8

Coherent State Path Integral Quantization of Quantum Field Theory

8.1 Coherent states and path integral quantization.

The path integral that we have used so far is a powerful tool but it can only be used in theories based on canonical quantization. For example, it cannot be used in theories of fermions (relativistic or not), among others. In this chapter we discuss a more general approach based on the concept of coherent states. Coherent states, and their application to path integrals, have been widely discussed. Excellent references include the 1975 lectures by Faddeev (Faddeev, 1976), and the books by Perelomov (Perelomov, 1986), Klauder (Klauder and Skagerstam, 1985), and Schulman (Schulman, 1981). The related concept of geometric quantization is insightfully presented in the work by Wiegmann (Wiegmann, 1989).

8.2 Coherent states

Let us consider a Hilbert space spanned by a complete set of harmonic oscillator states $\{|n\rangle\}$, with $n = 0, \dots, \infty$. Let $\hat{a}^\dagger$ and $\hat{a}$ be a pair of creation and annihilation operators acting on this Hilbert space, and satisfying the commutation relations

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 , \quad \left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0 , \quad [\hat{a}, \hat{a}] = 0 \tag{8.1}$$

These operators generate the harmonic oscillators states $\{|n\rangle\}$ in the usual way,

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n \; |0\rangle, \qquad \hat{a}|0\rangle = 0 \tag{8.2}$$

where $|0\rangle$ is the vacuum state of the oscillator.

Let us denote by $|z\rangle$ the *coherent state*

$$|z\rangle = e^{z\hat{a}^\dagger}\ |0\rangle, \qquad \langle z| = \langle 0|\ e^{\bar{z}\hat{a}} \tag{8.3}$$

where z is an arbitrary complex number and $\bar{z}$ is the complex conjugate. The coherent state $|z\rangle$ has the defining property of being a wave packet with optimal spread, i.e. the Heisenberg uncertainty inequality is an equality for these coherent states.

How does $\hat{a}$ act on the coherent state $|z\rangle$?

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!}\ \hat{a}\left(\hat{a}^\dagger\right)^n\ |0\rangle \tag{8.4}$$

Since

$$\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = n\ \left(\hat{a}^\dagger\right)^{n-1}, \tag{8.5}$$

we find

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!}\ n\left(\hat{a}^\dagger\right)^{n-1}\ |0\rangle \equiv z\ |z\rangle \tag{8.6}$$

Therefore $|z\rangle$ is a right eigenvector of $\hat{a}$ and z is the (right) eigenvalue.

Likewise we get

$$\hat{a}^\dagger|z\rangle = \hat{a}^\dagger\ \sum_{n=0}^{\infty} \frac{z^n}{n!}\ \left(\hat{a}^\dagger\right)^n\ |0\rangle = \sum_{n=1}^{\infty} n\ \frac{z^{n-1}}{n!}\ \left(\hat{a}^\dagger\right)^n\ |0\rangle \tag{8.7}$$

Thus,

$$\hat{a}^\dagger|z\rangle = \partial_z\ |z\rangle \tag{8.8}$$

Therefore the operators z and ∂_z provide a representation of the algebra of creation and annihilation operators.

Another quantity of interest is the overlap of two coherent states, $\langle z|z'\rangle$,

$$\langle z|z'\rangle = \langle 0|e^{\bar{z}\hat{a}}\ \ e^{z'\hat{a}^\dagger}\ |0\rangle \tag{8.9}$$

We will calculate this matrix element using the Baker-Hausdorff formulas

$$e^{\hat{A}}\ e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}\left[\hat{A},\hat{B}\right]} = e^{\left[\hat{A},\hat{B}\right]}\ e^{\hat{B}}\ e^{\hat{A}} \tag{8.10}$$

which holds provided the commutator $\left[\hat{A}, \hat{B}\right]$ is a c-number, i.e. it is proportional to the identity operator. Since $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, we find

$$\langle z|z'\rangle = e^{\bar{z}z'}\ \langle 0|e^{z'\hat{a}^\dagger}\ e^{\bar{z}\hat{a}}|0\rangle \tag{8.11}$$

But

$$e^{\bar{z}\hat{a}}\;|0\rangle = |0\rangle, \qquad \langle 0|\; e^{z'\hat{a}^\dagger} = \langle 0| \tag{8.12}$$

Hence we get

$$\langle z|z'\rangle = e^{\bar{z}z'} \tag{8.13}$$

An arbitrary state $|\psi\rangle$ of this Hilbert space can be expanded in the harmonic oscillator basis states $\{|n\rangle\}$,

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}}\;|n\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!}\;\left(\hat{a}^\dagger\right)^n\;|0\rangle \tag{8.14}$$

The projection of the state $|\psi\rangle$ onto the coherent state $|z\rangle$ is

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!}\;\langle z|\left(\hat{a}^\dagger\right)^n|0\rangle \tag{8.15}$$

Since

$$\langle z|\;\hat{a}^\dagger = \bar{z}\;\langle z| \tag{8.16}$$

we find

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{n!}\;\bar{z}^n \equiv \psi(\bar{z}) \tag{8.17}$$

Therefore the projection of $|\psi\rangle$ onto $|z\rangle$ is the *anti-holomorphic* (i.e. *anti-analytic*) function $\psi(\bar{z})$. In other words, in this representation, the space of states $\{|\psi\rangle\}$ are in one-to-one correspondence with the space of anti-analytic functions.

In summary, the coherent states $\{|z\rangle\}$ satisfy the following properties

$$\begin{aligned} \hat{a}|z\rangle &= z|z\rangle & \langle z|\hat{a} &= \partial_{\bar{z}}\langle z| \\ \hat{a}^\dagger|z\rangle &= \partial_z|z\rangle & \langle z|\hat{a}^\dagger &= \bar{z}\langle z| \\ \langle z|\psi\rangle &= \psi(\bar{z}) & \langle\psi|z\rangle &= \bar{\psi}(z) \end{aligned} \tag{8.18}$$

Next we will prove the *resolution of identity*

$$\hat{I} = \int \frac{dzd\bar{z}}{2\pi i}\;e^{-z\bar{z}}|z\rangle\langle z| \tag{8.19}$$

Let $|\psi\rangle$ and $|\phi\rangle$ be two arbitrary states

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{\psi_n}{\sqrt{n!}}|n\rangle, \qquad |\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi_n}{\sqrt{n!}}\;|n\rangle \tag{8.20}$$

such that their inner product is

$$\langle\phi|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\bar{\phi}_n\psi_n \tag{8.21}$$

Let us compute the matrix element of the operator $\hat{I}$ given in Eq.(8.19),

$$\langle\phi|\hat{I}|\psi\rangle = \sum_{m.n} \frac{\bar{\phi}_n\psi_n}{n!}\langle n|\hat{I}|m\rangle \tag{8.22}$$

Thus we need to compute

$$\langle n|\hat{I}|m\rangle = \int \frac{dzd\bar{z}}{2\pi i}\, e^{-|z|^2}\langle n|z\rangle\langle z|m\rangle \tag{8.23}$$

Recall that the integration measure is defined to be given by

$$\frac{dzd\bar{z}}{2\pi i} = \frac{d\mathrm{Re}z\; d\mathrm{Im}z}{\pi} \tag{8.24}$$

The overlaps are given by

$$\langle n|z\rangle = \frac{1}{\sqrt{n!}}\langle 0|\left(\hat{a}\right)^n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle \tag{8.25}$$

and

$$\langle z|m\rangle = \frac{1}{\sqrt{m!}}\langle z|\left(\hat{a}^\dagger\right)^m|0\rangle = \frac{\bar{z}^m}{\sqrt{m!}}\langle z|0\rangle \tag{8.26}$$

Now, since $|\langle 0|z\rangle|^2 = 1$, we get

$$\begin{aligned}\langle n|\hat{I}|m\rangle &= \int \frac{dzd\bar{z}}{2\pi i}\, \frac{e^{-|z|^2}}{\sqrt{n!m!}} z^n\bar{z}^m \\ &= \int_0^\infty \rho d\rho \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{e^{-\rho^2}}{\sqrt{n!m!}}\rho^{n+m} e^{i(n-m)\varphi}\end{aligned} \tag{8.27}$$

Thus,

$$\langle n|\hat{I}|m\rangle = \frac{\delta_{n,m}}{n!}\int_0^\infty dx\; x^n e^{-x} = \langle n|m\rangle \tag{8.28}$$

Hence, we have found that

$$\langle\phi|\hat{I}|\psi\rangle = \langle\phi|\psi\rangle \tag{8.29}$$

for any pair of states $|\psi\rangle$ and $|\phi\rangle$. Therefore $\hat{I}$ is the identity operator in this Hilbert space. We conclude that the set of coherent states $\{|z\rangle\}$ is an *over-complete* set of states.

Furthermore, since

$$\langle z| \left(\hat{a}^{\dagger}\right)^{n} (\hat{a})^{m} |z'\rangle = \bar{z}^{n} z'^{m} \langle z|z'\rangle = \bar{z}^{n} z'^{m} e^{\bar{z}z'} \tag{8.30}$$

we conclude that the matrix elements in generic coherent states $|z\rangle$ and $|z'\rangle$ of any arbitrary normal ordered operator of the form

$$\hat{A} = \sum_{n,m} A_{n,m} \left(\hat{a}^{\dagger}\right)^{n} (\hat{a})^{m} \tag{8.31}$$

are equal to

$$\langle z|\hat{A}|z'\rangle = \left(\sum_{n,m} A_{n,m} \bar{z}^{n} z'^{m}\right) e^{\bar{z}z'} \tag{8.32}$$

Therefore, if $\hat{A}(\hat{a}, \hat{a}^{\dagger})$ is an arbitrary *normal ordered* operator (relative to the state $|0\rangle$), its matrix elements are given by

$$\langle z|\hat{A}(\hat{a}, \hat{a}^{\dagger})|z'\rangle = A(\bar{z}, z')\ e^{\bar{z}z'} \tag{8.33}$$

where $A(\bar{z}, z')$ is a function of two complex variables $\bar{z}$ and z' obtained from $\hat{A}$ by the formal replacement

$$\hat{a} \leftrightarrow z', \qquad \hat{a}^{\dagger} \leftrightarrow \bar{z} \tag{8.34}$$

The complex function $A(\bar{z}, z')$ is oftentimes called the symbol of the (normal-ordered) operator $\hat{A}$.

For example, the matrix elements of the number operator $\hat{N} = \hat{a}^{\dagger}\hat{a}$, which measures the number of excitations, is

$$\langle z|\hat{N}|z'\rangle = \langle z|\hat{a}^{\dagger}\hat{a}|z'\rangle = \bar{z}z'\ e^{\bar{z}z'} \tag{8.35}$$

8.3 Path integrals and coherent states

The concept of coherent states have been applied to broad areas of Quantum Mechanics (Perelomov, 1986; Klauder and Skagerstam, 1985). Here we will focus on its application to path integrals (Faddeev, 1976).

We want to compute the matrix elements of the evolution operator $\mathcal{U}$,

$$\mathcal{U} = e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^{\dagger}, \hat{a})} \tag{8.36}$$

where $\hat{H}(\hat{a}^{\dagger}, \hat{a})$ is a normal ordered operator and $T = t_f - t_i$ is the total time lapse. Thus, if $|i\rangle$ and $|f\rangle$ denote two arbitrary initial and final states, we can write the matrix element of $\mathcal{U}$ as

$$\langle f| e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})} |i\rangle = \lim_{\epsilon\to 0, N\to\infty} \langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})\right)^N |i\rangle \tag{8.37}$$

However, instead of inserting a complete set of states at each intermediate time t_j (with $j = 1, \ldots, N$), we will now insert an over-complete set of coherent states $\{|z_j\rangle\}$ at each time t_j through the insertion of the resolution of the identity,

$$\begin{aligned}\langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})\right)^N |i\rangle =& \\ = \int \Big(\prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i}\Big) e^{-\sum_{j=1}^{N} |z_j|^2} & \left[\prod_{k=1}^{N-1} \langle z_{k+1}| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})\right) |z_k\rangle\right] \\ \times \langle f| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})\right) |z_N\rangle & \langle z_1| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})\right) |z_i\rangle\end{aligned} \tag{8.38}$$

In the limit $\epsilon \to 0$ these matrix elements become

$$\begin{aligned}\langle z_{k+1}| \left(1 - i\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})\right) |z_k\rangle =& \langle z_{k+1}|z_k\rangle - i\frac{\epsilon}{\hbar}\langle z_{k+1}|\hat{H}(\hat{a}^\dagger,\hat{a})|z_k\rangle \\ =& \langle z_{k+1}|z_k\rangle \left[1 - i\frac{\epsilon}{\hbar}H(\bar{z}_{k+1}, z_k)\right]\end{aligned} \tag{8.39}$$

where $H(\bar{z}_{k+1}, z_k)$ is a *function* obtained from the *normal-ordered* Hamiltonian by performing the substitutions $\hat{a}^\dagger \to \bar{z}_{k+1}$ and $\hat{a} \to z_k$.

