# 7
# Quantization of the Free Dirac Field

## 7.1 The Dirac Equation and Quantum Field Theory

The Dirac equation is a relativistic wave equation that describes the quantum dynamics of spinors. We will see in this section that a consistent description of this theory cannot be done outside the framework of (local) relativistic Quantum Field Theory.

The Dirac Equation

$$(i\partial\!\!\!/ - m)\psi = 0 \qquad \bar{\psi}(i\partial\!\!\!/ + m) = 0 \tag{7.1}$$

can be regarded as the equations of motion of a complex field $\psi$. Much as in the case of the scalar field, and also in close analogy to the theory of non-relativistic many particle systems discussed in the last chapter, the Dirac field is an operator which acts on a Fock space. We have already discussed that the Dirac equation also follows from a least-action-principle. Indeed the Lagrangian

$$\mathcal{L} = \frac{i}{2}[\bar{\psi}\partial\!\!\!/\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi \equiv \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{7.2}$$

has the Dirac equation for its equation of motion. Also, the momentum $\Pi_\alpha(x)$ canonically conjugate to $\psi_\alpha(x)$ is

$$\Pi^\psi_\alpha(x) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi_\alpha(x)} = i\psi^\dagger_\alpha \tag{7.3}$$

Thus, they obey the equal-time Poisson Brackets

$$\{\psi_\alpha(\boldsymbol{x}), \Pi^\psi_\beta(\boldsymbol{y})\}_{PB} = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{7.4}$$

Thus

$$\{\psi_\alpha(\boldsymbol{x}), \psi^\dagger_\beta(\boldsymbol{y})\}_{PB} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{7.5}$$

In other words the field $\psi_\alpha$ and its adjoint $\psi^\dagger_\alpha$ are a canonical pair. This result follows from the fact that the Dirac Lagrangian is *first order* in time derivatives. Notice that the field theory of non-relativistic many-particle systems (for both fermions on bosons) also has a Lagrangian which is first order in time derivatives. We will see that, because of this property, the quantum field theory of both types of systems follows rather similar lines, at least at a formal level. As in the case of the many-particle systems, two types of statistics are available to us: Fermi and Bose. We will see that only the choice of *Fermi statistics* leads to a physically meaningful theory of the Dirac equation.

The Hamiltonian for the Dirac theory is

$$H = \int d^3x \, \bar{\psi}_\alpha(\boldsymbol{x}) \Big[ - i\boldsymbol{\gamma} \cdot \nabla + m \Big]_{\alpha\beta} \psi_\beta(\boldsymbol{x}) \tag{7.6}$$

where the fields $\psi(x)$ and $\bar{\psi} = \psi^\dagger \gamma_0$ are operators which act on a Hilbert space to be specified below. Notice that the one-particle operator in Eq. (7.6) is just the one-particle Dirac Hamiltonian obtained if we regard the Dirac Equation as a Schrödinger Equation for spinors. We will leave the issue of their commutation relations (*i.e.,* Fermi or Bose) open for the time being. In any event, the equations of motion are *independent* of that choice (*i.e.,* do not depend on the statistics).

In the Heisenberg representation, we find

$$i\gamma_0 \partial_0 \psi = \Big[ \gamma_0 \psi, H \Big] = (-i\boldsymbol{\gamma} \cdot \nabla + m)\psi \tag{7.7}$$

which is just the Dirac equation.

We will solve these equations by means of a Fourier expansion in modes of the form

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{\omega(p)} \left( \tilde{\psi}_+(p) e^{-ip\cdot x} + \tilde{\psi}_-(p) e^{ip\cdot x} \right) \tag{7.8}$$

where $\omega(p)$ is a quantity with units of energy, and which will turn out to be equal to $p_0 = \sqrt{\boldsymbol{p}^2 + m^2}$, and $p \cdot x = p_0 x_0 - \boldsymbol{p} \cdot \boldsymbol{x}$. In terms of $\tilde{\psi}_\pm(p)$, the Dirac equation becomes

$$(p_0 \gamma_0 - \boldsymbol{\gamma} \cdot \boldsymbol{p} \pm m) \, \tilde{\psi}_\pm(p) = 0 \tag{7.9}$$

In other words, $\tilde{\psi}_\pm(p)$ creates one-particle states with energy $\pm p_0$. Let us make the substitution

$$\tilde{\psi}_\pm(p) = (\pm \not{p} + m) \tilde{\phi} \tag{7.10}$$

We get

$$(\not p \mp m)(\pm \not p + m)\tilde{\phi} = \pm(p^2 - m^2)\tilde{\phi} = 0 \tag{7.11}$$

This equation has non-trivial solutions only if the mass-shell condition is obeyed

$$p^2 - m^2 = 0 \tag{7.12}$$

Thus, we can identify $p_0 = \omega(\boldsymbol{p}) = \sqrt{\vec{p}^2 + m^2}$. At zero momentum these states become

$$\tilde{\psi}_{\pm}(p_0, \boldsymbol{p} = 0) = (\pm p_0 \gamma_0 + m)\tilde{\phi} \tag{7.13}$$

where $\tilde{\phi}$ is an arbitrary 4-spinor. Let us choose $\tilde{\phi}$ to be an eigenstate of $\gamma_0$. Recall that in the Dirac representation $\gamma_0$ is diagonal

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{7.14}$$

Thus the spinors $u^{(1)}(m, \boldsymbol{0})$ and $u^{(2)}(m, \boldsymbol{0})$

$$u^{(1)}(m, \boldsymbol{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u^{(2)}(m, \boldsymbol{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.15}$$

have $\gamma_0$-eigenvalue $+1$

$$\gamma_0 u^{(i)}(m, \boldsymbol{0}) = +u^{(\sigma)}(m, \boldsymbol{0}) \qquad \sigma = 1, 2 \tag{7.16}$$

and the spinors $v^{(\sigma)}(m, \boldsymbol{0})(\sigma = 1, 2)$

$$v^{(1)}(m, \boldsymbol{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v^{(2)}(m, \boldsymbol{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.17}$$

have $\gamma_0$-eigenvalue $-1$,

$$\gamma_0 v^{(\sigma)}(m, \boldsymbol{0}) = -v^{(\sigma)}(m, \boldsymbol{0}) \qquad \sigma = 1, 2 \tag{7.18}$$

Let $\varphi^{(i)}(m, 0)$ be the 2-spinors $(\sigma = 1, 2)$

$$\varphi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \varphi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7.19}$$

