charge operator Q commutes with the Dirac Hamiltonian $\hat{H}$

$$[Q, H] = 0 \tag{7.59}$$

Hence, the eigenstates of the Hamiltonian H have a well defined charge.

In terms of the creation and annihilation operators, the total charge operator Q becomes

$$Q = -e\int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\sum_{\sigma=1,2}\left(b_\sigma^\dagger(\boldsymbol{p})b_\sigma(\boldsymbol{p}) + d_\sigma(\boldsymbol{p})d_\sigma^\dagger(\boldsymbol{p})\right) \tag{7.60}$$

which is not normal-ordered relative to $|\text{vac}\rangle$. The normal-ordered charge operator $:Q:$ is

$$:Q: = -e\int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\sum_{\sigma=1,2}\left[b_\sigma^\dagger(\boldsymbol{p})b_\sigma(\boldsymbol{p}) - d_\sigma^\dagger(\boldsymbol{p})d_\sigma(\boldsymbol{p})\right] \tag{7.61}$$

and we can write

$$Q = :Q: + Q_{\text{vac}} \tag{7.62}$$

where Q_{vac} is the unobservable (and divergent) vacuum charge

$$Q_{\text{vac}} = -eV\int \frac{d^3p}{(2\pi)^3} \tag{7.63}$$

V being the volume of space. From now on we will *define* the charge to be the subtracted charge operator

$$:Q: = Q - Q_{\text{vac}} \tag{7.64}$$

which annihilates the vacuum state

$$:Q:|\text{vac}\rangle = 0 \tag{7.65}$$

the vacuum is neutral. In other words, we measure the *charge of a state* relative to the vacuum charge which we define to be zero. Equivalently, this amounts to a definition of the order of the operators in $:Q:$

$$:Q: = -e\int d^3x \frac{1}{2}[\psi^\dagger(x), \psi(x)] \tag{7.66}$$

The one-particle states $b^\dagger_\sigma|\text{vac}\rangle$ and $d^\dagger_\sigma|\text{vac}\rangle$ have well defined charge:

$$\begin{aligned} :Q: b^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle &= -\,eb^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle \\ :Q: d^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle &= +\,ed^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle \end{aligned} \tag{7.67}$$

Hence we identify the state $b^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle$ with an *electron* of charge $-e$, spin σ, momentum $\boldsymbol{p}$ and energy $p_0 = \sqrt{\boldsymbol{p}^2 + m^2}$. Similarly, the state $d^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle$ is a *positron* with the same quantum numbers and energy of the electron but with *positive charge* $+e$.

7.1.5 The Dirac energy-momentum tensor

The energy-momentum tensor of the Dirac theory is obtained following the standard approach presented earlier. On general grounds, we expect that the energy-momentum tensor should be given by

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \mathcal{L} \tag{7.68}$$

Using the equation of motion of the free Dirac filed, i.e the Dirac equation, we find that the energy-momentum tensor for the Dirac theory reduces to

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi \tag{7.69}$$

From this expression it follows that the Hamiltonian is

$$H = \int d^3x\, T^{00} i\bar{\psi}\gamma^0\partial^0\psi = \int d^3x \psi^\dagger\gamma^0(-i\boldsymbol{\gamma}\cdot\boldsymbol{\partial} + m)\psi \tag{7.70}$$

which is indeed the Hamiltonian of the Dirac field.

7.1.6 Causality and the spin-statistics connection

Let us finally discuss the question of causality and the spin-statistics connection in the Dirac theory. To this end we will consider the *anticommutator* of two Dirac fields at different times

$$i\Delta_{\alpha\beta}(x-y) = \{\psi_\alpha(x), \psi_\beta(y)\} \tag{7.71}$$

By using the field expansion we obtain the expression

$$i\Delta_{\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\sum_{\sigma=1,2}\left[e^{-ip\cdot(x-y)}u_\alpha^{(\sigma)}(p)\bar{u}_\beta^\sigma(p)+e^{ip\cdot(x-y)}v_\alpha^{(\sigma)}(p)\bar{v}_\beta^{(\sigma)}(p)\right] \tag{7.72}$$