Hence, we can write the following expression for the matrix element of the evolution operator of the form

$$\begin{aligned}&\langle f| e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})} |i\rangle = \\ &= \lim_{\epsilon\to 0, N\to\infty} \int \left(\prod_{j=1}^{N} \frac{dz_j d\bar{z}_j}{2\pi i}\right) e^{-\sum_{j=1}^{N} |z_j|^2} e^{\sum_{j=1}^{N-1} \bar{z}_{j+1} z_j} \prod_{j=1}^{N-1} \left[1 - i\frac{\epsilon}{\hbar}H(\bar{z}_{k+1}, z_k)\right] \\ &\times \langle f|z_N\rangle\langle z_1|i\rangle \left[1 - i\frac{\epsilon}{\hbar}\frac{\langle f|\hat{H}|z_N\rangle}{\langle f|z_N\rangle}\right]\left[1 - i\frac{\epsilon}{\hbar}\frac{\langle z_1|\hat{H}|i\rangle}{\langle z_1|i\rangle}\right]\end{aligned} \tag{8.40}$$

By further expanding the initial and final states in coherent states

$$\langle f| = \int \frac{dz_f d\bar{z}_f}{2\pi i} e^{-|z_f|^2} \bar{\psi}_f(z_f) \langle z_f|$$
$$|i\rangle = \int \frac{dz_i d\bar{z}_i}{2\pi i} e^{-|z_i|^2} \psi_i(\bar{z}_i) |z_i\rangle \tag{8.41}$$

we find the (formal) result

$$\langle f| e^{-i\frac{T}{\hbar}\hat{H}(\hat{a}^\dagger, \hat{a})} |i\rangle = $$
$$= \int \mathcal{D}z\mathcal{D}\bar{z} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\frac{\hbar}{2i}\left(z\partial_t \bar{z} - \bar{z}\partial_t z\right) - H(z,\bar{z})\right]}$$
$$\times e^{\frac{1}{2}(|z_i|^2 + |z_f|^2)} \bar{\psi}_f(z_f)\psi_i(\bar{z}_i) \tag{8.42}$$

This is the coherent-state form of the path integral. We can identify in this expression the Lagrangian L as the quantity

$$L = \frac{\hbar}{2i}\left(z\partial_t \bar{z} - \bar{z}\partial_t z\right) - H(z,\bar{z}) \tag{8.43}$$

It is easy to check that this expression is equivalent to the phase-space path integral derived in Section 5.1.

Notice that the Lagrangian for the coherent-state representation presented in Eq.(8.43) is first order in time derivatives. Because of this feature we are not guaranteed that the paths are necessarily differentiable. This property leads to all kinds of subtleties that for the most part we will ignore in what follows.

8.4 Path integral for a non-relativistic Bose gas

The field theoretic description of a gas of (spinless) non-relativistic bosons is given in terms of the creation and annihilation field operators $\hat{\phi}^\dagger(\boldsymbol{x})$ and $\hat{\phi}(\boldsymbol{x})$, that satisfy the equal time commutation relations (in d space dimensions)

$$\left[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^\dagger(\boldsymbol{y})\right] = \delta^d(\boldsymbol{x} - \boldsymbol{y}) \tag{8.44}$$

Relative to the empty state $|0\rangle$, such that

$$\hat{\phi}(\boldsymbol{x})|0\rangle = 0 \tag{8.45}$$

the normal ordered Hamiltonian (in the Grand canonical Ensemble) is

$$\hat{H} = \int d^d x \, \hat{\phi}^\dagger(\boldsymbol{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 - \mu + V(\boldsymbol{x}) \right] \hat{\phi}(\boldsymbol{x}) + \frac{1}{2} \int d^d x \int d^d y \, \hat{\phi}^\dagger(\boldsymbol{x}) \hat{\phi}^\dagger(\boldsymbol{y}) U(\boldsymbol{x} - \boldsymbol{y}) \hat{\phi}(\boldsymbol{y}) \hat{\phi}(\boldsymbol{x}) \tag{8.46}$$

where m is the mass of the bosons, μ is the chemical potential, $V(\boldsymbol{x})$ is an external potential and $U(\boldsymbol{x} - \boldsymbol{y})$ is the interaction potential between pairs of bosons.

Following our discussion of the coherent state path integral we see that it is immediate to write down a path integral for a thermodynamically large system of bosons. The boson coherent states are now labelled by a *complex field* $\phi(\boldsymbol{x})$ and its *complex conjugate* $\bar{\phi}(\boldsymbol{x})$.

$$|\{\phi(\boldsymbol{x})\}\rangle = e^{\int d\boldsymbol{x} \, \phi(\boldsymbol{x}) \hat{\phi}^\dagger(\boldsymbol{x})} |0\rangle \tag{8.47}$$

which has the coherent state property of being a right eigenstate of the field operator $\hat{\phi}(\mathbf{x})$,

$$\hat{\phi}(\boldsymbol{x}) |\{\phi\}\rangle = \phi(\boldsymbol{x}) |\{\phi\}\rangle \tag{8.48}$$

as well as obeying the resolution of the identity in this space of states

$$\mathcal{I} = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{-\int d\boldsymbol{x} \, |\phi(\boldsymbol{x})|^2} |\{\phi\}\rangle \langle \{\phi\}| \tag{8.49}$$

The matrix element of the evolution operator of this system between an arbitrary initial state $|i\rangle$ and an arbitrary final state $|f\rangle$, separated by a time lapse $T = t_f - t_i$ (not to be confused with the temperature!), now takes the form

$$\langle f | e^{-\frac{i}{\hbar} \hat{H} T} | i \rangle = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \, \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \, \left(\int d^d x \, \frac{\hbar}{i} \left[\phi(\boldsymbol{x}, t) \partial_t \bar{\phi}(\boldsymbol{x}, t) - \bar{\phi}(\boldsymbol{x}, t) \partial_t \phi(\boldsymbol{x}, t) \right] - H[\phi, \bar{\phi}] \right) \right\} \times \bar{\Psi}_f(\phi(\boldsymbol{x}, t_f)) \Psi_i(\bar{\phi}(\boldsymbol{x}, t_i)) \, e^{\frac{1}{2} \int d\boldsymbol{x} (|\phi(\boldsymbol{x}, t_f)|^2 + |\phi(\boldsymbol{x}, t_i)|^2)} \tag{8.50}$$

where the functional $H[\phi,\bar\phi]$ is

$$H[\phi,\bar\phi] = \int d^dx\, \bar\phi(\boldsymbol{x})\left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + V(\boldsymbol{x})\right]\phi(\boldsymbol{x}) + \frac{1}{2}\int d^dx \int d^dy\, |\phi(\boldsymbol{x})|^2|\phi(\boldsymbol{y})|^2 U(\boldsymbol{x}-\boldsymbol{y}) \tag{8.51}$$

It is also possible to write the action S in the less symmetric but simpler form (where $d^Dx \equiv dt d^dx$)

$$S = \int d^Dx\, \bar\phi(x)\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\vec\nabla^2 + \mu - V(\boldsymbol{x})\right)\phi(x) - \frac{1}{2}\int d^Dx \int d^Dy\, |\phi(x)|^2|\phi(y)|^2 U(x-y) \tag{8.52}$$

where $U(x-y) = U(\boldsymbol{x}-\boldsymbol{y})\delta(t_x - t_y)$.

Therefore the path integral for a system of non-relativistic bosons, with chemical potential μ, has the same form os the path integral of the charged scalar field we discussed before except that the action is first order in time derivatives. The fact that the field is complex follows from the requirement that the number of bosons is a globally conserved quantity, which is why one is allowed to introduce a chemical potential.

This formulation is useful to study superfluid Helium and similar problems. Suppose for instance that we want to compute the partition function Z for this system of bosons at finite temperature T,

$$Z = \text{tr} e^{-\beta\hat H} \tag{8.53}$$

where $\beta = 1/T$ (in units where $k_B = 1$). The coherent-state path integral representation of the partition function is obtained by 1) restricting the initial and final states to be the same $|i\rangle = |f\rangle$ and arbitrary, 2) summing over all possible states, and 3) a Wick rotation to imaginary time $t \to -i\tau$, with the *time-span* $T \to -i\beta\hbar$, with periodic boundary conditions in imaginary time.

The result is the (imaginary time) path integral

$$Z = \int \mathcal{D}\phi\mathcal{D}\bar\phi\; e^{-S_E(\phi,\bar\phi)} \tag{8.54}$$

where S_E is the Euclidean action

$$S_E(\phi,\bar{\phi}) = \frac{1}{\hbar}\int_0^{\beta} d\tau \int d\boldsymbol{x}\; \bar{\phi}\left[\hbar\partial_\tau - \mu - \frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\phi + \frac{1}{2\hbar}\int_0^{\beta} d\tau \int d\boldsymbol{x} \int d\boldsymbol{y}\; U(x-y)\; |\phi(x)|^2 |\phi(y)|^2 \tag{8.55}$$

The fields $\phi(x) = \phi(\boldsymbol{x},\tau)$ satisfy periodic boundary conditions in imaginary time

$$\phi(\boldsymbol{x},\tau) = \phi(\boldsymbol{x},\tau+\beta\hbar) \tag{8.56}$$

This requirement suggests an expansion of the field $\phi(x)$ in Fourier modes of the form

$$\phi(\boldsymbol{x},\tau) = \sum_{n=-\infty}^{\infty} e^{i\omega_n \tau}\phi(\vec{x},\omega_n) \tag{8.57}$$

where the frequencies ω_n (the *Matsubara* frequencies) must be chosen so that ϕ obeys the required PBCs. We find

$$\omega_n = \frac{2\pi}{\beta\hbar}n = \frac{2\pi T}{\hbar}n\,, \qquad n \in \mathbb{Z} \tag{8.58}$$

where n is an arbitrary integer. In one of the problems at the end of this chapter you will evaluate this path integral using a semiclassical approximation.

8.5 Fermion coherent states

In this section we will develop a formalism for *fermions* which follows closely what we have done for bosons while taking into account the anti-commuting nature of fermionic operators.

Let $\{c_i^\dagger\}$ be a set of *fermion* creation operators, with $i = 1,\ldots,N$, and $\{c_i\}$ the set of their N adjoint operators, the associated annihilation operators. The number operator for the i-th fermion is $n_i = c_i^\dagger c_i$. Let us define the basis states states of the i-th fermion by the kets $|0_i\rangle$ and $|1_i\rangle$, which obey the obvious definitions:

$$c_i|0_i\rangle = 0, \qquad c_i^\dagger|0_i\rangle = |1_i\rangle, \qquad c_i^\dagger c_i|0_i\rangle = 0, \qquad c_i^\dagger c_i|1_i\rangle = |1_i\rangle \tag{8.59}$$

For N fermions the Hilbert space is spanned by the anti-symmetrized states $|n_1,\ldots,n_N\rangle$. Let

$$|0\rangle \equiv |0_1,\ldots,0_N\rangle \tag{8.60}$$

be the empty state. A general state in this Hilbert space is

$$|n_1, \dots, n_N\rangle = \left(c_1^\dagger\right)^{n_1} \dots \left(c_N^\dagger\right)^{n_N} |0\rangle \tag{8.61}$$

As we saw before, the wave function $\langle n_1, \dots, n_N|\Psi\rangle$ is fully antisymmetric. If the state $|\Psi\rangle$ is a product state, the wave function is a Slater determinant.