In terms of $\varphi^{(i)}$ the solutions are

$$\tilde{\psi}_+(p) = u^{(\sigma)}(p) = \frac{(\not p + m)}{\sqrt{2m(p_0+m)}} u^{(\sigma)}(m, \mathbf{0}) = \begin{pmatrix} \sqrt{\frac{p_0+m}{2m}} & \varphi^{(\sigma)}(m,0) \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(p_0+m)}} & \varphi^{(\sigma)}(m,0) \end{pmatrix} \tag{7.20}$$

and

$$\tilde{\psi}_-(p) = v^{(\sigma)}(p) = \frac{(-\not p + m)}{\sqrt{2m(p_0+m)}} v^{(\sigma)}(m, \mathbf{0}) = \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(p_0+m)}} & \varphi^{(\sigma)}(m,\mathbf{0}) \\ \sqrt{\frac{p_0+m}{2m}} & \varphi^{(\sigma)}(m,\mathbf{0}) \end{pmatrix} \tag{7.21}$$

where the two solutions $\tilde{\psi}_+(p)$ have energy $+p_0 = +\sqrt{\boldsymbol{p}^2 + m^2}$ while $\tilde{\psi}_-(p)$ have energy $-p_0 = -\sqrt{\boldsymbol{p}^2 + m^2}$.

Therefore, the one-particle states of the Dirac theory can have both *positive* and *negative* energy and, as it stands, the spectrum of the one-particle Dirac Hamiltonian, shown schematically in Fig. 10.15, is *not* positive. In addition, each Dirac state has a two-fold degeneracy due to the spin of the particle.

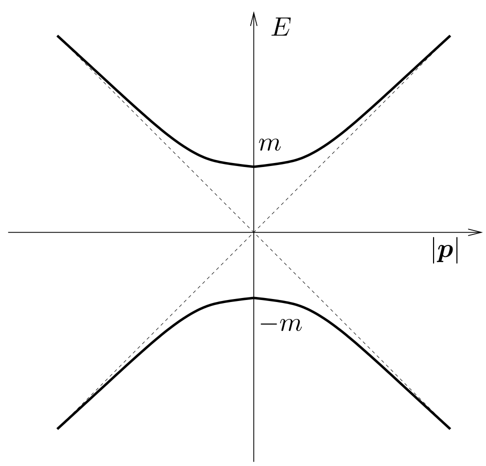


Figure 7.1 Single particle specrum of the Dirac theory.

The spinors $u^{(i)}$ and $v^{(i)}$ have the normalization

$$\begin{aligned} \bar{u}^{(\sigma)}(p) u^{(\nu)}(p) &= \delta_{\sigma\nu} \\ \bar{v}^{(\sigma)}(p) v^{(\nu)}(p) &= -\delta_{\sigma\nu} \\ \bar{u}^{(\sigma)}(p) v^{(\nu)}(p) &= 0 \end{aligned} \tag{7.22}$$

where $\bar{u} = u^\dagger \gamma_0$ and $\bar{v} = v^\dagger \gamma_0$. It is straightforward to check that the operators $\Lambda_\pm(p)$

$$\Lambda_\pm(p) = \frac{1}{2m}(\pm \not{p} + m) \tag{7.23}$$

are *projection operators* that project the spinors onto the subspaces with positive ($\Lambda_+$) and negative ($\Lambda_-$) energy respectively. These operators satisfy

$$\Lambda_\pm^2 = \Lambda_\pm \qquad \text{Tr}\ \Lambda_\pm = 2 \qquad \Lambda_+ + \Lambda_- = 1 \tag{7.24}$$

Hence, the four 4-spinors $u^{(\sigma)}$ and $v^{(\sigma)}$ are orthonormal and complete, and provide a natural basis of the Hilbert space of single-particle states.

We can use these results to write the expansion of the field operator

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \left[ a_{\sigma,+}(\boldsymbol{p}) u_+^{(\sigma)}(p) e^{-ip\cdot x} + a_{\sigma,-}(\boldsymbol{p}) v_-^{(\sigma)}(p) e^{ip\cdot x} \right] \tag{7.25}$$

where the coefficients $a_{\sigma,\pm}(\boldsymbol{p})$ are operators with as yet unspecified commutation relations. The (formal) Hamiltonian for this system is

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} p_0 [a_{\sigma,+}^\dagger(\boldsymbol{p}) a_{\sigma,+}(p) - a_{\sigma,-}^\dagger(\boldsymbol{p}) a_{\sigma,-}(\boldsymbol{p})] \tag{7.26}$$

Since the single-particle spectrum does not have a lower bound, any attempt to quantize the theory with *canonical commutation relations* will have the problem that the *total energy of the system is not bounded from below.* In other words "Dirac bosons" do not have a ground state and the system is unstable since we can put as many bosons as we wish in states with arbitrarily large but negative energy.

Dirac realized that the simple and elegant way out of this problem was to *require* the electrons to obey the Pauli Exclusion Principle since, in that case, there is a natural and stable ground state. However, this assumption implies that the Dirac theory must be quantized as a theory of *fermions.* Hence we are led to quantize the theory with *canonical anticommutation relations*

$$\begin{aligned} \{a_{s,\sigma}(\boldsymbol{p}), a_{s',\sigma'}(\boldsymbol{p}')\} &= 0 \\ \{a_{s,\sigma}(\boldsymbol{p}), a_{s',\sigma'}^\dagger(\boldsymbol{p}')\} &= (2\pi)^3 \frac{p_0}{m} \delta^3(\boldsymbol{p} - \boldsymbol{p}') \delta_{ss'} \delta_{\sigma\sigma'} \end{aligned} \tag{7.27}$$

where $s = \pm$. Let us denote by $|0\rangle$ the state anihilated by the operators $a_{s,\sigma}(\boldsymbol{p})$,

$$a_{s,\sigma}(\boldsymbol{p})|0\rangle = 0 \tag{7.28}$$

We will see now that this state is not the vacuum (or ground state) of the Dirac theory. Let us now discuss the construction of the ground state and of the excitation spectrum.