By using the (completeness) identities

$$\begin{aligned}\sum_{\sigma=1,2} u_\alpha^{(\sigma)}(p)\bar{u}_\beta^{(\sigma)}(p) &= \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta}\\ \sum_{\sigma=1,2} v_\alpha^{(\sigma)}(p)\bar{v}_\beta^{(\sigma)}(p) &= \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta}\end{aligned} \tag{7.73}$$

we can write the anticommutator in the form

$$i\Delta_{\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\left[\left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta}e^{-ip\cdot(x-y)} + \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta}e^{ip\cdot(x-y)}\right] \tag{7.74}$$

After some straightforward algebra, we obtain the result

$$\begin{aligned}i\Delta_{\alpha\beta}(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}(i\not{\partial}_x+m)\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right]\\ &= (i\not{\partial}_x+m)_{\alpha\beta}\int\frac{d^3p}{(2\pi)^3 2p_0}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right]\end{aligned} \tag{7.75}$$

We recognize the integral on the r.h.s. of Eq.(7.75) to be the commutator of two free scalar (Klein-Gordon) fields, $\Delta_{KG}(x-y)$.

Hence, the anticommutator two Dirac fields of the Dirac theory is

$$i\Delta_{\alpha\beta}(x-y) = (i\not{\partial}+m)_{\alpha\beta}\, i\Delta_{KG}(x-y) \tag{7.76}$$

Since $\Delta_{KG}(x-y)$ *vanishes* at space-like separations, so does $\Delta_{\alpha\beta}(x-y)$. Hence, the Dirac theory quantized with anticommutators obeys causality.

On the other hand, had we had quantized the Dirac theory with commutators (which, as we saw, leads to a theory without a ground state) we would have also found a violation of causality. Indeed, we would have obtained instead the result

$$\Delta_{\alpha\beta}(x-y) = (i\not{\partial}+m)_{\alpha\beta}\tilde{\Delta}(x-y) \tag{7.77}$$

where $\tilde{\Delta}(x-y)$ is given by

$$\tilde{\Delta}(x-y) = \int \frac{d^3p}{(2\pi)^3 2p_0} \left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)} \right) \tag{7.78}$$

which does not vanish at space-like separations. Instead, at equal times and at long distances $\widetilde{\Delta}(x-y)$ decays as

$$\tilde{\Delta}(R,0) \simeq \frac{e^{-mR}}{R^2}, \qquad \text{for } mR \gg 1 \tag{7.79}$$

Thus, if the Dirac theory were to be quantized with commutators, the field operators *would not commute* at equal times at distances shorter than the Compton wavelength. This would be a violation of locality. The same result holds in the theory of the scalar field if it is quantized with anticommutators.

These results can be summarized in the *Spin-Statistics Theorem*: fields with *half-integer spin* must be quantized as *fermions*, obey canonical *anti-commutation* relations, whereas fields with *integer spin* must be quantized as *bosons*, obey canonical *commutation* relations. If a field theory is quantized with the wrong spin-statistics connection, either the theory becomes non-local, with violations of causality, and/or it does not have a ground state, or it contains states in its spectrum with negative norm. Notice that the arguments we have used were derived for free local theories. It is a highly non-trivial task to prove that the spin-statistics connection also remains valid for interacting theories. Although this can be done by making sufficiently strong assumptions of the behavior of perturbation theory, in reality it must the spin-statistics connection must be regarded as an *axiom* of local relativistic quantum field theories.

7.2 The Propagator of the Dirac spinor field

We will now compute the propagator for a spinor field $\psi_\alpha(x)$. We will find that it is essentially the Green function for the the Dirac operator. The propagator is defined by

$$S_{\alpha\beta}(x-x') = -i\langle \text{vac}|T\psi_\alpha(x)\bar{\psi}_\beta(x')|\text{vac}\rangle \tag{7.80}$$

where we have used the *time ordered product* of two fermionic field operators, which is defined by

$$T\psi_\alpha(x)\bar{\psi}_\beta(x') = \theta(x_0 - x_0')\psi_\alpha(x)\bar{\psi}_\beta(x') - \theta(x_0' - x_0)\bar{\psi}_\beta(x')\psi_\alpha(x) \tag{7.81}$$

Notice the change in sign with respect to the time ordered product of bosonic operators. The sign change reflects the anticommutation properties of the field. In other words, inside a time ordered product, Fermi fields behave as if they were anticommuting c-numbers.