8.5.1 Definition of fermion coherent states

We now define fermion coherent states. Let $\{\bar{\xi}_i, \xi_i\}$, with $i = 1, \dots, N$, be a set of $2N$ *Grassmann variables*, a set of symbols also known as the *generators* of a Grassmann algebra. By definition, Grassmann variables satisfy the following properties

$$\left\{\xi_i, \xi_j\right\} = \left\{\bar{\xi}_i, \bar{\xi}_j\right\} = \left\{\xi_i, \bar{\xi}_j\right\} = \xi_i^2 = \bar{\xi}_i^2 = 0 \tag{8.62}$$

Therefore, Grassmann variables behave a a set of time-ordered fermion operators.

We will also require that the Grassmann variables anti-commute with the fermion operators:

$$\left\{\xi_i, c_j\right\} = \left\{\bar{\xi}_i, c_j^\dagger\right\} = \left\{\bar{\xi}_i, c_{,j}\right\} = \left\{\bar{\xi}_i, c_j^\dagger\right\} = 0 \tag{8.63}$$

Let us define the *fermion coherent states* to be

$$|\xi\rangle \equiv e^{-\xi c^\dagger} |0\rangle \tag{8.64}$$

$$\langle\xi| \equiv \langle 0|\, e^{\bar{\xi} c} \tag{8.65}$$

As a consequence of these definitions we have:

$$e^{-\xi c^\dagger} = 1 - \xi c^\dagger \tag{8.66}$$

Similarly, if ψ is a Grassmann variable, then

$$\langle\xi|\psi\rangle = \langle 0|e^{\bar{\xi}c} e^{-\psi c^\dagger}|0\rangle = 1 + \bar{\xi}\psi = e^{\bar{\xi}\psi} \tag{8.67}$$

For N fermions we have,

$$|\xi\rangle \equiv |\xi_1, \dots, \xi_N\rangle = \Pi_{i=1}^N e^{-\xi_i c_i^\dagger}|0\rangle \equiv e^{-\sum_{i=1}^N \xi_i c_i^\dagger}|0\rangle \tag{8.68}$$

since the following commutator vanishes,

$$\left[\xi_i c_i^\dagger, \xi_j c_j^\dagger\right] = 0 \tag{8.69}$$

8.5.2 Analytic functions of Grassmann variables

We will define $\psi(\xi)$ to be an *analytic* function of the Grassmann variable if it has a power series expansion in ξ,

$$\psi(\xi) = \psi_0 + \psi_1\xi + \psi_2\xi^2 + \dots \tag{8.70}$$

where $\psi_n \in \mathbb{C}$. Since

$$\xi^n = 0, \qquad \forall n \geq 2 \tag{8.71}$$

then, all analytic functions of a Grassmann variable reduce to a first degree polynomial,

$$\psi(\xi) \equiv \psi_0 + \psi_1\xi \tag{8.72}$$

Similarly, we define complex conjugation by

$$\overline{\psi(\xi)} \equiv \bar{\psi}_0 + \bar{\psi}_1\bar{\xi} \tag{8.73}$$

where $\bar{\psi}_0$ and $\bar{\psi}_1$ are the complex conjugates of ψ_0 and ψ_1 respectively.

We can also define functions of two Grassmann variables ξ and $\bar{\xi}$,

$$A(\bar{\xi},\xi) = a_0 + a_1\xi + \bar{a}_1\bar{\xi} + a_{12}\bar{\xi}\xi \tag{8.74}$$

where a_1, $\bar{a}_1$ and a_{12} are complex numbers; a_1 and $\bar{a}_1$ are *not* necessarily complex conjugates of each other.

8.5.3 Differentiation over Grassmann variables

Since analytic functions of Grassmann variables have such a simple structure, differentiation is just as simple. Indeed, we define the *derivative* as the coefficient of the linear term

$$\partial_\xi\psi(\xi) \equiv \psi_1 \tag{8.75}$$

Likewise we also have

$$\partial_{\bar{\xi}}\overline{\psi(\xi)} \equiv \bar{\psi}_1 \tag{8.76}$$

Clearly, using this rule we can write

$$\partial_\xi(\bar{\xi}\xi) = -\partial_\xi(\xi\bar{\xi}) = -\bar{\xi} \tag{8.77}$$

A similar argument shows that

$$\partial_\xi A(\bar{\xi},\xi) = a_1 - a_{12}\,\bar{\xi} \tag{8.78}$$

$$\partial_{\bar{\xi}} A(\bar{\xi},\xi) = \bar{a}_1 + a_{12}\,\xi \tag{8.79}$$

$$\partial_{\bar{\xi}}\partial_\xi A(\bar{\xi},\xi) = -a_{12} = -\partial_\xi\partial_{\bar{\xi}} A(\bar{\xi},\xi) \tag{8.80}$$

from where we conclude that ∂_ξ and $\partial_{\bar{\xi}}$ *anti-commute*,

$$\left\{\partial_{\bar{\xi}}, \partial_\xi\right\} = 0, \qquad \text{and} \qquad \partial_\xi \partial_\xi = \partial_{\bar{\xi}} \partial_{\bar{\xi}} = 0 \tag{8.81}$$

8.5.4 Integration over Grassmann variables

The basic differentiation rule of Eq. (8.75) implies that

$$1 = \partial_\xi \xi \tag{8.82}$$

which suggests the following *definitions*:

$$\int d\xi \; 1 = 0, \qquad \int d\xi \; \partial_\xi \xi = 0, \qquad \int d\xi \; \xi = 1 \tag{8.83}$$

Analogous rules also apply for the conjugate variables $\bar{\xi}$.

It is instructive to compare the differentiation and integration rules:

$$\begin{array}{lcl} \int d\xi \; 1 = 0 & \leftrightarrow & \partial_\xi 1 = 0 \\ \int d\xi \; \xi = 1 & \leftrightarrow & \partial_\xi \xi = 1 \end{array} \tag{8.84}$$

Thus, for Grassmann variables differentiation and integration are exactly equivalent

$$\partial_\xi \Longleftrightarrow \int d\xi \tag{8.85}$$

These rules imply that the integral of an analytic function $f(\xi)$ is

$$\int d\xi \; f(\xi) = \int d\xi \; (f_0 + f_1 \xi) = f_1 \tag{8.86}$$

and

$$\begin{aligned} \int d\xi A(\bar{\xi}, \xi) &= \int d\xi \left(a_0 + a_1 \xi + \bar{a}_1 \bar{\xi} + a_{12} \bar{\xi} \xi\right) = a_1 - a_{12} \bar{\xi} \\ \int d\bar{\xi} A(\bar{\xi}, \xi) &= \int d\xi \left(a_0 + a_1 \xi + \bar{a}_1 \bar{\xi} + a_{12} \bar{\xi} \xi\right) = \bar{a}_1 + a_{12} \xi \\ \int d\bar{\xi} d\xi A(\bar{\xi}, \xi) &= - \int d\xi d\bar{\xi} A(\bar{\xi}, \xi) = -a_{12} \end{aligned} \tag{8.87}$$

It is straightforward to show that with these definitions, the following expression is a consistent definition of a delta-function:

$$\delta(\xi', \xi) = \int d\eta \; e^{-\eta(\xi - \xi')} \tag{8.88}$$

where ξ, ξ' and η are Grassmann variables.

Finally, given that we have a vector space of analytic functions we can define an *inner product* as follows:

$$\langle f|g\rangle = \int d\bar{\xi}d\xi\; e^{-\bar{\xi}\xi}\; \bar{f}(\xi)\; g(\bar{\xi}) = \bar{f}_0 g_0 + \bar{f}_1 g_1 \tag{8.89}$$

as expected.

8.5.5 Properties of fermion coherent states

We defined above the fermion bra and ket coherent states

$$|\{\xi_j\}\rangle = e^{-\sum_j \xi_j c_j^\dagger}|0\rangle, \qquad \langle\{\xi_j\}| = \langle 0|\; e^{\sum_j \bar{\xi}_j c_j} \tag{8.90}$$

After a little algebra, using the rules defined above, it is easy to see that the following identities hold:

$$c_i|\{\xi_j\}\rangle = \xi_i\; |\{\xi_j\}\rangle \qquad c_i^\dagger|\{\xi_j\}\rangle = -\partial_{\xi_i}\; |\{\xi_j\}\rangle \tag{8.91}$$

$$\langle\{\xi_j\}|\; c_i = \partial_{\bar{\xi}_i}\; \langle\{\xi_j\}| \qquad \langle\{\xi_j\}|\; c_i^\dagger = \bar{\xi}_i\; \langle\{\xi_j\}| \tag{8.92}$$

The inner product of two coherent states, $|\{\xi_j\}\rangle$ and $|\{\xi_j'\}\rangle$, is

$$\langle\{\xi_j\}|\{\xi_j'\}\rangle = e^{\sum_j \bar{\xi}_j \xi_j} \tag{8.93}$$

Similarly, we also have the *Resolution of the Identity* (which is easy to prove)

$$I = \int \left(\Pi_{i=1}^N d\bar{\xi}_i d\xi_i\right)\; e^{-\sum_{i=1}^N \bar{\xi}_i\xi_i}\; |\{\xi_i\}\rangle\langle\{\xi_i\}| \tag{8.94}$$

Let $|\Psi\rangle$ be some state. Then, we can use Eq. (8.94) to expand the state $|\Psi\rangle$ in fermion coherent states $|\xi\rangle$,

$$|\Psi\rangle = \int \left(\Pi_{i=1}^N d\bar{\xi}_i d\xi_i\right)\; e^{-\sum_{i=1}^N \bar{\xi}_i\xi_i}\; \Psi(\xi)|\{\xi_i\}\rangle \tag{8.95}$$

where

$$\Psi(\bar{\xi}) \equiv \Psi(\bar{\xi}_1, \dots, \bar{\xi}_N) \tag{8.96}$$

We can use the rules derived above to compute the following matrix elements

$$\langle\xi|c_j|\Psi\rangle = \partial_{\bar{\xi}_j}\Psi(\bar{\xi}), \qquad \langle\xi|c_j^\dagger|\Psi\rangle = \bar{\xi}_j\; \Psi(\bar{\xi}) \tag{8.97}$$

which is consistent with what we concluded above.

Let $|0\rangle$ be the "empty state". We will not call it the "vacuum" since it is not in the sector of the ground state of the systems of interest. Let $A\left(\{c_j^\dagger\},\{c_j\}\right)$ be a *normal ordered operator* (with respect to the state $|0\rangle$). By using the formalism worked out above one can show without difficulty that its matrix elements in the coherent states $|\xi\rangle$ and $|\xi'\rangle$ are

$$\langle\xi|A\left(\{c_j^\dagger\},\{c_j\}\right)|\xi'\rangle = e^{\sum_i \bar{\xi}_i \xi'_i} A\left(\{\bar{\xi}_j\},\{\xi'_j\}\right) \tag{8.98}$$

For example, the expectation value of the fermion number operator $\hat{N}$,

$$\hat{N} = \sum_j c_j^\dagger c_j \tag{8.99}$$

in the coherent state $|\xi\rangle$ is

$$\frac{\langle\xi|\hat{N}|\xi\rangle}{\langle\xi|\xi\rangle} = \sum_j \bar{\xi}_j \xi_j \tag{8.100}$$

8.5.6 Grassmann gaussian integrals

Let us consider a Gaussian integral over Grassmann variables of the form

$$\mathcal{Z}[\bar{\zeta},\zeta] = \int \left(\prod_{i=1}^{N} d\bar{\xi}_i d\xi_i\right) e^{-\sum_{i,j} \bar{\xi}_i M_{ij} \xi_j + \bar{\xi}_i \zeta_i + \bar{\zeta}_i \xi_i} \tag{8.101}$$

where $\{\zeta_i\}$ and $\{\bar{\zeta}_i\}$ are a set of $2N$ Grassmann variables, and the matrix M_{ij} is a complex Hermitian matrix. We will now show that

$$\mathcal{Z}[\bar{\zeta},\zeta] = (\det M)\; e^{\sum_{ij} \bar{\zeta}_i \left(M^{-1}\right)_{ij} \zeta_j} \tag{8.102}$$

Before showing that Eq. (8.102) is correct, let us make a few observations:

1. Eq. (8.102) looks like the familiar expression for Gaussian integrals for bosons except that, instead of a factor of $(\det M)^{-1/2}$, we get a factor of $\det M$ in the numerator. Except for the absence of a square root, that the fluctuation determinant appears in the numerator is the main effect of the Fermi statistics!.
2. Moreover, if we had considered a system of N Grassmann variables (instead of $2N$) we would have obtained instead a factor of

$$(\det M)^{1/2} = \text{Pf}(M) \tag{8.103}$$

where M would now be an $N \times N$ *real anti-symmetric* matrix, and $\text{Pf}(M)$ denotes the Pfaffian of the matrix M.