### 7.1.1 Ground State and Normal Ordering

We will show now that the *ground state* or *vacuum* $|\text{vac}\rangle$ is the state in which *all the negative energy states are filled* (as shown in Fig.7.2) *i.e.*,

$$|\text{vac}\rangle = \prod_{\sigma,\text{p}} \text{a}^{\dagger}_{-,\sigma}(\tilde{\text{p}})|0\rangle \tag{7.29}$$

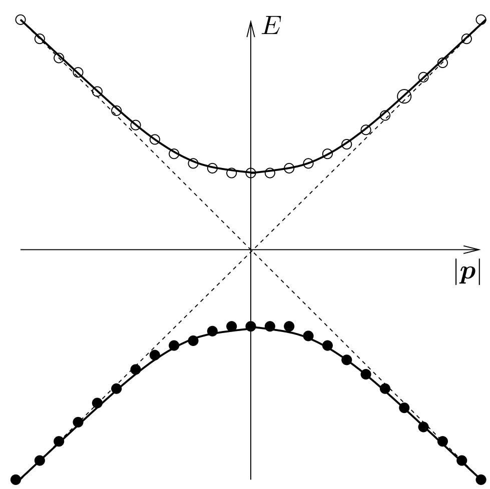


Figure 7.2 Ground State of the Dirac theory.

We will normal-order all the operators relative to the vacuum state $|\text{vac}\rangle$. This amounts to a particle-hole transformation for the negative energy states. Thus, we define the fermion creation and annihilation operators $b_\sigma(\boldsymbol{p}), b^\dagger_\sigma(\boldsymbol{p})$ and $d_\sigma(\boldsymbol{p}), d^\dagger_\sigma(\boldsymbol{p})$ to be

$$\begin{aligned} b_\sigma(\boldsymbol{p}) &= a_{\sigma,+}(\boldsymbol{p}) \\ d_\sigma(\boldsymbol{p}) &= a^\dagger_{\sigma,-}(\boldsymbol{p}) \end{aligned} \tag{7.30}$$

which obey

$$b_\sigma(\boldsymbol{p})|\text{vac}\rangle = \text{d}_\sigma(\text{p})|\text{vac}\rangle = 0 \tag{7.31}$$

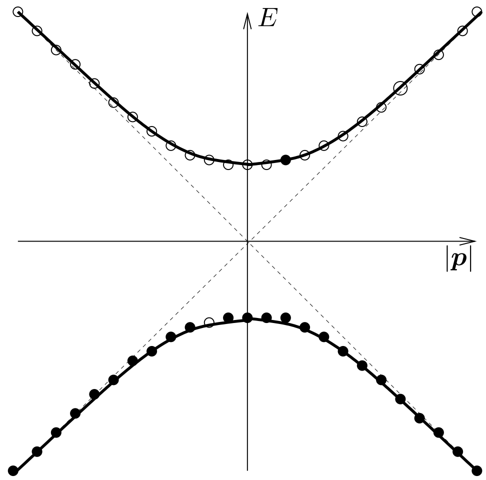


Figure 7.3 An electron-hole pair.

The Hamiltonian now reads

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} p_0 \sum_{\sigma=1,2} [b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) - d_\sigma(\boldsymbol{p}) d_\sigma^\dagger(\boldsymbol{p})] \tag{7.32}$$

We now normal order $\hat{H}$ relative to the vacuum state

$$H =: H : + E_0 \tag{7.33}$$

with a normal-ordered Hamiltonian

$$: H := \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} p_0 \Big[ b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) + d_\sigma^\dagger(\boldsymbol{p}) d_\sigma(\boldsymbol{p}) \Big] \tag{7.34}$$

The constant $E_0$ is the (negative and divergent) ground state energy

$$E_0 = -2V \int d^3p \, \sqrt{\boldsymbol{p}^2 + m^2} \tag{7.35}$$

similar to the expression we already countered in the Klein-Gordon theory, but with opposite sign. The factor of 2 is due to spin.

In terms of the operators $b_\sigma$ and $d_\sigma$ the Dirac field has the mode expansion

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \Big[ b_\sigma(\boldsymbol{p}) u^{(\sigma)}(\boldsymbol{p}) e^{-ip\cdot x} + d_\sigma^\dagger(\boldsymbol{p}) v^{(\sigma)}(\boldsymbol{p}) e^{ip\cdot x} \Big] \tag{7.36}$$

which satisfy equal-time canonical anticommutation relations

$$\{\psi(\boldsymbol{x}), \psi^\dagger(\boldsymbol{x}')\} = \delta^3(\boldsymbol{x} - \boldsymbol{x}')$$
$$\{\psi(\boldsymbol{x}), \psi(\boldsymbol{x}')\} = \{\psi^\dagger(\boldsymbol{x}), \psi^\dagger(\boldsymbol{x}')\} = 0 \tag{7.37}$$

### 7.1.2 One-particle states

The excitations of this theory can be constructed by using the same methods employed for non-relativistic many-particle systems. Let us first construct the total four-momentum operator $P^\mu$

$$P^\mu = \int d^3x\, T^{0\mu} = \int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}p^\mu \sum_{\sigma=1,2} : b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) - d_\sigma(\boldsymbol{p}) d_\sigma^\dagger(\boldsymbol{p}) : \tag{7.38}$$

Hence

$$: P^\mu := \int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}p^\mu \sum_{\sigma=1,2} \left[ b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) + d_\sigma^\dagger(\boldsymbol{p}) d_\sigma(\boldsymbol{p}) \right] \tag{7.39}$$

The states $b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$ and $d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$ have energy $p_0 = \sqrt{\boldsymbol{p}^2 + m^2}$ and momentum $\boldsymbol{p}$, *i.e.*,

$$: H : b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle = p_0 b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$$
$$: H : d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle = p_0 d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$$
$$: P^i : b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle = p^i b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$$
$$: P^i : d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle = p^i d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle \tag{7.40}$$

We see that there are four different states which have the same energy and momentum. Let us find quantum numbers to classify these states.

### 7.1.3 Spin

The angular momentum tensor $\mathcal{M}_{\mu\nu\lambda}$ for the Dirac theory is

$$\mathcal{M}_{\mu\nu\lambda} = \int d^3x\, i\bar\psi(x)\, \gamma^\mu (x^\nu \partial^\lambda - x^\lambda \partial^\nu + \Sigma^{\nu\lambda})\, \psi(x) \tag{7.41}$$

where $\Sigma^{\nu\lambda}$ is the matrix

$$\Sigma^{\nu\lambda} = \frac{1}{2}\sigma^{\nu\lambda} = \frac{1}{4}[\gamma^\nu, \gamma^\lambda] \tag{7.42}$$

The conserved angular momentum $J^{\nu\lambda}$ is

$$J^{\nu\lambda} = \mathcal{M}^{0\nu\lambda} = \int d^3x\, i\psi^\dagger(x)(x^\nu \partial^\lambda - x^\lambda \partial^\nu + \Sigma^{\nu\lambda})\, \psi(x) \tag{7.43}$$

In particular, out of its space components $J^{ij}$, we can construct the total angular momentum three-vector $\boldsymbol{J}$

$$J^i = \frac{1}{2}\epsilon^{ijk} J^{jk} = \int d^3x \ \psi^\dagger \left( \epsilon^{ijk} x^j \partial^k + \frac{1}{2}\epsilon^{ijk}\Sigma^{jk} \right) \psi \tag{7.44}$$

It is easy to check that, in the Dirac representation, the last term represents the spin.