We will show now that this propagator is closely connected to the propagator of the free scalar field, the Green function for the Klein-Gordon operator $\partial^2 + m^2$.

By acting with the Dirac operator on $S_{\alpha\beta}(x-x')$ we find

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} S_{\beta\lambda}(x-x') = -i\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} \langle \text{vac}|T\psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle \tag{7.82}$$

We now use that

$$\frac{\partial}{\partial x_0}\theta(x_0 - x_0') = \delta(x_0 - x_0') \tag{7.83}$$

and the fact that the equation of motion of the Heisenberg field operators ψ_α is the Dirac equation,

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} \psi_\beta(x) = 0 \tag{7.84}$$

to show that

$$\begin{aligned}\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} S_{\beta\lambda}(x-x') &= -i\langle\text{vac}|T\left(i\partial\!\!\!/ - m\right)_{\alpha\beta}\psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle \\ &+ \delta(x_0 - x_0')\Big(\langle\text{vac}|\gamma^0_{\alpha\beta}\psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle \\ &+ \langle\text{vac}|\bar{\psi}_\lambda(x')\gamma^0_{\alpha\beta}\psi_\beta(x)|\text{vac}\rangle\Big) \\ &= \delta(x_0 - x_0')\gamma^0_{\alpha\beta}\langle\text{vac}|\left\{\psi_\beta(x), \psi^\dagger_\nu(x')\right\}|\text{vac}\rangle\gamma^0_{\nu\lambda} \\ &= \delta(x_0 - x_0')\delta^3(\boldsymbol{x} - \boldsymbol{x}')\delta_{\alpha\lambda}\end{aligned} \tag{7.85}$$

Therefore we find that $S_{\beta\lambda}(x-x')$ is the solution of the equation

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} S_{\beta\lambda}(x-x') = \delta^4(x-x')\delta_{\alpha\lambda} \tag{7.86}$$

Hence $S_{\beta\lambda}(x-x') = -i\langle\text{vac}|T\psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle$ is the Green function of the Dirac operator.

We saw before that there is a close connection between the Dirac and the Klein-Gordon operators. We will now use this connection to relate their propagators. Let us write the Green function $S_{\alpha\lambda}(x-x')$ in the form

$$S_{\alpha\lambda}(x-x') = \left(i\partial\!\!\!/ + m\right)_{\alpha\beta} G_{\beta\lambda}(x-x') \tag{7.87}$$

Since $S_{\alpha\lambda}(x-x')$ satisfies Eq.(7.86), we find that

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} S_{\beta\lambda}(x-x') = \left(i\partial\!\!\!/ - m\right)_{\alpha\beta}\left(i\partial\!\!\!/ + m\right)_{\beta\nu} G_{\nu\lambda}(x-x') \tag{7.88}$$

But

$$\left(i\not{\partial} - m\right)_{\alpha\beta} \left(i\not{\partial} + m\right)_{\beta\nu} = -\left(\partial^2 + m^2\right) \delta_{\alpha\nu} \tag{7.89}$$

Hence, $G_{\alpha\nu}(x - x')$ must satisfy

$$-\left(\partial^2 + m^2\right) G_{\alpha\nu}(x - x') = \delta^4(x - x')\delta_{\alpha\nu} \tag{7.90}$$

Therefore $G_{\alpha\nu}(x - x')$ is given by

$$G_{\alpha\nu}(x - x') = -G^{(0)}(x - x')\delta_{\alpha\nu} \tag{7.91}$$

where $G^{(0)}(x-x')$ is the propagator for a free massive scalar field, the Green function of the Klein-Gordon equation

$$\left(\partial^2 + m^2\right) G^{(0)}(x - x') = \delta^4(x - x') \tag{7.92}$$

We then conclude that the Dirac propagator $S_{\alpha\beta}(x - x')$, and the Klein-Gordon propagator $G^{(0)}(x - x')$ are related by

$$S_{\alpha\beta}(x - x') = -\left(i\not{\partial} + m\right)_{\alpha\beta} G^{(0)}(x - x') \tag{7.93}$$

In particular, this relationship implies that the have essentially the same asymptotic behaviors that we discussed for the free scalar field, a power-law behavior at short distances (albeit with a different power), and exponential (or oscillatory) behavior at large distances. The spinor structure of the Dirac propagator is determined by the operator in front of $G^{(0)}(x-x')$ in Eq.(7.93).