To prove that Eq. (8.102) is correct it will be sufficient to consider the case $\zeta_i = \bar{\zeta}_i = 0$, since the contribution from these sources is identical to the bosonic case (except for the ordering of the Grassmann variables). Using the Grassmann identities we can write the exponential factor as

$$e^{-\sum\limits_{i,j} \bar{\xi}_i M_{ij} \xi_j} = \prod_{ij} \left(1 - \bar{\xi}_i M_{ij} \xi_j\right) \tag{8.104}$$

The integral that we need to do is

$$\mathcal{Z}[0,0] = \int \left(\prod_{i=1}^{N} d\bar{\xi}_i \xi_i \right) \prod_{ij} \left(1 - \bar{\xi}_i M_{ij} \xi_j\right) \tag{8.105}$$

From the integration rules, we can easily see that the only non-vanishing terms in this expression are those that have the just one ξ_i and one $\bar{\xi}_i$ (for each i). Hence we can write

$$\begin{aligned}
\mathcal{Z}[0,0] =& (-1)^N \int \left(\prod_{i=1}^{N} d\bar{\xi}_i \; d\xi_i \right) \bar{\xi}_1 M_{12} \xi_2 \bar{\xi}_2 M_{23} \xi_3 \ldots + \text{permutations} \\
=& (-1)^N M_{12} M_{23} M_{34} \ldots \int \left(\prod_{i=1}^{N} d\bar{\xi}_i \; d\xi_i \right) \bar{\xi}_1 \xi_2 \bar{\xi}_2 \xi_3 \bar{\xi}_3 \ldots \bar{\xi}_N \xi_1 \\
& + \text{permutations} \\
=& (-1)^{2N} M_{12} M_{23} M_{34} \ldots M_{N,1} + \text{permutations}
\end{aligned} \tag{8.106}$$

What is the contribution of the terms labeled "permutations"? It is easy to see that if we permute any pair of labels, say 2 and 3, we will get a contribution of the form

$$(-1)^{2N} (-1) M_{13} M_{32} M_{24} \ldots \tag{8.107}$$

Hence we conclude that the Gaussian Grassmann integral is just the determinant of the matrix M,

$$\mathcal{Z}[0,0] = \int \left(\prod_{i=1}^{N} d\bar{\xi}_i d\xi_i \right) e^{-\sum\limits_{ij} \bar{\xi}_i M_{ij} \xi_j} = \det M \tag{8.108}$$

Alternatively, we can diagonalize the quadratic form and notice that the Jacobian is "upside-down" compared with the result we found for bosons.

8.6 Path integrals for fermions

We are now ready to give a prescription for the construction of a fermion path integral in a general system. Let H the a *normal-ordered* Hamiltonian, with respect to *some* reference state $|0\rangle$, of a system of *fermions*. Let $|\Psi_i\rangle$ be the ket at the initial time t_i and $|\Psi_f\rangle$ be the final state at time t_f. The matrix element of the evolution operator can be written as a Grassmann path integral

$$\begin{aligned}\langle\Psi_f, t_f|\Psi_i, t_i\rangle =&\langle\Psi_f| e^{-\frac{i}{\hbar}H(t_f - t_i)}|\Psi_i\rangle\\ \equiv& \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{\frac{i}{\hbar}S(\bar{\psi},\psi)} \times \text{projection operators}\end{aligned} \tag{8.109}$$

where we have not written down the explicit form of the projection operators onto the initial and final states. The action $S(\bar{\psi},\psi)$ is

$$S(\bar{\psi},\psi) = \int_{t_i}^{t_f} dt\ \left[i\hbar\bar{\psi}\partial_t\psi - H(\bar{\psi},\psi)\right] \tag{8.110}$$

This expression of the fermion path integral holds for any theory of fermions, relativistic or not. Notice that it has *the same form* as the bosonic path integral. The only change is that for fermions the determinant appears in the numerator while for bosons it is in the denominator!

8.7 Path integral quantization of the Dirac field

We will now apply the methods we just developed to the case of the Dirac theory. Let $\psi_\alpha(x)$, with $\alpha = 1,\dots,4$ be a free massive Dirac field in 3+1 space-time dimensions. This field satisfies the Dirac Equation as an equation of motion,

$$\left(i\partial\!\!\!/ - m\right)\psi = 0 \tag{8.111}$$

where ψ is a 4-spinor and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. Recall that the Dirac γ-matrices satisfy the algebra

$$\left\{\gamma^\mu,\gamma^\nu\right\} = 2g^{\mu\nu} \tag{8.112}$$

where $g^{\mu\nu}$ is the Minkowski space metric tensor (in the Bjorken-Drell form).

We saw before that in the quantum field theory description of the Dirac theory, ψ is an operator acting on the Fock space of (fermionic) states. We also saw that the Dirac equation can be regarded as the classical equation of motion of the Lagrangian density

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{8.113}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$. We also noted that the momentum canonically conjugate to the field ψ is $i\psi^\dagger$, from where the standard fermionic equal time anti-commutation relations follow

$$\left\{\psi_\alpha(\boldsymbol{x}, x_0), \psi^\dagger_\beta(\boldsymbol{y}, x_0)\right\} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{8.114}$$

The Lagrangian density $\mathcal{L}$ for a Dirac fermion coupled to sources η_α and $\bar{\eta}_\alpha$ is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \bar{\psi}\eta + \bar{\eta}\psi \tag{8.115}$$

where the sources are "classical" (Grassmann) anticommuting fields.

We can follow the same steps described in the preceding sections to find the following expression for the path-integral of the Dirac field in terms of Grassmann fields $\psi(x)$ and $\bar{\psi}(x)$ (which here are independent variables!)

$$\begin{aligned}\mathcal{Z}[\bar{\eta}, \eta] =& \frac{1}{\langle 0|0\rangle}\langle 0|T\, e^{i\int d^4x\left(\bar{\psi}\eta + \bar{\eta}\psi\right)}|0\rangle \\ \equiv& \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS + i\int d^4x(\bar{\psi}\eta + \bar{\eta}\psi)}\end{aligned} \tag{8.116}$$

where $S = \int d^4x\mathcal{L}$.

From this result it follows that the Dirac propagator is given by

$$\begin{aligned}iS_{\alpha\beta}(x-y) =& \langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle \\ =& \frac{(-i)^2}{\mathcal{Z}[0,0]}\left.\frac{\delta^2\mathcal{Z}[\bar{\eta},\eta]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}\right|_{\bar{\eta}=\eta=0} \\ =& \langle x, \alpha|\frac{1}{i\partial\!\!\!/ - m}|y, \beta\rangle\end{aligned} \tag{8.117}$$

Similarly, the partition function for a theory of free Dirac fermions is

$$\mathcal{Z}_{\text{Dirac}} = \text{Det}\left(i\partial\!\!\!/ - m\right) \tag{8.118}$$

In contrast, in the case of a free real scalar field we found

$$\mathcal{Z}_{\text{scalar}} = \left[\text{Det}\left(\partial^2 + m^2\right)\right]^{-1/2} \tag{8.119}$$

Therefore for the case of the Dirac field the partition function is a determinant whereas for the scalar field is the inverse of a determinant (actually, of its square root). The fact that one result is the inverse of the other is a consequence that Dirac fields are quantized as fermions whereas scalar fields are quantized as bosons. As we saw, this is a general result for Fermi and Bose fields regardless of whether they are relativistic or not. Moreover, from this result it follows that the vacuum (ground state) energy of a Dirac field is negative whereas the vacuum energy for a scalar field is positive. We will see that in interacting field theories the result of Eq.(8.118) leads to the Feynman diagram rule that each fermion loop carries a minus sign which reflects the Fermi-Dirac statistics, and hence this rule holds even in the absence of relativistic invariance.

We now note that, due to the charge-conjugation symmetry of the Dirac theory, the spectrum of the Dirac operator is symmetric, i.e for every positive eigenvalue of the Dirac operator there is a negative eigenvalue of equal magnitude. More formally, since $\{\gamma^5, \gamma^\mu\} = 0$,

$$\gamma_5 \left(i\partial\!\!\!/ - m \right) \gamma_5 = \left(-i\partial\!\!\!/ - m \right) \tag{8.120}$$

and $\gamma_5^2 = I$ (the 4×4 identity matrix), it follows that

$$\text{Det} \left(i\partial\!\!\!/ - m \right) = \text{Det} \left(i\partial\!\!\!/ + m \right) \tag{8.121}$$

Hence

$$\begin{aligned} \text{Det} \left(i\partial\!\!\!/ - m \right) =& \Big[\text{Det} \left(i\partial\!\!\!/ - m \right) \text{Det} \left(i\partial\!\!\!/ + m \right) \Big]^{1/2} \\ =& \Big[\text{Det} \left(\partial^2 + m^2 \right) \Big]^2 \end{aligned} \tag{8.122}$$

where we used that the "square" of the Dirac operator is the Klein-Gordon operator multiplied by the 4×4 identity matrix, which is why the exponent of the r.h.s of the equation above is $2 = 4 \times \frac{1}{2}$. It is easy to see that this result implies that the vacuum energy for the Dirac fermion E_0^{Dirac} and the vacuum energy of a scalar field E_0^{scalar}, with the same mass m, are related by

$$E_0^{\text{Dirac}} = -4 E_0^{\text{scalar}} \tag{8.123}$$

Here we have ignored the fact that both the l.h.s. and the r.h.s. of this equation are divergent, as we saw before. However since they have the same divergence, or what is the same is they are regularized in the same way, the comparison is meaningful.

On the other hand, if instead of Dirac fermions, which are charged (and hence complex) fields, we consider Majorana fermions, which are charge neutral, and hence are real fields, we would have obtained instead the results

$$\mathcal{Z}_{\text{Majorana}} = \text{Pf}\left(i\not{\partial} - m\right) = \left[\text{Det}\left(i\not{\partial} - m\right)\right]^{1/2} \tag{8.124}$$

where Pf is the Pfaffian or, what is the same, the square root of the determinant. Thus the vacuum energy of a Majorana fermion is half the vacuum energy of a Dirac fermion.

Finally, since a massless Dirac fermion is equivalent to two Weyl fermions, one for each chirality, it follows that the vacuum energy of a Majorana Weyl fermion is equal and opposite to the vacuum energy of a scalar field. Moreover this relation holds for all the states of their spectra which are identical. This observation is the origin of the concept of supersymmetry in which the fermionic states and the bosonic states are precisely matched. As a result, the vacuum energy of a supersymmetric theory is zero.

We end by discussing briefly the theory of Dirac fermions in Euclidean space-time. We will focus on the theory in four dimensions, but this can be done in any dimension. There are two equivalent ways to do this analytic continuation. One option is to define a set of four anti-hermitian Dirac gamma matrices

$$\gamma_j = \gamma^j, \qquad \gamma_4 = -i\gamma^0 \tag{8.125}$$

with $j = 1, 2, 3$, that satisfy the algebra

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu} \tag{8.126}$$

with $\mu = 1, \dots, 4$. Similarly, we define the hermitian γ_5 matrix

$$\gamma_5 = \gamma^5 = \gamma_1\gamma_2\gamma_3\gamma_4 \tag{8.127}$$

In this notation the partition function is

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp(-S_E(\psi,\bar{\psi})) \tag{8.128}$$

where S_E is the Euclidean for a Dirac spinor which coupled to an abelian gauge field becomes

$$S_E = \int d^4x\bar{\psi}(i\not{D} + m)\psi \tag{8.129}$$

where $D_\mu = \partial_\mu + iA_\mu$, again with $\mu = 1, 2, 3, 4$, with $A_4 = -iA_0$. The Euclidean Dirac propagator in momentum space is given by (with $p_0 = -ip_4$)

$$S(p) = \frac{1}{-\not{p} + m} = \frac{\not{p} + m}{p^2 + m^2} \tag{8.130}$$

Alternatively, we can define the four gamma matrices to be hermitian and satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu nu}$. In this notation the Euclidean action is

$$S_E = \int d^4x \bar{\psi}(\not{D} + m)\psi \tag{8.131}$$

where the operator $\not\partial$ is anti-hermitian. In this notation the propagator is

$$S(p) = \frac{1}{-i\not{p} + m} = \frac{i\not{p} + m}{p^2 + m^2} \tag{8.132}$$

8.8 Functional determinants

We will now face the problem of how to compute functional determinants. We have discussed before how to do that for path-integrals with a few degrees of freedom (i.e. in Quantum Mechanics). We will now generalize these ideas to Quantum Field Theory. We will begin by discussing some simple determinants that show up in systems of fermions and bosons at finite temperature and density.