In the quantized theory, the angular momentum operator is

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{7.45}$$

where $\boldsymbol{L}$ is the *orbital angular momentum*

$$\boldsymbol{L} = \int d^3x \ \psi^\dagger(\boldsymbol{x}) \boldsymbol{x} \times \boldsymbol{\partial} \psi(\boldsymbol{x}) \tag{7.46}$$

while $\boldsymbol{S}$ is the *spin*

$$\boldsymbol{S} = \int d^3x \ \psi^\dagger(\boldsymbol{x}) \ \boldsymbol{\Sigma} \ \psi(\boldsymbol{x}) \tag{7.47}$$

where $\boldsymbol{\Sigma}$ is the $4 \times 4$ matrix

$$\boldsymbol{\Sigma} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \equiv \frac{1}{2}\boldsymbol{\sigma} \tag{7.48}$$

In order to measure the spin polarization of a state we first go to the rest frame in which $\boldsymbol{p} = 0$. In this frame we can consider the four-vector $W^\mu$

$$W^\mu = (0, m\boldsymbol{\Sigma}) \tag{7.49}$$

Let $n^\mu$ be the space-like 4-vector

$$n^\mu = (0, \boldsymbol{n}) \tag{7.50}$$

where $\boldsymbol{n}$ has unit length. Thus, $n^\mu n_\mu = -1$. We will use $n^\mu$ to fix the direction of polarization in the rest frame.

The scalar product $W_\mu n^\mu$ is a Lorentz invariant scalar. Thus its values are independent of the choice of frame. In the rest frame we have

$$W_\mu n^\mu = -m\boldsymbol{n} \cdot \boldsymbol{\Sigma} \equiv -\frac{m}{2}\vec{n} \cdot \boldsymbol{\sigma} = -\frac{m}{2} \begin{pmatrix} \boldsymbol{n} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{n} \cdot \boldsymbol{\sigma} \end{pmatrix} \tag{7.51}$$

In particular, if $\boldsymbol{n} = e_z$ then $W_\mu n^\mu$ is

$$W_\mu n^\mu = -\frac{m}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{7.52}$$

which is diagonal. The operator $-\frac{1}{m}W\cdot n$ is a Lorentz scalar which measures the spin polarization:

$$\begin{aligned}
-\frac{1}{m}W\cdot n u_+^{(1)}(p) &= +\frac{1}{2}u_+^{(1)}(p)\\
-\frac{1}{m}W\cdot n u_+^{(2)}(p) &= -\frac{1}{2}u_+^{(2)}(p)\\
-\frac{1}{m}W\cdot n v_-^{(1)}(p) &= +\frac{1}{2}v_-^{(1)}(p)\\
-\frac{1}{m}W\cdot n v_-^{(2)}(p) &= -\frac{1}{2}v_-^{(2)}(p)
\end{aligned} \tag{7.53}$$

It is straightforward to check that $-\frac{1}{m}W\cdot n$ is the Lorentz scalar

$$-\frac{1}{m}W\cdot n = \frac{1}{4m}\epsilon_{\mu\nu\lambda\rho}n^\mu p^\nu \sigma^{\lambda\rho} = \frac{1}{2m}\gamma_5 \not{n} \not{p} \tag{7.54}$$

which enables us to write the spin projection operator $P(n)$

$$P(n) = \frac{1}{2}(I + \gamma_5 \not{n}) \tag{7.55}$$

where we used that

$$\begin{aligned}
\frac{1}{2m}\gamma_5 \not{n} \not{p}\, u^{(\sigma)}(p) &= \ \ \frac{1}{2}\gamma_5 \not{n}\, u^{(\sigma)}(p) = (-1)^\sigma \frac{1}{2}u^{(\sigma)}(p)\\
\frac{1}{2m}\gamma_5 \not{n} \not{p}\, v^{(\sigma)}(p) &= -\frac{1}{2}\gamma_5 \not{n}\, v^{(\sigma)}(p) = (-1)^\sigma \frac{1}{2}v^{(\sigma)}(p)
\end{aligned} \tag{7.56}$$

### *7.1.4 Charge*

The Dirac Lagrangian is invariant under the global (phase) transformation

$$\begin{aligned}
\psi &\to \psi' = e^{i\alpha}\psi\\
\bar\psi &\to \bar\psi' = e^{-i\alpha}\bar\psi
\end{aligned} \tag{7.57}$$

Consequently, it has a locally conserved current $j^\mu$

$$j^\mu = \bar\psi\gamma^\mu\psi \tag{7.58}$$

which is also locally gauge invariant. As a result it has a conserved total charge $Q = -e\int d^3x j^0(x)$. The corresponding operator in the quantized theory $Q$ is

$$Q = -e\int d^3x\; j^0(x) = -e\int d^3x\; \psi^\dagger(x)\psi(x) \tag{7.59}$$

The total charge operator $Q$ commutes with the Dirac Hamiltonian $\hat{H}$

$$[Q, H] = 0 \tag{7.60}$$

Hence, the eigenstates of the Hamiltonian $H$ can have well defined charge.