In momentum space, the Feynman propagator for the Dirac field, given by Eq. (7.93), becomes

$$S_{\alpha\beta}(p) = \left(\frac{\not{p} + m}{p^2 - m^2 + i\epsilon}\right)_{\alpha\beta} \tag{7.94}$$

Hence we get the same pole structure in the time-ordered propagator as we did for the Klein-Gordon field.

7.3 Discrete symmetries of the Dirac theory

We will now discuss three important discrete symmetries in relativistic field theories: *charge conjugation*, *parity* and *time reversal*. These discrete symmetries have a different role, and a different standing, than the continuous symmetries discussed before. In a relativistic quantum field theory the ground state, the *vacuum*, must be invariant under continuous Lorentz transformations, but it may not be invariant under $\mathcal{C}$, $\mathcal{P}$ or $\mathcal{T}$. However, in a local relativistic quantum field theory the product $\mathcal{CPT}$ is always a good symmetry. This is in fact an *axiom* of relativistic local quantum field theory. Thus, although $\mathcal{C}$, $\mathcal{P}$ or $\mathcal{T}$ may or may not to be good symmetries of the vacuum state, $\mathcal{CPT}$ *must* be a good symmetry. As in the case of the symmetries we discussed before, these symmetries must also be realized in the *Fock* space of the quantum field theory.

7.3.1 Charge conjugation

Charge conjugation is a symmetry that exchanges *particles* and *antiparticles* (or *holes*). Consider a Dirac field minimally coupled to an external electromagnetic field A_μ. The equation of motion for the Dirac field ψ is

$$\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi = 0 \tag{7.95}$$

We will *define* the charge conjugate field ψ^c

$$\psi^c(x) = \mathcal{C}\psi(x)\mathcal{C}^{-1} \tag{7.96}$$

where $\mathcal{C}$ is the (unitary) charge conjugation operator, $\mathcal{C}^{-1} = \mathcal{C}^\dagger$, such that ψ^c obeys

$$\left(i\partial\!\!\!/ + eA\!\!\!/ - m\right)\psi^c = 0 \tag{7.97}$$

Since $\bar\psi = \psi^\dagger\gamma^0$ obeys

$$\bar\psi\left[\gamma^\mu\left(-i\overleftarrow{\partial}_\mu - eA_\mu\right) - m\right] = 0 \tag{7.98}$$

which, when transposed, becomes

$$\left[\gamma^{\mu\,T}\left(-i\partial_\mu - eA_\mu\right) - m\right]\bar\psi^{\,T} = 0 \tag{7.99}$$

where T is the transpose, and

$$\bar\psi^{\,T} = \gamma^{0\,T}\psi^* \tag{7.100}$$

Let C be an invertible 4×4 matrix, where C^{-1} is its inverse. Then, we can write

$$C\left[\gamma^{\mu\,t}\left(-i\partial_\mu - eA_\mu\right) - m\right]C^{-1}C\bar\psi^{\,t} = 0 \tag{7.101}$$

such that

$$C\left(\gamma^{\,\mu}\right)^t C^{-1} = -\gamma^\mu \tag{7.102}$$

Hence,

$$\left[\left(i\partial\!\!\!/ + eA\!\!\!/\right) - m\right]C\bar\psi^{\,T} = 0 \tag{7.103}$$

For Eq. (7.103) to hold, we must have

$$\mathcal{C}\psi\mathcal{C}^\dagger = \psi^c = C\bar{\psi}^T = C\gamma^{0T}\psi^* \tag{7.104}$$

Hence the field ψ^c thus defined has *positive* charge $+e$.