8.8.1 Functional determinants for coherent states

Consider a system of fermions (or bosons) with one-body Hamiltonian $\hat{h}$ at non-zero temperature T and chemical potential μ. The partition function

$$Z = \text{tr } e^{-\beta\left(\hat{H} - \mu\hat{N}\right)} \tag{8.133}$$

where $\beta = 1/k_B T$,

$$\hat{H} = \int dx\ \hat{\psi}^\dagger(x)\ \hat{h}\ \hat{\psi}(x) \tag{8.134}$$

and

$$\hat{N} = \int dx\ \hat{\psi}^\dagger(x)\ \hat{\psi}(x) \tag{8.135}$$

is the number operator. Here x denotes both spacial and internal (spin) labels.

The functional (or path) integral expression for the partition function is

$$Z = \int \mathcal{D}\psi^* \mathcal{D}\psi\ e^{\frac{i}{\hbar}\int dt\ \psi^*\left(i\hbar\partial_t - \hat{h} + \mu\right)\psi} \tag{8.136}$$

In imaginary time set $t \to -i\tau$, with $0 \le \tau \le \beta\hbar$:

$$Z = \int \mathcal{D}\psi^* \mathcal{D}\psi\ e^{-\int_0^{\beta\hbar} d\tau\ \psi^*\left(\partial_\tau + \hat{h} - \mu\right)\psi} \tag{8.137}$$

The fields $\psi(\tau)$ can represent either be *bosons*, in which case they are just complex functions of x and τ, or *fermions*, in which case they are complex Grassmann functions of x and τ. The only subtlety resides in the choice of boundary conditions

1. *Bosons*:
   Since the partition function is a trace, in this case the fields (be complex or real) must obey the usual *periodic boundary conditions* in imaginary time,
$$\psi(\tau) = \psi(\tau + \beta\hbar) \tag{8.138}$$
2. *Fermions*:
   In the case of fermions the fields are complex Grassmann variables. However, if we want to compute a trace it turns out that, due to the anti-commutation rules, it is necessary to require the fields to obey instead *anti-periodic boundary conditions*,
$$\psi(\tau) = -\psi(\tau + \beta\hbar) \tag{8.139}$$

Let $\{|\lambda\rangle\}$ be a complete set of eigenstates of the one-body Hamiltonian $\hat{h}$, $\{\varepsilon_\lambda\}$ be its eigenvalue spectrum with λ a spectral parameter. Hence, we have a suitable set of quantum numbers spanning the spectrum of $\hat{h}$, and let $\{\phi_\lambda(\tau)\}$ be the associated complete set of eigenfunctions. We now expand the field configurations in the basis of eigenfunctions of $\hat{h}$,
$$\psi(\tau) = \sum_\lambda \psi_\lambda \, \phi_\lambda(\tau) \tag{8.140}$$
The eigenfunctions of $\hat{h}$ are complete and orthonormal.

Thus, if we expand the fields, the path-integral of Eq. (8.136) becomes (with $\hbar = 1$)
$$Z = \int \left(\prod_\lambda d\psi_\lambda^* d\psi_\lambda \right) e^{-\int d\tau \sum_\lambda \psi_\lambda^* \left(-\partial_\tau - \varepsilon_\lambda + \mu\right) \psi_\lambda} \tag{8.141}$$
which becomes (after absorbing all uninteresting constant factors in the integration measure)
$$Z = \prod_\lambda \left[\mathrm{Det}\left(-\partial_\tau - \varepsilon_\lambda + \mu\right)\right]^\sigma \tag{8.142}$$
where $\sigma = +1$ for *fermions* and $\sigma = -1$ for *bosons*.

Let $\psi_n^\lambda(\tau)$ be the solution of the equation
$$\left(-\partial_\tau - \varepsilon_\lambda + \mu\right) \psi_n^\lambda(\tau) = \alpha_n \psi_n^\lambda(\tau) \tag{8.143}$$

where α_n is the (generally complex) eigenvalue. The eigenfunctions $\psi_n^\lambda(\tau)$ will be required to satisfy either periodic or anti-periodic boundary conditions,

$$\psi_n^\lambda(\tau) = -\sigma\psi_n^\lambda(\tau+\beta) \tag{8.144}$$

where, once again, $\sigma = \pm 1$.

The eigenvalue condition, Eq. (8.143) is solved by

$$\psi_n(\tau) = \psi_n \, e^{i\alpha_n\tau} \tag{8.145}$$

provided α_n satisfies

$$\alpha_n = -i\omega_n + \mu - \varepsilon_\lambda \tag{8.146}$$

where the Matsubara frequencies are given by (with $k_B = 1$)

$$\omega_n = \begin{cases} 2\pi T\left(n+\frac{1}{2}\right), & \text{for fermions} \\ 2\pi T n, & \text{for bosons} \end{cases} \tag{8.147}$$

Let us consider now the function $\varphi_\alpha(\tau)$ which is an eigenfunction of $-\partial_\tau - \varepsilon_\lambda + \mu$,

$$(-\partial_\tau - \varepsilon_\lambda + \mu)\,\varphi_\alpha(\tau) = \alpha\,\varphi_\alpha(\tau) \tag{8.148}$$

which satisfies only an *initial condition* for $\varphi_\alpha^\lambda(0)$, such as

$$\varphi_\alpha^\lambda(0) = 1 \tag{8.149}$$

Notice that since the operator is linear in ∂_τ we cannot impose additional conditions on the derivative of φ_α.

The solution of

$$\partial_\tau \ln \varphi_\alpha^\lambda(\tau) = \mu - \varepsilon - \alpha \tag{8.150}$$

is

$$\varphi_\alpha^\lambda(\tau) = \varphi_\alpha^\lambda(0)\, e^{(\mu - \varepsilon_\lambda - \alpha)\,\tau} \tag{8.151}$$

After imposing the initial condition of Eq. (8.149), we find

$$\varphi_\alpha^\lambda(\tau) = e^{-\,(\alpha + \varepsilon_\lambda - \mu)\,\tau} \tag{8.152}$$

But, although this function $\varphi_\alpha^\lambda(\tau)$ satisfies all the requirements, it does not have the same zeros as the determinant $\mathrm{Det}(-\partial_\tau + \mu - \varepsilon_\lambda - \alpha)$. However, the function

$${}_\sigma F_\alpha^\lambda(\tau) = 1 + \sigma\,\varphi_\alpha^\lambda(\tau) \tag{8.153}$$

does satisfy all the requirements. Indeed,

$$ {}_{\sigma}F_{\alpha}^{\lambda}(\beta) = 1 + \sigma\, e^{-(\alpha + \varepsilon_\lambda - \mu)\,\beta} \tag{8.154}$$

which vanishes for $\alpha = \alpha_n$. Then, a version of Coleman's argument, discussed in Sec.5.2.2, tells us that

$$ \frac{\mathrm{Det}\,(-\partial_\tau + \mu - \varepsilon_\lambda - \alpha)}{{}_{\sigma}F_{\alpha}^{\lambda}(\beta)} = \text{constant} \tag{8.155}$$

where the right hand side is a constant in the sense that it does not depend on the choice of the eigenvalues $\{\varepsilon_\lambda\}$.

Hence,

$$ \mathrm{Det}\,(-\partial_\tau + \mu - \varepsilon_\lambda) = \text{const.}\ {}_{\sigma}F_{0}^{\lambda}(\beta) \tag{8.156}$$

The partition function is

$$ Z = e^{-\beta F} = \prod_{\lambda} \left[\mathrm{Det}\,(-\partial_\tau + \mu - \varepsilon_\lambda)\right]^{\sigma} \tag{8.157}$$

where F is the free energy, which we find it is given by

$$ \begin{aligned} F &= -\sigma T \sum_{\lambda} \ln \mathrm{Det}\,(-\partial_\tau + \mu - \varepsilon_\lambda - \alpha) \\ &= -\sigma T \sum_{\lambda} \ln\left(1 + \sigma e^{\beta(\mu - \varepsilon_\lambda)}\right) + f(\beta\mu) \end{aligned} \tag{8.158}$$

which is the correct result for non-interacting fermions and bosons. Here, we have set

$$ f(\beta\mu) = \begin{cases} 0 & \text{fermions} \\ -2T\mathcal{N}\ \ln\left(1 - e^{\beta\mu}\right) & \text{bosons} \end{cases} \tag{8.159}$$

where $\mathcal{N}$ is the number of states in the spectrum $\{\lambda\}$.

In some cases the spectrum has the symmetry $\varepsilon_\lambda = -\varepsilon_{-\lambda}$, *e. g.* the Dirac theory whose spectrum is $\varepsilon_\pm = \pm\sqrt{p^2 + m^2}$, and these expressions can be simplified further,

$$ \begin{aligned} \prod_{\lambda} \mathrm{Det}\,(-\partial_t + i\varepsilon_\lambda) &= \prod_{\lambda>0} \left[\mathrm{Det}\,(-\partial_t + i\varepsilon_\lambda)\,\mathrm{Det}\,(-\partial_t - i\varepsilon_\lambda)\right] \\ &= \prod_{\lambda>0} \mathrm{Det}\left(\partial_t^2 + \varepsilon_\lambda^2\right) \\ &\equiv \prod_{\lambda>0} \mathrm{Det}\left(-\partial_\tau^2 + \varepsilon_\lambda^2\right) \end{aligned} \tag{8.160}$$

where, in the last step, we performed a Wick rotation. This last expression we have encountered before. The result is

$$\prod_{\lambda>0} \mathrm{Det}\left(-\partial_\tau^2 + \varepsilon_\lambda^2\right) = \text{const.}\ \psi_0(\beta) \tag{8.161}$$

where $\psi_0(\tau)$ is the solution of the differential equation

$$\left(-\partial_\tau^2 + \varepsilon_\lambda^2\right)\psi_0(\tau) = 0 \tag{8.162}$$

which satisfies the initial conditions

$$\psi_0(0) = 0 \qquad \partial_\tau \psi_0(0) = 1 \tag{8.163}$$

The solution is

$$\psi_0(\tau) = \frac{\sinh(|\varepsilon_\lambda|\tau)}{|\varepsilon_\lambda|} \tag{8.164}$$

Hence

$$\psi_0(\beta) = \frac{\sinh(|\varepsilon_\lambda|\beta)}{|\varepsilon_\lambda|} \longrightarrow \frac{e^{|\varepsilon_\lambda|\beta}}{2|\varepsilon_\lambda|} \qquad \text{as } \beta \to \infty, \tag{8.165}$$

In particular, since

$$\prod_{\lambda>0} \frac{e^{|\varepsilon_\lambda|\beta}}{2|\varepsilon_\lambda|} = e^{\beta \sum_{\lambda>0} |\varepsilon_\lambda|} = e^{-\beta \sum_{\lambda<0} \varepsilon_\lambda} \tag{8.166}$$

we get that the ground state energy E_G is the sum of the single particle energies of the occupied (negative energy) states:

$$E_G = \sum_{\lambda<0} \varepsilon_\lambda \tag{8.167}$$

8.8.2 Functional determinants, heat kernels and ζ-function regularization

We have seen before that the evaluation of the effects of quantum fluctuations involves the calculation of the determinant of a differential operator. In the case of non-relativistic single particle Quantum Mechanics, is Section 5.2.2 we discussed in detail how to calculate a functional determinant of the form $\mathrm{Det}\left[-\partial_t^2 + W(t)\right]$. However the method we used for that purpose becomes unmanageably cumbersome if applied to the calculation of determinants of partial differential operators of the form $\mathrm{Det}\left[-D^2 + W(x)\right]$, where $x \equiv x_\mu$ and D is some differential operator. Fortunately there are better and more efficient ways of doing such calculations.