In terms of the creation and annihilation operators, we find

$$Q = -e \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \left( b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) + d_\sigma(\boldsymbol{p}) d_\sigma^\dagger(\boldsymbol{p}) \right) \tag{7.61}$$

which is not normal-ordered relative to $|\text{vac}\rangle$. The normal-ordered charge operator $: Q :$ is

$$: Q := -e \int \frac{d^3p}{(2\pi)^3} \frac{m}{p_0} \sum_{\sigma=1,2} \left[ b_\sigma^\dagger(\boldsymbol{p}) b_\sigma(\boldsymbol{p}) - d_\sigma^\dagger(\boldsymbol{p}) d_\sigma(\boldsymbol{p}) \right] \tag{7.62}$$

and we can write

$$Q =: Q : + Q_{\text{vac}} \tag{7.63}$$

where $Q_{\text{vac}}$ is the unobservable (and divergent) vacuum charge

$$Q_{\text{vac}} = -eV \int \frac{d^3p}{(2\pi)^3} \tag{7.64}$$

$V$ being the volume of space. From now on we will *define* the charge to be the subtracted charge operator

$$: Q := Q \ - Q_{\text{vac}} \tag{7.65}$$

which annihilates the vacuum state

$$: Q : |\text{vac}\rangle = 0 \tag{7.66}$$

*i.e.,* the vacuum is neutral. In other words, we measure the *charge of a state* relative to the vacuum charge which we define to be zero. Equivalently, this amounts to a definition of the order of the operators in $: Q :$

$$: Q : = -e \int d^3x \frac{1}{2} [\psi^\dagger(x), \psi(x)] \tag{7.67}$$

The one-particle states $b_\sigma^\dagger |\text{vac}\rangle$ and $d_\sigma^\dagger |\text{vac}\rangle$ have well defined charge:

$$\begin{aligned} : Q : b_\sigma^\dagger(\boldsymbol{p}) |\text{vac}\rangle &= - e b_\sigma^\dagger(\boldsymbol{p}) |\text{vac}\rangle \\ : Q : d_\sigma^\dagger(\boldsymbol{p}) |\text{vac}\rangle &= + e d_\sigma^\dagger(\boldsymbol{p}) |\text{vac}\rangle \end{aligned} \tag{7.68}$$

Hence we identify the state $b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$ with an *electron* of charge $-e$, spin $\sigma$, momentum $\boldsymbol{p}$ and energy $p_0 = \sqrt{\boldsymbol{p}^2 + m^2}$. Similarly, the state $d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle$ is a *positron* with the same quantum numbers and energy of the electron but with *positive charge* $+e$.

### *7.1.5 Causality and the Spin-Statistics Connection*

Let us finally discuss the question of causality and the spin-statistics connection in the Dirac theory. To this end we will consider the *anticommutator* of two Dirac fields at different times

$$i\Delta_{\alpha\beta}(x-y) = \{\psi_\alpha(x), \psi_\beta(y)\} \tag{7.69}$$

By using the field expansion we obtain the expression

$$i\Delta_{\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\sum_{\sigma=1,2}\left[e^{-ip\cdot(x-y)}u_\alpha^{(\sigma)}(p)\bar{u}_\beta^{\sigma}(p)+e^{ip\cdot(x-y)}v_\alpha^{(\sigma)}(p)\bar{v}_\beta^{(\sigma)}(p)\right] \tag{7.70}$$

By using the (completeness) identities

$$\sum_{\sigma=1,2} u_\alpha^{(\sigma)}(p)\bar{u}_\beta^{(\sigma)}(p) = \left(\frac{\not p + m}{2m}\right)_{\alpha\beta}$$
$$\sum_{\sigma=1,2} v_\alpha^{(\sigma)}(p)\bar{v}_\beta^{(\sigma)}(p) = \left(\frac{\not p - m}{2m}\right)_{\alpha\beta} \tag{7.71}$$

we can write the anticommutator in the form

$$i\Delta_{\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{m}{p_0}\left[\left(\frac{\not p + m}{2m}\right)_{\alpha\beta} e^{-ip\cdot(x-y)} + \left(\frac{\not p - m}{2m}\right)_{\alpha\beta} e^{ip\cdot(x-y)}\right] \tag{7.72}$$

After some straightforward algebra, we get

$$\begin{aligned} i\Delta_{\alpha\beta}(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}(i\not\partial_x + m)\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] \\ &= (i\not\partial_x + m)_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3 2p_0}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right] \end{aligned} \tag{7.73}$$

We recognize the integral on the r.h.s. of Eq.(7.73) to be the commutator of two free scalar (Klein-Gordon) fields, $\Delta_{KG}(x-y)$.

Hence, the anticommutator two Dirac fields of the Dirac theory is

$$i\Delta_{\alpha\beta}(x-y) = (i\not\partial + m)_{\alpha\beta}\; i\Delta_{KG}(x-y) \tag{7.74}$$

Since $\Delta_{KG}(x-y)$ *vanishes* at space-like separations, so does $\Delta_{\alpha\beta}(x-y)$. Hence, the theory is causal.

On the other hand, had we had quantized the Dirac theory with commutators (which, as we saw, leads to a theory without a ground state) we would have also found a violation of causality. Indeed, we would have obtained instead the result

$$\Delta_{\alpha\beta}(x-y) = (i\partial\!\!\!/ + m)_{\alpha\beta}\tilde{\Delta}(x-y) \tag{7.75}$$

where $\tilde{\Delta}(x-y)$ is given by

$$\tilde{\Delta}(x-y) = \int \frac{d^3p}{(2\pi)^3 2p_0}\left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right) \tag{7.76}$$

which does not vanish at space-like separations. Instead, at equal times and at long distances $\widetilde{\Delta}(x-y)$ decays as

$$\tilde{\Delta}(R,0) \simeq \frac{e^{-MR}}{R^2} \qquad MR \to \infty \tag{7.77}$$

Thus, if the Dirac theory were to be quantized with commutators, the field operators *would not commute* at equal times at distances shorter than the Compton wavelength. This would be a violation of locality. The same result holds in the theory of the scalar field if it is quantized with anticommutators.

These results can be summarized in the *Spin-Statistics Theorem*: fields with *half-integer spin* must be quantized as *fermions*, *i.e.* obey canonical *anti-commutation* relations, whereas fields with *integer spin* must be quantized as *bosons*, *i.e.* obey canonical *commutation* relations. If a field theory is quantized with the wrong spin-statistics connection, either the theory becomes non-local, with violations of causality, and/or it does not have a ground state, or it contains states in its spectrum with negative norm. Notice that the arguments we have used were derived for free local theories. It is a highly non-trivial task to prove that the spin-statistics connection also remains valid for interacting theories. Although this can be done by making sufficiently strong assumptions of the behavior of perturbation theory, in reality it must the spin-statistics connection must be regarded as an *axiom* of local relativistic quantum field theories.

### *7.1.6 The Propagator of the spinor field*

Finally, we will compute the propagator for a spinor field $\psi_\alpha(x)$. We will find that it is essentially the Green function for the the Dirac operator $i\partial\!\!\!/ - m$. The propagator is defined by

$$S_{\alpha\beta}(x-x') = -i\langle \text{vac}|T\psi_\alpha(x)\bar{\psi}_\beta(x')|\text{vac}\rangle \tag{7.78}$$

where we have used the *time ordered product* of two fermionic field operators, which is defined by

$$T\psi_\alpha(x)\bar{\psi}_\beta(x') = \theta(x_0-x_0')\psi_\alpha(x)\bar{\psi}_\beta(x') - \theta(x_0'-x_0)\bar{\psi}_\beta(x')\psi_\alpha(x) \tag{7.79}$$

Notice the change in sign with respect to the time ordered product of bosonic operators. The sign change reflects the anticommutation properties of the field.