We can find the charge conjugation matrix C explicitly:

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} = C^{-1} \tag{7.105}$$

In particular this means that ψ^c is given by

$$\psi^c = C\bar{\psi}^T = C\gamma^{0t}\psi^* = i\gamma^2\psi^* \tag{7.106}$$

Eq. (7.106) provides us with a definition for a charge neutral Dirac fermion, i.e. ψ represents a neutral fermion if $\psi = \psi^c$. Hence the condition is

$$\psi = i\gamma^2\psi^* \tag{7.107}$$

A Dirac fermion that satisfies the neutrality condition is known as a Majorana fermion.

To understand the action of C on physical states we can look for instance, at the charge conjugate u^c of the positive energy, up spin, and charge $-e$, spinor in the rest frame

$$u = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} e^{-imt} \tag{7.108}$$

which is

$$u^c = \begin{pmatrix} 0 \\ -i\gamma^2\varphi^* \end{pmatrix} e^{imt} \tag{7.109}$$

which has negative energy, down spin and charge $+e$.

At the level of the full quantum field theory, we will require the vacuum state $|\text{vac}\rangle$ to be invariant under charge conjugation:

$$\mathcal{C}|\text{vac}\rangle = |\text{vac}\rangle \tag{7.110}$$

How do one-particle states transform? To determine that we look at the action of charge conjugation $\mathcal{C}$ on the one-particle states, and demand that particle and anti-particle states to be exchanged under charge conjugation

$$\begin{aligned} \mathcal{C}b_\sigma^\dagger(p)|\text{vac}\rangle &= \mathcal{C}b_\sigma^\dagger(p)\mathcal{C}^{-1}\mathcal{C}|\text{vac}\rangle \equiv d_\sigma^\dagger(p)|\text{vac}\rangle \\ \mathcal{C}d_\sigma^\dagger(p)|\text{vac}\rangle &= \mathcal{C}d_\sigma^\dagger(p)\mathcal{C}^{-1}\mathcal{C}|\text{vac}\rangle \equiv b_\sigma^\dagger(p)|\text{vac}\rangle \end{aligned} \tag{7.111}$$

Hence, for the one-particle states to satisfy these rules it is sufficient to require that the field operators $b_\sigma(\boldsymbol{p})$ and $d_\sigma(\boldsymbol{p})$ satisfy

$$\mathcal{C} b_\sigma(\boldsymbol{p}) \mathcal{C}^\dagger = d_\sigma(\boldsymbol{p}); \qquad \mathcal{C} d_\sigma(\boldsymbol{p}) \mathcal{C}^\dagger = b_\sigma(\boldsymbol{p}) \tag{7.112}$$

Using that

$$u_\sigma(\boldsymbol{p}) = -i\gamma^2 \left(v_\sigma(\boldsymbol{p})\right)^*; \qquad v_\sigma(\boldsymbol{p}) = -i\gamma^2 \left(u_\sigma(\boldsymbol{p})\right)^* \tag{7.113}$$

we find that the field operator $\psi(x)$ transforms as

$$\mathcal{C}\psi(x)\mathcal{C}^\dagger = \left(-i\bar{\psi}\gamma^0\gamma^2\right)^T \tag{7.114}$$

and

$$\mathcal{C}\bar{\psi}(x)\mathcal{C}^\dagger = \left(-i\gamma^0\gamma^2\psi\right)^T \tag{7.115}$$

In particular the fermionic bilinears we discussed before satisfy the transformation laws:

$$\begin{aligned} \mathcal{C}\bar{\psi}\psi\mathcal{C}^\dagger &= +\bar{\psi}\psi, & \mathcal{C}i\bar{\psi}\gamma^5\psi\mathcal{C}^\dagger &= i\bar{\psi}\gamma^5\psi, \\ \mathcal{C}\bar{\psi}\gamma^\mu\psi\mathcal{C}^\dagger &= -\bar{\psi}\gamma^\mu\psi, & \mathcal{C}\bar{\psi}\gamma^\mu\gamma^5\psi\mathcal{C}^\dagger &= +\bar{\psi}\gamma^\mu\gamma^5\psi \end{aligned} \tag{7.116}$$