Let $\hat{A}$ be an operator, and $\{f_n(x)\}$ be a complete set of eigenstates of $\hat{A}$, with the eigenvalue spectrum $\{a_n\}$, such that

$$\hat{A} f_n(x) = a_n f_n(x) \tag{8.168}$$

We will assume that $\hat{A}$ has a discrete spectrum of real positive eigenvalues, and hence it is bounded from below. For the case of a continuous spectrum we will put the system in a finite box, which makes the spectrum discrete, and take limits at the end of the calculation.

The function $\zeta(s)$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{for Re } s > 1 \tag{8.169}$$

is the well known Riemann ζ-function. We will now use the eigenvalue spectrum of the operator $\hat{A}$ to define the *generalized ζ-function*

$$\zeta_A(s) = \sum_n \frac{1}{a_n^s} \tag{8.170}$$

where the sum runs over the labels (here denoted by n) of the spectrum of the operator $\widehat{A}$. We will assume that the sum (infinite series) is convergent which, in practice, will require that we introduce some sort of regularization at the high end (high energies) of the spectrum.

Upon differentiation we find

$$\frac{d\zeta_A}{ds} = \sum_n \frac{d}{ds}\, e^{-s \ln a_n} = -\sum_n \frac{\ln a_n}{a_n^s} \tag{8.171}$$

where we have assumed convergence. Then, in the limit $s \to 0^+$ we find that the following identity holds

$$\lim_{s\to 0^+} \frac{d\zeta_A}{ds} = -\sum_n \ln a_n = -\ln \prod_n a_n \equiv -\ln \operatorname{Det} A \tag{8.172}$$

Hence, we can formally relate the generalized ζ-function, ζ_A, to the functional determinant of the operator A

$$\left.\frac{d\zeta_A}{ds}\right|_{s\to 0^+} = -\ln \operatorname{Det} A \tag{8.173}$$

We have thus reduced the computation of a determinant to the computation of a function with specific properties.

Let us define now the generalized *Heat Kernel*

$$G_A(x, y; \tau) = \sum_n \, e^{-a_n \tau} \, f_n(x) f_n^*(y) \equiv \langle x | e^{-\tau \hat{A}} | y \rangle \tag{8.174}$$

where $\tau > 0$. The Heat Kernel $G_A(x, y; \tau)$ clearly obeys the differential equation

$$-\partial_\tau G_A(x, y; \tau) = \hat{A} G_A(x, y; \tau) \tag{8.175}$$

which can be regarded as a generalized Heat Equation. Indeed, for $\hat{A} = -D\ \nabla^2$, this is the regular Heat Equation (where D is the diffusion constant) and, in this case τ represents time. In general we will refer to τ as *proper time.*

The Heat Kernel $G_A(x, y; \tau)$ satisfies the initial condition

$$\lim_{\tau \to 0^+} G_A(x, y; \tau) = \sum_n f_n(x) f_n^*(y) = \delta(x - y) \tag{8.176}$$

where we have used the completeness relation of the eigenfunctions $\{f_n(x)\}$. Hence, $G_A(x, y; \tau)$ is the solution of a generalized Heat Equation with kernel $\hat{A}$. It defines a generalized random walk or Markov process.

We will now show that $G_A(x, y; \tau)$ is related to the generalized ζ-function $\zeta_A(s)$. Indeed, let us consider the Heat Kernel $G_A(x, y; \tau)$ at *short distances*, $y \to x$, and compute the integral (below we denote by D is the dimensionality of space-time)

$$\begin{aligned} \int d^D x\ \lim_{y \to x} G_A(x, y; \tau) &= \sum_n e^{-a_n \tau} \int d^D x\ f_n(x) f_n^*(x) \\ &= \sum_n e^{-a_n \tau} \equiv \operatorname{tr} e^{-\tau \hat{A}} \end{aligned} \tag{8.177}$$

where we assumed that the eigenfunctions are normalized to unity

$$\int d^D x\ |f_n(x)|^2 = 1 \tag{8.178}$$

i.e. normalized inside a box.

We will now use that, for $s > 0$ and $a_n > 0$ (or at least that it has a positive real part) we can write

$$\int_0^\infty d\tau\ \tau^{s-1}\ e^{-a_n \tau} = \frac{\Gamma(s)}{a_n^s} \tag{8.179}$$

where $\Gamma(s)$ is the Euler Gamma function:

$$\Gamma(s) = \int_0^\infty d\tau\ \tau^{s-1}\ e^{-\tau} \tag{8.180}$$

Then, we obtain the identity

$$\int_0^\infty d\tau\ \tau^{s-1} \int d^D x\ \lim_{y \to x} G_A(x, y; \tau) = \sum_n \frac{\Gamma(s)}{a_n^s} \tag{8.181}$$

Therefore, we find that the generalized ζ-function, $\zeta_A(s)$, can be obtained from the generalized Heat Kernel $G_A(x,y;\tau)$:

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\ \tau^{s-1} \int d^D x\ \lim_{y\to x} G_A(x,y;\tau) \tag{8.182}$$

This result suggests the following strategy for the computation of determinants. Given the hermitian operator $\hat{A}$, we solve the generalized Heat Equation

$$\hat{A}\, G_A = -\partial_\tau\, G_A \tag{8.183}$$

subject to the initial condition

$$\lim_{\tau\to 0^+} G_A(x,y;\tau) = \delta^D(x-y) \tag{8.184}$$

Next we find the associated ζ-function, $\zeta_A(s)$, using the expression

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\ \tau^{s-1} \int d^D x\ \lim_{y\to x} G_A(x,y;\tau) \tag{8.185}$$

where we recognize the identity

$$\text{tr}\ e^{-\tau\hat{A}} = \int d^D x\ \lim_{y\to x} G_A(x,y;\tau) \tag{8.186}$$

We next take the limit $s \to 0^+$ to relate the generalized ζ-function to the determinant:

$$\lim_{s\to 0^+} \frac{d\zeta_A(s)}{ds} = -\ln\det\hat{A} \tag{8.187}$$

In practice we will have to exercise some care in this step since we will find singularities as we take this limit. Most often we will keep the points x and y apart by a small but finite distance a, which we will eventually attempt to take to zero. Hence, we will need to understand in detail the short distance behavior of the Heat Kernel.

Furthermore, the propagator of the theory

$$S_A(x,y) = \langle x|\ \hat{A}^{-1}\ |y\rangle \tag{8.188}$$

can also be related to the Heat Kernel. Indeed, by expending Eq.(8.188) in the eigenstates of $\hat{A}$, we find

$$S_A(x,y) = \sum_n \frac{\langle x|n\rangle\ \langle n|y\rangle}{a_n} = \sum_n \frac{f_n(x) f_n^*(y)}{a_n} \tag{8.189}$$

We can now write the following integral of the Heat Kernel as

$$\int_0^\infty d\tau\; G_A(x,y;\tau) = \sum_n f_n(x) f_n^*(y) \int_0^\infty d\tau\; e^{-a_n \tau} = \sum_n \frac{f_n(x) f_n^*(y)}{a_n} \tag{8.190}$$

Hence, the propagator $S_A(x,y)$ can be expressed as an integral of the Heat Kernel

$$S_A(x,y) = \int_0^\infty d\tau\; G_A(x,y;\tau) \tag{8.191}$$

Equivalently, we can say that since $G_A(x,y;\tau)$ satisfies the Heat Equation

$$\hat{A} G_A = -\partial_\tau G_A, \qquad \text{then} \qquad G_A(x,y;\tau) = \langle x|\, e^{-\tau \hat{A}}\, |y\rangle \tag{8.192}$$

Thus,

$$S_A(x,y) = \int_0^\infty d\tau\; \langle x|\, e^{-\tau \hat{A}}\, |y\rangle = \langle x|\, \hat{A}^{-1}\, |y\rangle \tag{8.193}$$

and that it is indeed the Green function of $\hat{A}$,

$$\hat{A}_x S(x,y) = \delta(x-y) \tag{8.194}$$

It is worth to note that the Heat Kernel $G_A(x,y;\tau)$, as can be seen from Eq. (8.192), is also the *Gibbs density matrix* of the bounded Hermitian operator $\hat{A}$. As such it has an imaginary time (τ!) path-integral representation. Here, to actually insure convergence, we must also require that the spectrum of $\hat{A}$ be positive. In that picture we view $G_A(x,y;\tau)$ as the amplitude for the imaginary-time (proper time) evolution from the initial state $|y\rangle$ to the final state $|x\rangle$. In other words, we picture $S_A(x,y)$ as the amplitude to go from y to x in an arbitrary time.

8.9 The determinant of the Euclidean Klein-Gordon operator

As an example of the use of the Heat Kernel method we will use it to compute the determinant of the Euclidean Klein-Gordon operator. Thus, we will take the hermitian operator $\hat{A}$ to be

$$\hat{A} = -\nabla^2 + m^2 \tag{8.195}$$

in D Euclidean space-time dimensions. This operator has a bounded positive spectrum. Here we will be interested in a system with infinite size $L \to \infty$, and a large volume $V = L^D$. We will follow the steps outlined above.

We begin by constructing the Heat Kernel $G(x,y;\tau)$. By definition it is the solution of the partial differential equation

$$\left(-\nabla^2+m^2\right)G(x,y;\tau)=-\partial_\tau G(x,y;\tau) \tag{8.196}$$

satisfying the initial condition

$$\lim_{\tau\to 0^+} G(x,y;\tau)=\delta^D(x-y) \tag{8.197}$$

We will find $G(x,y;\tau)$ by Fourier transforms,

$$G(x,y;\tau)=\int\frac{d^Dp}{(2\pi)^D}\, G(\boldsymbol{p},\tau)\, e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \tag{8.198}$$

We find that in order for $G(x,y;\tau)$ to satisfy Eq.(8.196), its Fourier transform $G(\boldsymbol{p};\tau)$ must satisfy the differential equation

$$-\partial_\tau G(\boldsymbol{p};\tau)=\left(\boldsymbol{p}^2+m^2\right)\, G(\boldsymbol{p};\tau) \tag{8.199}$$

The solution of this equation, consistent with the initial condition of Eq.(8.197) is

$$G(\boldsymbol{p};\tau)=e^{-\left(\boldsymbol{p}^2+m^2\right)\tau} \tag{8.200}$$

We can now easily find $G(x,y;\tau)$ by simply finding the anti-transform of $G(\boldsymbol{p};\tau)$:

$$\begin{aligned} G(x,y;\tau)&=\int\frac{d^Dp}{(2\pi)^D}\, e^{-\left(\boldsymbol{p}^2+m^2\right)\tau+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\\ &=\frac{1}{(4\pi\tau)^{D/2}}\, e^{-\left(m^2\tau+\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\tau}\right)} \end{aligned} \tag{8.201}$$

Notice that for $m\to 0$, $G(x,y;\tau)$ reduces to the usual diffusion kernel (with unit diffusion constant)

$$\lim_{m\to 0} G(x,y;\tau)=\frac{1}{(4\pi\tau)^{D/2}}\, e^{-\frac{|x-y|^2}{4\tau}} \tag{8.202}$$

Next we construct the ζ-function

$$\zeta_{-\nabla^2+m^2}(s)=\frac{1}{\Gamma(s)}\int_0^\infty d\tau\, \tau^{s-1}\int d^Dx\, \lim_{y\to x} G(x,y;\tau) \tag{8.203}$$

We first do the integral

$$\int_0^\infty d\tau\ \tau^{s-1} \int d^D x\ G(x,y;\tau)$$

$$= \frac{V}{(4\pi)^{D/2}} \int_0^\infty d\tau\ \tau^{s-1-D/2}\ e^{-\left(m^2\tau + \frac{R^2}{4\tau}\right)} \qquad (8.204)$$

where $R = |\boldsymbol{x} - \boldsymbol{y}|$. Upon scaling the variable $\tau = \lambda t$, with $\lambda = R/2m$, we find that

$$\int_0^\infty d\tau\ \tau^{s-1}\ G(x,y;\tau) = \frac{2}{(4\pi)^{D/2}} \left(\frac{R}{2m}\right)^{s-\frac{D}{2}}\ K_{\frac{D}{2}-s}(mR) \qquad (8.205)$$

where $K_\nu(z)$,

$$K_\nu(z) = \frac{1}{2}\int_0^\infty dt\ t^{\nu-1} e^{-\frac{z}{2}\left(t+\frac{1}{t}\right)} \qquad (8.206)$$

is a modified Bessel function. Its short argument behavior is

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu + \dots \qquad (8.207)$$

As a check, we notice that for $s = 1$ the integral of Eq.(8.204) does reproduce the Euclidean Klein-Gordon propagator that we discussed earlier in these lectures.