We will show now that this propagator is closely connected to the propagator of the free scalar field, *i.e.,* the Green function for the Klein-Gordon operator $\partial^2 + m^2$.

By acting with the Dirac operator on $S_{\alpha\beta}(x-x')$ we find

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} S_{\beta\lambda}(x-x') = -i\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} \langle \text{vac}|T\psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle \tag{7.80}$$

We now use that

$$\frac{\partial}{\partial x_0}\theta(x_0 - x_0') = \delta(x_0 - x_0') \tag{7.81}$$

and the fact that the equation of motion of the Heisenberg field operators $\psi_\alpha$ is the Dirac equation,

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} \psi_\beta(x) = 0 \tag{7.82}$$

to show that

$$\begin{aligned}
\Big(i\partial\!\!\!/ - m\Big)_{\alpha\beta} & S_{\beta\lambda}(x-x') = \\
& -i\langle \text{vac}|T\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} \psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle \\
& +\delta(x_0 - x_0')\Big(\langle \text{vac}|\gamma^0_{\alpha\beta}\psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle \\
& +\langle \text{vac}|\bar{\psi}_\lambda(x')\gamma^0_{\alpha\beta}\psi_\beta(x)|\text{vac}\rangle\Big) \\
& =\delta(x_0 - x_0')\gamma^0_{\alpha\beta}\langle \text{vac}|\left\{\psi_\beta(x), \psi^\dagger_\nu(x')\right\}|\text{vac}\rangle\gamma^0_{\nu\lambda} \\
& =\delta(x_0 - x_0')\delta^3(\boldsymbol{x} - \boldsymbol{x}')\delta_{\alpha\lambda}
\end{aligned} \tag{7.83}$$

Therefore we find that $S_{\beta\lambda}(x-x')$ is the solution of the equation

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} S_{\beta\lambda}(x-x') = \delta^4(x-x')\delta_{\alpha\lambda} \tag{7.84}$$

Hence $S_{\beta\lambda}(x-x') = -i\langle \text{vac}|T\psi_\beta(x)\bar{\psi}_\lambda(x')|\text{vac}\rangle$ is the Green function of the Dirac operator.

We saw before that there is a close connection between the Dirac and the Klein-Gordon operators. We will now use this connection to relate their propagators. Let us write the Green function $S_{\alpha\lambda}(x-x')$ in the form

$$S_{\alpha\lambda}(x-x') = \left(i\partial\!\!\!/ + m\right)_{\alpha\beta} G_{\beta\lambda}(x-x') \tag{7.85}$$

Since $S_{\alpha\lambda}(x-x')$ satisfies Eq.(7.84), we find that

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta} S_{\beta\lambda}(x-x') = \left(i\partial\!\!\!/ - m\right)_{\alpha\beta}\left(i\partial\!\!\!/ + m\right)_{\beta\nu} G_{\nu\lambda}(x-x') \tag{7.86}$$

But

$$\left(i\partial\!\!\!/ - m\right)_{\alpha\beta}\left(i\partial\!\!\!/ + m\right)_{\beta\nu} = -\left(\partial^2 + m^2\right)\delta_{\alpha\nu} \tag{7.87}$$

Hence, $G_{\alpha\nu}(x-x')$ must satisfy

$$-\left(\partial^2 + m^2\right)G_{\alpha\nu}(x-x') = \delta^4(x-x')\delta_{\alpha\nu} \tag{7.88}$$

Therefore $G_{\alpha\nu}(x-x')$ is given by

$$G_{\alpha\nu}(x-x') = G(x-x')\delta_{\alpha\nu} \tag{7.89}$$

where $G(x-x')$ is the propagator for a scalar field, *i.e.,* the Green function of the Klein-Gordon equation

$$-\left(\partial^2 + m^2\right)G(x-x') = \delta^4(x-x') \tag{7.90}$$

We then conclude that the Dirac propagator $S_{\alpha\beta}(x-x')$, and the Klein-Gordon propagator $G(x-x')$ are related by

$$S_{\alpha\beta}(x-x') = \left(i\partial\!\!\!/ + m\right)_{\alpha\beta}G(x-x') \tag{7.91}$$

In particular, this relationship implies that the have exactly the same asymptotic behaviors that we discussed before. The spinor structure of the Dirac propagator is determined by the operator in front of $G(x-x')$ in Eq.(7.91).

The Feynman propagator for the Dirac field, given by Eq. (7.91), in momentum space becomes

$$S_{\alpha\beta}(p) = \left(\frac{p\!\!\!/ + m}{p^2 - m^2 + i\epsilon}\right)_{\alpha\beta} \tag{7.92}$$

Hence we get the same pole structure in the time-ordered propagator as we did for the Klein-Gordon field.

## 7.2 Discrete Symmetries of the Dirac Theory

We will now discuss three important discrete symmetries in relativistic field theories: *charge conjugation*, *parity* and *time reversal*. These discrete symmetries have a different role, and a different standing, than the continuous symmetries discussed before. In a relativistic quantum field theory the ground state, *i.e.,* the *vacuum*, must be invariant under continuous Lorentz transformations, but it may not be invariant under $\mathcal{C}$, $\mathcal{P}$ or $\mathcal{T}$. However, in a local relativistic quantum field theory the product $\mathcal{CPT}$ is always a good symmetry. This is in fact an *axiom* of relativistic local quantum field theory. Thus, although $\mathcal{C}$, $\mathcal{P}$ or $\mathcal{T}$ may or may not to be good symmetries of the vacuum state, $\mathcal{CPT}$ *must* be a good symmetry. As in the case of the symmetries we discussed before, these symmetries must also be realized unitarily in the *Fock* space of the quantum field theory.