7.3.2 Parity

We will define as *parity* the transformation $\mathcal{P} = \mathcal{P}^{-1}$ which reverses the momentum of a particle but not its spin. Once again, the vacuum state is invariant under parity. Thus, we must require

$$\begin{aligned} \mathcal{P} b^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle &= \mathcal{P} b^\dagger_\sigma(\boldsymbol{p})\mathcal{P}^{-1}\mathcal{P}|\text{vac}\rangle \equiv b^\dagger_\sigma(-\boldsymbol{p})|\text{vac}\rangle \\ \mathcal{P} d^\dagger_\sigma(\boldsymbol{p})|\text{vac}\rangle &= \mathcal{P} d^\dagger_\sigma(\boldsymbol{p})\mathcal{P}^{-1}\mathcal{P}|\text{vac}\rangle \equiv d^\dagger_\sigma(-\boldsymbol{p})|\text{vac}\rangle \end{aligned} \tag{7.117}$$

In real space this transformation is equivalent to

$$\mathcal{P}\psi(\boldsymbol{x}, x_0)\mathcal{P}^{-1} = \gamma^0\psi(-\boldsymbol{x}, x_0), \quad \mathcal{P}\bar{\psi}(\boldsymbol{x}, x_0)\mathcal{P}^{-1} = \bar{\psi}(-\boldsymbol{x}, x_0)\gamma_0 \tag{7.118}$$

7.3.3 Time reversal

Finally, we discuss time reversal $\mathcal{T}$. We will define $\mathcal{T}$ as the operator

$$\mathcal{T} e^{iHx_0}\mathcal{T}^{-1} = e^{-iHx_0}, \tag{7.119}$$

We will require the time reversal operator to be unitary in the sense that

$$\mathcal{T}^{-1} = \mathcal{T}^\dagger \tag{7.120}$$

However we will also require the operator to act on c-numbers as complex conjugation, i.e.

$$\mathcal{T}\ (\text{c-number}) = (\text{c-number})^*\ \mathcal{T} \tag{7.121}$$

An operator with these properties is said to be anti-linear (anti-unitary).

As a result, time reversal is the operator that which reverses the momentum *and* the spin of the particles:

$$\mathcal{T} b_\sigma(\boldsymbol{p})\mathcal{T}^\dagger = b_{-\sigma}(-\boldsymbol{p}), \qquad \mathcal{T} d_\sigma(\boldsymbol{p})\mathcal{T}^\dagger = d_{-\sigma}(-\boldsymbol{p}) \tag{7.122}$$

while leaving the vacuum state invariant:

$$\mathcal{T}|\text{vac}\rangle = |\text{vac}\rangle \tag{7.123}$$

In real space this implies:

$$\mathcal{T}\psi(\boldsymbol{x}, x_0)\mathcal{T}^\dagger = -\gamma^1\gamma^3\ \psi^*(\boldsymbol{x}, -x_0) \tag{7.124}$$

7.4 Chiral symmetry

We will now discuss a global symmetry specific of theories of spinors known as chiral symmetry. Let us consider again the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{7.125}$$

Let us define the *chiral transformation*

$$\psi' = e^{i\gamma_5\theta}\psi \tag{7.126}$$

where θ is *constant phase* in the range $0 \leq \theta < 2\pi$. We wish to find the Lagrangian for the new transformed field ψ. From

$$\mathcal{L} = \bar{\psi}e^{i\gamma_5\theta}\left(i\gamma^\mu\partial_\mu - m\right)e^{i\gamma_5\theta}\psi \tag{7.127}$$

and the fact that

$$\{\gamma_\mu, \gamma_5\} = 0 \tag{7.128}$$

after some simple algebra, which uses the identity

$$\gamma^\mu e^{i\gamma_5\theta} = e^{-i\gamma_5\theta}\gamma^\mu, \tag{7.129}$$

we find that the field ψ satisfies a modified Dirac Lagrangian of the form

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - me^{2i\gamma_5\theta}\right)\psi \tag{7.130}$$

Thus for $m \neq 0$, the form of the Dirac Lagrangian changes under a chiral transformation. However, if the theory is massless, the Dirac theory has an exact global chiral symmetry.