The next step is to take the short distance limit

$$\lim_{R\to 0}\int_0^\infty d\tau\ \tau^{s-1} G(\boldsymbol{x},\boldsymbol{y};\tau) = \lim_{R\to 0} \frac{2^{1-s}}{(2\pi)^{D/2}} \frac{m^{D-2s}}{(mR)^{\frac{D}{2}-s}} K_{\frac{D}{2}-s}(mR)$$

$$= \frac{\Gamma\left(s-\frac{D}{2}\right)}{(4\pi)^{D/2} m^{2s-D}} \qquad (8.208)$$

Notice that we have exchanged the order of the limit and the integration. Also, after we took the short distance limit $R \to 0$, the expression above acquired a factor of $\Gamma(s - D/2)$, which is singular as $s-D/2$ approaches zero (or any negative integer). Thus, a small but finite R smears this singularity.

Finally we find the ζ-function by doing the (trivial) integration over space

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\tau \int d^D x \lim_{\boldsymbol{y}\to\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y};\tau)$$

$$= V\mu^{-2s}\frac{m^D}{(4\pi)^{D/2}}\frac{\Gamma\left(s-\frac{D}{2}\right)}{\Gamma(s)}\left(\frac{m}{\mu}\right)^{-2s} \qquad (8.209)$$

where $\mu = 1/R$ plays the role of a cutoff mass (or momentum) scale that we will need to make some quantities dimensionless. The appearance of this quantity is also a consequence of the singularities.

We will now consider the specific case of $D = 4$ dimensions. For $D = 4$ the ζ-function is

$$\zeta(s) = V \frac{m^4}{16\pi^2} \frac{\mu^{-2s}}{(s-1)(s-2)} \left(\frac{m}{\mu}\right)^{-2s} \tag{8.210}$$

We can now compute the desired (logarithm of the) determinant for $D = 4$ dimensions:

$$\ln \mathrm{Det}\left[-\nabla^2 + m^2\right] = -\lim_{s\to 0^+} \frac{d\zeta}{ds} = \frac{m^4}{16\pi^2}\left[\ln\frac{m}{\mu} - \frac{3}{4}\right] V \tag{8.211}$$

where $V = L^4$. A similar calculation for $D = 2$ yields the result

$$\ln \mathrm{Det}\left[-\nabla^2 + m^2\right] = -\frac{m^2}{2\pi}\left[\ln\frac{m}{\mu} - \frac{1}{2}\right] V \tag{8.212}$$

where $V = L^2$.

8.10 Path integral for spin

We will now discuss the use of path integral methods to describe a quantum mechanical spin. Consider a quantum mechanical system which consists of a spin in the spin-S representation of the group $SU(2)$. The space of states of the spin-S representation is $2S+1$-dimensional, and it is spanned by the basis $\{|S, M\rangle\}$ which are the eigenstates of the operators $\boldsymbol{S}^2$ and S_3, i.e.

$$\begin{aligned} \boldsymbol{S}^2 \; |S, M\rangle &= S(S+1) \; |S, M\rangle \\ S_3 \; |S, M\rangle &= M \; |S, M\rangle \end{aligned} \tag{8.213}$$

with $|M| \leq S$ (in integer-spaced intervals). This set of states is complete ad it forms a basis of this Hilbert space. The operators S_1, S_2 and S_3 obey the $SU(2)$ algebra,

$$[S_a, S_b] = i\epsilon_{abc} S_c \tag{8.214}$$

where $a, b, c = 1, 2, 3$.

The simplest physical problem involving spin is the coupling to an external magnetic field $\boldsymbol{B}$ through the Zeeman interaction

$$H_{\mathrm{Zeeman}} = \mu \, \boldsymbol{B} \cdot \boldsymbol{S} \tag{8.215}$$

where μ is the Zeeman coupling constant (i.e. the product of the Bohr magneton and the gyromagnetic factor).

Let us denote by $|0\rangle$ the *highest weight state* $|S, S\rangle$. Let us define the spin raising and lowering operators $S^{\pm}$,

$$S^{\pm} = S_1 \pm iS_2 \tag{8.216}$$

The highest weight state $|0\rangle$ is annihilated by S^+,

$$S^+|0\rangle = S^+|S, S\rangle = 0 \tag{8.217}$$

Clearly, we also have

$$\begin{aligned} \boldsymbol{S}^2|0\rangle =& S(S+1)|0\rangle \\ S_3|0\rangle =& S|0\rangle \end{aligned} \tag{8.218}$$

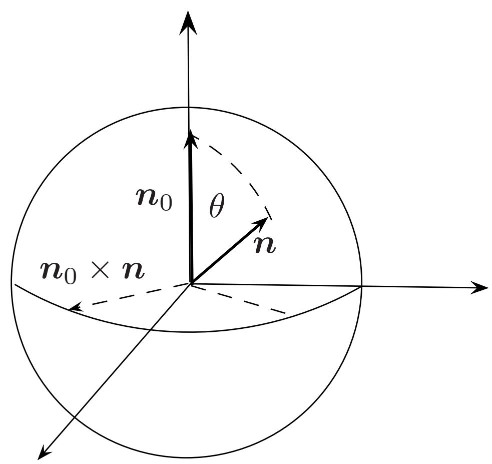

Figure 8.1 Geometry for a spin coherent state $|\boldsymbol{n}\rangle$ (see text).

Let us consider now the spin coherent state $|\boldsymbol{n}\rangle$, (Perelomov, 1986)

$$|\boldsymbol{n}\rangle = e^{i\theta \boldsymbol{n}_0 \times \boldsymbol{n} \cdot \boldsymbol{S}}|0\rangle \tag{8.219}$$

where $\boldsymbol{n}$ is a three-dimensional unit vector ($\boldsymbol{n}^2 = 1$), $\boldsymbol{n}_0$ is a unit vector pointing along the direction of the quantization axis (i.e. the "North Pole" of the unit sphere) and θ is the *colatitude*, (see Fig. 8.1)

$$\boldsymbol{n} \cdot \boldsymbol{n}_0 = \cos\theta \tag{8.220}$$

As we will see the state $|\boldsymbol{n}\rangle$ is a coherent spin state which represents a spin polarized along the $\boldsymbol{n}$ axis. The state $|\boldsymbol{n}\rangle$ can be expanded in the basis $|S, M\rangle$,

$$|\boldsymbol{n}\rangle = \sum_{M=-S}^{S} D_{MS}^{(S)}(\boldsymbol{n})\, |S, M\rangle \tag{8.221}$$

Here $D_{MS}^{(S)}(\boldsymbol{n})$ are the representation matrices in the spin-S representation.

It is important to note that there are many rotations that lead to the same state $|\boldsymbol{n}\rangle$ from the highest weight $|0\rangle$. For example any rotation along the direction $\boldsymbol{n}$ results only in a change in the phase of the state $|\boldsymbol{n}\rangle$. These rotations are equivalent to a multiplication on the right by a rotation about the z axis. However, in Quantum Mechanics this phase has no physically observable consequence. Hence we will regard all of these states as being physically equivalent.

In other words, the states form *equivalence classes* (or *rays*) and we must pick one and only one state from each class. These rotations are generated by S_3, the (only) diagonal generator of $SU(2)$. Hence, the physical states are not in one-to-one correspondence with the elements of $SU(2)$ but instead with the elements of the right *coset* $SU(2)/U(1)$, with the $U(1)$ generated by S_3. In the case of of coherent state of a more general Lie group, the coset is obtained by dividing out the maximal torus generated by all the diagonal generators of the group. In mathematical language, if we consider all the rotations at once, the spin coherent states are said to form a hermitian line bundle.

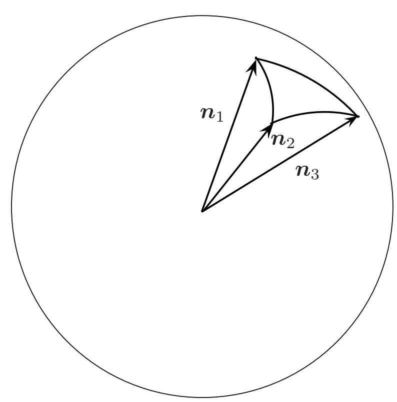

Figure 8.2 Spherical triangle with vertices at the unit vectors, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$.

A consequence of these observations is that the D matrices do not form a group under matrix multiplication. Instead they satisfy

$$D^{(S)}(\boldsymbol{n}_1)D^{(S)}(\boldsymbol{n}_2) = D^{(S)}(\boldsymbol{n}_3)\, e^{i\Phi(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)S_3} \tag{8.222}$$

where the phase factor is usually called a *cocycle*. Here $\Phi(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)$ is the (oriented) area of the spherical triangle with vertices at $\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3$.

However, since the sphere is a closed surface, which area do we actually mean? "Inside" or "outside"? Thus, the phase factor is ambiguous by an amount determined by 4π, the total area of the sphere,

$$e^{i4\pi M} \tag{8.223}$$

However, since M is either an integer or a half-integer this ambiguity in Φ has no consequence whatsoever,

$$e^{i4\pi M} = 1 \tag{8.224}$$

We can also regard this result as a requirement that M be quantized to be an integer or a half-integer, i.e. the representations of $SU(2)$.

The states $|\boldsymbol{n}\rangle$ are coherent states which satisfy the following properties (Perelomov, 1986). The overlap of two coherent states $|\vec{n}_1\rangle$ and $|\boldsymbol{n}_2\rangle$ is

$$\begin{aligned}\langle \boldsymbol{n}_1|\boldsymbol{n}_2\rangle &= \langle 0|D^{(S)}(\boldsymbol{n}_1)^\dagger D^{(S)}(\boldsymbol{n}_2)|0\rangle \\ &= \langle 0|D^{(S)}(\boldsymbol{n}_0)e^{i\Phi(\boldsymbol{n}_1,\boldsymbol{n}_2,\vec{n}_0)S_3}|0\rangle \\ &= \left(\frac{1+\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{2}\right)^S e^{i\Phi(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_0)S}\end{aligned} \tag{8.225}$$

The (diagonal) matrix element of the spin operator is

$$\langle \boldsymbol{n}|\boldsymbol{S}|\boldsymbol{n}\rangle = S\,\boldsymbol{n} \tag{8.226}$$

Finally, the (over-complete) set of coherent states $\{|\boldsymbol{n}\rangle\}$ have a resolution of the identity of the form

$$\hat{I} = \int d\mu(\boldsymbol{n})\; |\boldsymbol{n}\rangle\langle \boldsymbol{n}| \tag{8.227}$$

where the integration measure $d\mu(\boldsymbol{n})$ is

$$d\mu(\boldsymbol{n}) = \left(\frac{2S+1}{4\pi}\right)\delta(\boldsymbol{n}^2-1)d^3n \tag{8.228}$$