### *7.2.1 Charge Conjugation*

Charge conjugation is a symmetry that exchanges *particles* and *antiparticles* (or *holes*). Consider a Dirac minimally coupled to an external electromagnetic field $A_\mu$. The equation of motion for the Dirac field $\psi$ is

$$\left(i\not{\partial} - e\not{A} - m\right)\psi = 0 \tag{7.93}$$

We will *define* the charge conjugate field $\psi^c$

$$\psi^c(x) = \mathcal{C}\psi(x)\mathcal{C}^{-1} \tag{7.94}$$

where $\mathcal{C}$ is the (unitary) charge conjugation operator, $\mathcal{C}^{-1} = \mathcal{C}^\dagger$, such that $\psi^c$ obeys

$$\left(i\not{\partial} + e\not{A} - m\right)\psi^c = 0 \tag{7.95}$$

Since $\bar{\psi} = \psi^\dagger\gamma^0$ obeys

$$\bar{\psi}\left[\gamma^\mu\left(-i\overleftarrow{\partial}_\mu - eA_\mu\right) - m\right] = 0 \tag{7.96}$$

which, when transposed, becomes

$$\left[\gamma^{\mu\ t}\left(-i\partial_\mu - eA_\mu\right) - m\right]\bar{\psi}^{\ t} = 0 \tag{7.97}$$

where

$$\bar{\psi}^{\ t} = \gamma^{0\ t}\psi^* \tag{7.98}$$

Let $C$ be an invertible $4\times 4$ matrix, where $C^{-1}$ is its inverse. Then, we can write

$$C\left[\gamma^{\mu\ t}\left(-i\partial_\mu - eA_\mu\right) - m\right]C^{-1}C\bar{\psi}^{\ t} = 0 \tag{7.99}$$

such that

$$C\left(\gamma^{\ \mu}\right)^t C^{-1} = -\gamma^\mu \tag{7.100}$$

Hence

$$\left[\left(i\partial_\mu + eA_\mu\right) - m\right]C\bar{\psi}^{\ t} = 0 \tag{7.101}$$

For Eq. 7.101 to hold, we must have

$$\mathcal{C}\psi\mathcal{C}^\dagger = \psi^c = C\bar{\psi}^{\ t} = C\gamma^{0t}\psi^* \tag{7.102}$$

Hence the field $\psi^c$ thus defined has *positive* charge $+e$.

We can find the charge conjugation matrix $C$ explicitly:

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} = C^{-1} \tag{7.103}$$

In particular this means that $\psi^c$ is given by

$$\psi^c = C\bar{\psi}^T = C\gamma^{0t}\psi^* = i\gamma^2\psi^* \tag{7.104}$$

Eq. (7.104) provides us with a definition for a charge neutral Dirac fermion, *i.e.,* $\psi$ represents a neutral fermion if $\psi = \psi^c$. Hence the condition is

$$\psi = i\gamma^2\psi^* \tag{7.105}$$

A Dirac fermion that satisfies the neutrality condition is known as a Majorana fermion,

To understand the action of $C$ on physical states we can look for instance, at the charge conjugate $u^c$ of the positive energy, up spin, and charge $-e$, spinor

$$u = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} e^{-imt} \tag{7.106}$$

which is

$$u^c = \begin{pmatrix} 0 \\ -i\gamma^2\varphi^* \end{pmatrix} e^{imt} \tag{7.107}$$

which has negative energy, down spin and charge $+e$.

At the level of the full quantum field theory, we will require the vacuum state $|\text{vac}\rangle$ to be invariant under charge conjugation:

$$\mathcal{C}|\text{vac}\rangle = |\text{vac}\rangle \tag{7.108}$$

How do one-particle states transform? To determine that we look at the action of charge conjugation $\mathcal{C}$ on the one-particle states, and demand that particle and anti-particle states to be exchanged under charge conjugation

$$\begin{aligned} \mathcal{C}b^\dagger_\sigma(p)|\text{vac}\rangle &= \mathcal{C}b^\dagger_\sigma(p)\mathcal{C}^{-1}\mathcal{C}|\text{vac}\rangle \equiv d^\dagger_\sigma(p)|\text{vac}\rangle \\ \mathcal{C}d^\dagger_\sigma(p)|\text{vac}\rangle &= \mathcal{C}d^\dagger_\sigma(p)\mathcal{C}^{-1}\mathcal{C}|\text{vac}\rangle \equiv b^\dagger_\sigma(p)|\text{vac}\rangle \end{aligned} \tag{7.109}$$

Hence, for the one-particle states to satisfy these rules it is sufficient to require that the field operators $b_\sigma(\boldsymbol{p})$ and $d_\sigma(\boldsymbol{p})$ satisfy

$$\mathcal{C}b_\sigma(\boldsymbol{p})\mathcal{C}^\dagger = d_\sigma(\boldsymbol{p}); \qquad \mathcal{C}d_\sigma(\boldsymbol{p})\mathcal{C}^\dagger = b_\sigma(\boldsymbol{p}) \tag{7.110}$$

Using that

$$u_\sigma(\boldsymbol{p}) = -i\gamma^2\left(v_\sigma(\boldsymbol{p})\right)^*; \qquad v_\sigma(\boldsymbol{p}) = -i\gamma^2\left(u_\sigma(\boldsymbol{p})\right)^* \tag{7.111}$$

we find that the field operator $\psi(x)$ transforms as

$$\mathcal{C}\psi(x)\mathcal{C}^\dagger = \left(-i\gamma^0\gamma^2\psi\right)^t \tag{7.112}$$

In particular the fermionic bilinears we discussed before satisfy the transformation laws:

$$\begin{aligned}
\mathcal{C}\bar\psi\psi\mathcal{C}^\dagger &= +\bar\psi\psi \\
\mathcal{C}i\bar\psi\gamma^5\psi\mathcal{C}^\dagger &= i\bar\psi\gamma^5\psi \\
\mathcal{C}\bar\psi\gamma^\mu\psi\mathcal{C}^\dagger &= -\bar\psi\gamma^\mu\psi \\
\mathcal{C}\bar\psi\gamma^\mu\gamma^5\psi\mathcal{C}^\dagger &= +\bar\psi\gamma^\mu\gamma^5\psi
\end{aligned} \tag{7.113}$$

### *7.2.2 Parity*

We will define as *parity* the transformation $\mathcal{P} = \mathcal{P}^{-1}$ which reverses the momentum of a particle but not its spin. Once again, the vacuum state is invariant under parity. Thus, we must require

$$\begin{aligned}
\mathcal{P}b_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle &= \mathcal{P}b_\sigma^\dagger(\boldsymbol{p})\mathcal{P}^{-1}\mathcal{P}|\text{vac}\rangle \equiv b_\sigma^\dagger(-\boldsymbol{p})|\text{vac}\rangle \\
\mathcal{P}d_\sigma^\dagger(\boldsymbol{p})|\text{vac}\rangle &= \mathcal{P}d_\sigma^\dagger(\boldsymbol{p})\mathcal{P}^{-1}\mathcal{P}|\text{vac}\rangle \equiv d_\sigma^\dagger(-\boldsymbol{p})|\text{vac}\rangle
\end{aligned} \tag{7.114}$$