It is also instructive to determine how various fermion bilinears transform under the chiral transformation. We find that the Dirac mass , $\bar{\psi}\psi$, and the axial mass, $i\bar{\psi}\gamma^5\psi$, under a chiral transformation transform as follows

$$\begin{aligned} \bar{\psi}'\psi' &= \cos(2\theta)\bar{\psi}\psi + \sin(2\theta) i\bar{\psi}\gamma_5\psi \\ i\bar{\psi}'\gamma^5\psi' &= -\sin(2\theta)\bar{\psi}\psi + \cos(2\theta) i\bar{\psi}\gamma_5\psi \end{aligned} \tag{7.131}$$

and, hence, are not invariant under the chiral transformation. Instead, under the chiral transformation the Dirac mass, $\bar{\psi}\psi$, and the γ_5 mass, $i\bar{\psi}\gamma_5\psi$, transform (rotate) into each other. Finally we note that, although the Dirac Lagrangian is not chiral invariant if $m \neq 0$, the Dirac current

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}\gamma^\mu\psi \tag{7.132}$$

is invariant under the chiral transformation, and so is the coupling of the Dirac field to a gauge field.

7.5 Massless fermions

Let us look at the massless limit of the Dirac equation in more detail. Historically this problem grew out of the study of neutrinos (which we now know are not necessarily massless, at least not all of them). For an eigenstate of 4-momentum p_μ, the Dirac equation is

$$\left(\not{p} - m\right)\psi(p) = 0 \tag{7.133}$$

In the *massless limit* $m = 0$, the Dirac equation simply becomes

$$\not{p}\psi(p) = 0 \tag{7.134}$$

which is equivalent to

$$\gamma_5\gamma_0\not{p}\psi(p) = 0 \tag{7.135}$$

Upon expanding in components we find

$$\gamma_5 p_0\psi(p) = \gamma_5\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi(p) \tag{7.136}$$

However, since

$$\gamma_5\gamma_0\boldsymbol{\gamma} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} = \boldsymbol{\Sigma} \tag{7.137}$$

we can write

$$\gamma_5 p_0 \psi(p) = \boldsymbol{\Sigma} \cdot \boldsymbol{p} \psi(p) \tag{7.138}$$

Thus, the *chirality* γ_5 is equivalent to the *helicity* $\boldsymbol{\Sigma} \cdot \boldsymbol{p}$ of the state (only in the massless limit!). This suggests the introduction of the chiral basis in which γ_5 is diagonal,

$$\gamma_0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{7.139}$$

In this basis the massless Dirac equation becomes

$$\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \cdot \boldsymbol{p} \psi(p) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} p_0 \psi(p) \tag{7.140}$$

Let us write the 4-spinor ψ in terms of two 2-spinors of the form

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{7.141}$$

in terms of which the Dirac equation decomposes into two separate equations for each chiral component, ψ_R and ψ_L. Thus the *right handed* (positive chirality) component ψ_R satisfies the Weyl equation

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p} - p_0)\, \psi_R = 0 \tag{7.142}$$

while the *left handed* (negative chirality) component satisfies instead

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p} + p_0)\, \psi_L = 0 \tag{7.143}$$

Hence, one massless Dirac spinor is equivalent to two Weyl spinors.

We conclude that a theory of free massless Dirac fermions has a global chiral symmetry. It is natural to assume that it must also have a locally conserved chiral (axial) current, that we will denote by j^5_μ. An elementary calculation derives the result

$$j^5_\mu = i\bar{\psi}\gamma^5\gamma_\mu\psi \tag{7.144}$$

However, we also know that the Dirac theory has a global $U(1)$ phase symmetry which, when coupled to the vector field A_μ, becomes local $U(1)$ gauge invariance, and has a locally conserved gauge current, $j_\mu = \bar{\psi}\gamma_\mu\psi$. We will see in chapter 20 that in an interacting theory its UV divergencies makes it impossible for both currents to be simultaneously conserved and that, if the theory is to be gauge-invariant, then the chiral current cannot not conserved. In other words, a classically conserved current may not be conserved at the quantum level. The phenomenon of non-conservation at the quantum level of a classically conserved current is known as a *quantum anomaly*. In this case, the non-conservation of the chiral current is known as the chiral (or axial) anomaly.