In real space this transformation should amount to

$$\mathcal{P}\psi(\boldsymbol{x},x_0)\mathcal{P}^{-1} = \gamma^0\psi(-\boldsymbol{x},x_0), \quad \mathcal{P}\bar\psi(\boldsymbol{x},x_0)\mathcal{P}^{-1} = \bar\psi(-\boldsymbol{x},x_0)\gamma_0 \tag{7.115}$$

### *7.2.3 Time Reversal*

Finally, we discuss time reversal $\mathcal{T}$. We will define $\mathcal{T}$ as the anti-linear unitary operator, *i.e.*,

$$\mathcal{T}e^{iHx_0}\mathcal{T}^{-1} = e^{-iHx_0}, \qquad \text{and} \qquad \mathcal{T}^{-1} = \mathcal{T}^\dagger \tag{7.116}$$

which reverses the momentum *and* the spin of the particles:

$$\begin{aligned}
\mathcal{T}b_\sigma(\boldsymbol{p})\mathcal{T}^\dagger &= b_{-\sigma}(-\boldsymbol{p}) \\
\mathcal{T}d_\sigma(\boldsymbol{p})\mathcal{T}^\dagger &= d_{-\sigma}(-\boldsymbol{p})
\end{aligned} \tag{7.117}$$

while leaving the vacuum state invariant:

$$\mathcal{T}|\text{vac}\rangle = |\text{vac}\rangle \tag{7.118}$$

In real space this implies:

$$\mathcal{T}\psi(\boldsymbol{x},x_0)\mathcal{T}^\dagger = -\gamma^1\gamma^3\ \psi^*(-\boldsymbol{x},x_0) \tag{7.119}$$

## 7.3 Chiral Symmetry

We will now discuss a global symmetry specific of theories of spinors: chiral symmetry. Let us consider again the Dirac equation

$$\left(i\partial\!\!\!/ - m\right)\psi = 0 \tag{7.120}$$

Let us define the *chiral transformation*

$$\psi' = e^{i\gamma_5\theta}\psi \tag{7.121}$$

where $\theta$ is a *constant* angle, $0 \leq \theta < 2\pi$. We wish to find an equation for $\psi$. From

$$(i\gamma^\mu\partial_\mu - m)\, e^{i\gamma_5\theta}\psi = 0 \tag{7.122}$$

and

$$\{\gamma_\mu, \gamma_5\} = 0 \tag{7.123}$$

after some simple algebra, which uses that

$$\gamma^\mu e^{i\gamma_5\theta} = e^{-i\gamma_5\theta}\gamma^\mu \tag{7.124}$$

we find

$$\left(i\partial\!\!\!/ - me^{2i\gamma_5\theta}\right)\psi = 0 \tag{7.125}$$

Thus for $m \neq 0$, if $\psi$ is a solution of the Dirac equation, $\psi'$ is not a solution. But, if the theory is massless, we find that the Dirac theory has an exact global chiral symmetry.

It is also instructive to determine how various fermion bilinears transform under the chiral transformation. We find that the fermion mass term

$$\bar{\psi}'\psi' = \bar{\psi}e^{2i\gamma_5\theta}\psi = \cos(2\theta)\bar{\psi}\psi + i\sin(2\theta)\bar{\psi}\gamma_5\psi \tag{7.126}$$

which, as expected, is not invariant under the chiral transformation. Instead, under the chiral transformation the mass term $\bar{\psi}\psi$ and the $\gamma_5$ mass term $\bar{\psi}\gamma_5\psi$ transform (rotate) into each other. Clearly, the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{7.127}$$

is not chiral invariant if $m \neq 0$. However, the fermion current

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}\gamma^\mu\psi \tag{7.128}$$

is chiral invariant, and so is the coupling to a gauge field.

## 7.4 Massless particles

Let us look at the massless limit of the Dirac equation in more detail. Historically this problem grew out of the study of neutrinos (which we now know are not massless). For an eigenstate of 4- momentum $p_\mu$, the Dirac equation is

$$\left(\not{p} - m\right)\psi(p) = 0 \tag{7.129}$$

In the *massless limit* $m = 0$, we get

$$\not{p}\psi(p) = 0 \tag{7.130}$$

which is equivalent to

$$\gamma_5\gamma_0\not{p}\psi(p) = 0 \tag{7.131}$$

Upon expanding in components we find

$$\gamma_5 p_0 \psi(p) = \gamma_5\gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p}\psi(p) \tag{7.132}$$

However,

$$\gamma_5\gamma_0\boldsymbol{\gamma} = \begin{bmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{bmatrix} = \boldsymbol{\Sigma} \tag{7.133}$$

Hence

$$\gamma_5 p_0\psi(p) = \boldsymbol{\Sigma}\cdot\boldsymbol{p}\psi(p) \tag{7.134}$$

Thus, the *chirality* $\gamma_5$ is equivalent to the *helicity* $\boldsymbol{\Sigma}\cdot\boldsymbol{p}$ of the state (in the massless limit only!). This suggests the introduction of a basis in which $\gamma_5$ is diagonal, the chiral basis, in which

$$\gamma_0 = \begin{pmatrix}0 & -I\\ -I & 0\end{pmatrix} \qquad \gamma_5 = \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix} \tag{7.135}$$

In this basis the massless Dirac equation becomes

$$\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}\cdot\boldsymbol{p}\psi(p) = \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}p_0\psi(p) \tag{7.136}$$

Let us write the 4-spinor $\psi$ in terms of two 2-spinors of the form

$$\psi = \begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix} \tag{7.137}$$

in terms of which the Dirac equation decomposes into two separate equations for each chiral component, $\psi_R$ and $\psi_L$. Thus the *right handed* (positive chirality) component $\psi_R$ satisfies the Weyl equation

$$\left(\boldsymbol{\sigma}\cdot\boldsymbol{p} - p_0\right)\psi_R = 0 \tag{7.138}$$

while the *left handed* (negative chirality) component satisfies instead

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p} + p_0)\, \psi_L = 0 \tag{7.139}$$

Hence one massless Dirac spinor is equivalent to two Weyl spinors.