Let us now use the coherent states $\{|\boldsymbol{n}\rangle\}$ to find the path integral for a spin. In imaginary time τ (and with periodic boundary conditions) the path integral is simply the partition function

$$Z = \mathrm{tr} e^{-\beta H} \tag{8.229}$$

where $\beta = 1/T$ (T is the temperature) and H is the Hamiltonian. As usual the path integral form of the partition function is found by splitting up the imaginary time interval $0 \leq \tau \leq \beta$ in N_τ steps each of length $\delta\tau$ such that $N_\tau \delta\tau = \beta$. Hence we have

$$Z = \lim_{N_\tau \to \infty, \delta\tau \to 0} \text{tr} \left(e^{-\delta\tau H} \right)^{N_\tau} \tag{8.230}$$

and insert the resolution of the identity at every intermediate time step,

$$\begin{aligned} Z &= \lim_{N_\tau \to \infty, \delta\tau \to 0} \left(\prod_{j=1}^{N_\tau} \int d\mu(\boldsymbol{n}_j) \right) \left(\prod_{j=1}^{N_\tau} \langle \boldsymbol{n}(\tau_j) | e^{-\delta\tau H} | \boldsymbol{n}(\tau_{j+1}) \rangle \right) \\ &\simeq \lim_{N_\tau \to \infty, \delta\tau \to 0} \left(\prod_{j=1}^{N_\tau} \int d\mu(\vec{n}_j) \right) \left(\prod_{j=1}^{N_\tau} \left[\langle \boldsymbol{n}(\tau_j) | \vec{n}(\tau_{j+1}) \rangle - \delta\tau \langle \boldsymbol{n}(\tau_j) | H | \boldsymbol{n}(\tau_{j+1}) \rangle \right] \right) \end{aligned} \tag{8.231}$$

However, since

$$\frac{\langle \boldsymbol{n}(\tau_j) | H | \boldsymbol{n}(\tau_{j+1}) \rangle}{\langle \boldsymbol{n}(\tau_j) | \boldsymbol{n}(\tau_{j+1}) \rangle} \simeq \langle \boldsymbol{n}(\tau_j) | H | \boldsymbol{n}(\tau_j) \rangle = \mu S \boldsymbol{B} \cdot \boldsymbol{n}(\tau_j) \tag{8.232}$$

and

$$\langle \boldsymbol{n}(\tau_j) | \boldsymbol{n}(\tau_{j+1}) \rangle = \left(\frac{1 + \boldsymbol{n}(\tau_j) \cdot \boldsymbol{n}(\tau_{j+1})}{2} \right)^S e^{i\Phi(\boldsymbol{n}(\tau_j), \boldsymbol{n}(\tau_{j+1}), \boldsymbol{n}_0) S} \tag{8.233}$$

we can write the partition function in the form

$$Z = \lim_{N_\tau \to \infty, \delta\tau \to 0} \int \mathcal{D}\boldsymbol{n} \; e^{-S_E[\boldsymbol{n}]} \tag{8.234}$$

where $S_E[\boldsymbol{n}]$ is given by

$$\begin{aligned} -S_E[\boldsymbol{n}] = iS \sum_{j=1}^{N_\tau} \Phi(\boldsymbol{n}(\tau_j), \vec{n}(\tau_{j+1}), \boldsymbol{n}_0) \\ + S \sum_{j=1}^{N_\tau} \ln \left(\frac{1 + \boldsymbol{n}(\tau_j) \cdot \vec{n}(\tau_{j+1})}{2} \right) - \sum_{j=1}^{N_\tau} (\delta\tau) \mu S \; \boldsymbol{n}(\tau_j) \cdot \boldsymbol{B} \end{aligned} \tag{8.235}$$

The first term of the right hand side of Eq. (8.235) contains the expression $\Phi(\boldsymbol{n}(\tau_j), \vec{n}(\tau_{j+1}), \boldsymbol{n}_0)$ which has a simple geometric interpretation: it is the sum of the areas of the N_τ contiguous spherical triangles. These triangles have the pole $\boldsymbol{n}_0$ as a common vertex, and their other pairs of vertices trace a spherical polygon with vertices at $\{\boldsymbol{n}(\tau_j)\}$. In the time continuum limit this spherical polygon becomes the *history* of the spin, which traces a closed oriented curve $\Gamma = \{\boldsymbol{n}(\tau)\}$ (with $0 \leq \tau \leq \beta$). Let us denote by Ω^+ the region of the sphere whose boundary is Γ and which contains the pole $\boldsymbol{n}_0$. The complement of this region is Ω^- and it contains the opposite pole $-\boldsymbol{n}_0$. Hence we find that

$$\lim_{N_\tau \to \infty, \delta\tau \to 0} \Phi(\boldsymbol{n}(\tau_j), \boldsymbol{n}(\tau_{j+1}), \boldsymbol{n}_0) = \mathcal{A}[\Omega^+] = 4\pi - \mathcal{A}[\Omega^-] \tag{8.236}$$

where $\mathcal{A}[\Omega]$ is the area of the region Ω. Once again, the ambiguity of the area leads to the requirement that S should be an integer or a half-integer.

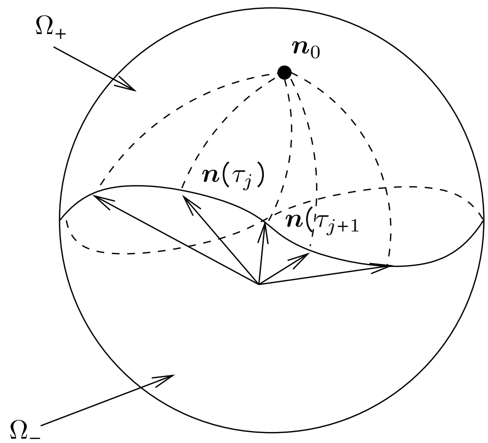

Figure 8.3 The function $\boldsymbol{n}(\tau, s)$ as arbitrary extension of the of the history $\boldsymbol{n}(\tau)$ to the upper cap Ω_+ of the sphere S_2.

There is a simple an elegant way to write the area enclosed by Γ. Let $\vec{n}(\tau)$ be a history and Γ be the set of points o the 2-sphere traced by $\vec{n}(\tau)$ for $0 \leq \tau \leq \beta$. Let us define $\boldsymbol{n}(\tau, s)$ (with $0 \leq s \leq 1$) to be an *arbitrary* extension of $\boldsymbol{n}(\tau)$ from the curve Γ to the interior of the upper cap Ω^+, as shown in Fig.8.3, such that

$$\boldsymbol{n}(\tau, 0) = \boldsymbol{n}(\tau), \quad \boldsymbol{n}(\tau, 1) = \boldsymbol{n}_0, \qquad \boldsymbol{n}(\tau, 0) = \boldsymbol{n}(\tau + \beta, 0) \tag{8.237}$$

Then the area can be written in the compact form

$$\mathcal{A}[\Omega^+] = \int_0^1 ds \int_0^\beta d\tau\; \boldsymbol{n}(\tau, s) \cdot \partial_\tau \boldsymbol{n}(\tau, s) \times \partial_s \boldsymbol{n}(\tau, s) \equiv S_{\rm WZ}[\boldsymbol{n}] \tag{8.238}$$

In Mathematics this expression for the area is called the (simplectic) 2-form, and in the literature is usually called a Wess-Zumino action (Witten, 1984), $S_{\rm WZ}$, or a *Berry phase*. (Berry, 1984; Simon, 1983) The coherent state path integral for spin is a special case of the method of geometric quantization (Wiegmann, 1989).

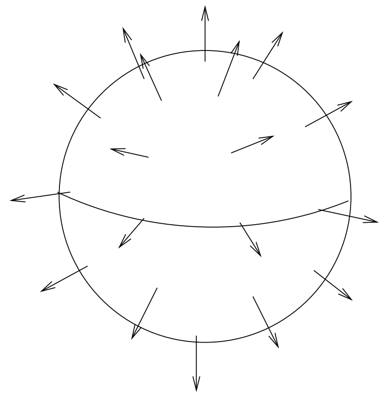

Figure 8.4 A hairy ball or monopole

Thus, in the (formal) time continuum limit, the action S_E becomes (Fradkin and Stone, 1988)

$$\mathcal{S}_E = -iS\ \mathcal{S}_{\rm WZ}[\boldsymbol{n}] + \frac{S\delta\tau}{2}\int_0^\beta d\tau\, (\partial_\tau \boldsymbol{n}(\tau))^2 + \int_0^\beta d\tau\ \mu S\ \boldsymbol{B}\cdot\boldsymbol{n}(\tau) \quad (8.239)$$

Notice that we have kept (temporarily) a term of order $\delta\tau$, which we will drop shortly.

How do we interpret Eq. (8.239)? Since $\boldsymbol{n}(\tau)$ is constrained to be a point on the surface of the unit sphere, i.e. $\boldsymbol{n}^2 = 1$, the action $S_E[\boldsymbol{n}]$ can be interpreted as the action of a particle of mass $M = S\delta\tau \to 0$ and $\vec{n}(\tau)$ is the position vector of the particle at (imaginary) time τ. Thus, the second term is a (vanishingly small) kinetic energy term, and the last term of Eq. (8.239) is a potential energy term.

What is the meaning of the first term? In Eq. (8.238) we saw that $\mathcal{S}_{\rm WZ}[\boldsymbol{n}]$, the the so-called Wess-Zumino or Berry phase term in the action, is the *area* of the (positively oriented) region $\mathcal{A}[\Omega_+]$ "enclosed" by the "path" $\boldsymbol{n}(\tau)$. In fact,

$$\mathcal{S}_{\rm WZ}[\boldsymbol{n}] = \int_0^1 ds \int_0^\beta d\tau \boldsymbol{n}\cdot\partial_\tau\boldsymbol{n}\times\partial_s\boldsymbol{n} \quad (8.240)$$

is the area of the oriented surface Ω^+ whose boundary is the oriented path $\Gamma = \partial\Omega^+$ (see Fig. 8.3). Using Stokes theorem we can write the the expression $S\mathcal{A}[\boldsymbol{n}]$ as the circulation of a vector field $\boldsymbol{A}[\boldsymbol{n}]$,

$$\oint_{\partial\Omega} d\boldsymbol{n}\cdot\boldsymbol{A}[\boldsymbol{n}(\tau)] = \iint_{\Omega^+} d\boldsymbol{S}\cdot\nabla_{\boldsymbol{n}}\times\boldsymbol{A}[\boldsymbol{n}(\tau)] \quad (8.241)$$

provided the "magnetic field" $\nabla_{\boldsymbol{n}} \times \boldsymbol{A}$ is "constant", namely

$$\boldsymbol{B} = \nabla_{\boldsymbol{n}} \times \boldsymbol{A}[\boldsymbol{n}(\tau)] = S\, \boldsymbol{n}(\tau) \tag{8.242}$$

In other words, this is the magnetic field of a magnetic monopole located at the center of the sphere. What is the total flux Φ of this magnetic field?

$$\Phi = \int_{\text{sphere}} d\boldsymbol{S} \cdot \nabla_{\boldsymbol{n}} \times \boldsymbol{A}[\boldsymbol{n}] = S \int d\boldsymbol{S} \cdot \boldsymbol{n} \equiv 4\pi S \tag{8.243}$$

Thus, the total number of flux quanta N_ϕ piercing the unit sphere is

$$N_\phi = \frac{\Phi}{2\pi} = 2S = \text{magnetic charge} \tag{8.244}$$

We reach the condition that the magnetic charge is *quantized*, a result known as the *Dirac quantization condition*.

Is this result consistent with what we know about charged particles in magnetic fields? In particular, how is this result related to the physics of spin? To answer these questions we will go back to real time and write the action

$$\mathcal{S}[\boldsymbol{n}] = \int_0^T dt \left[\frac{M}{2} \left(\frac{d\boldsymbol{n}}{dt} \right)^2 + \boldsymbol{A}[\boldsymbol{n}(t)] \cdot \frac{d\boldsymbol{n}}{dt} - \mu S \boldsymbol{n}(t) \cdot \boldsymbol{B} \right] \tag{8.245}$$

with the constraint $\boldsymbol{n}^2 = 1$ and where the limit $M \to 0$ is implied.

The classical hamiltonian associated to the action of Eq. (8.245) is

$$H = \frac{1}{2M} \Big[\boldsymbol{n} \times (\boldsymbol{p} - \boldsymbol{A}[\boldsymbol{n}]) \Big]^2 + \mu S \boldsymbol{n} \cdot \boldsymbol{B} \equiv H_0 + \mu S \boldsymbol{n} \cdot \boldsymbol{B} \tag{8.246}$$

It is easy to check that the vector $\boldsymbol{\Lambda}$,

$$\boldsymbol{\Lambda} = \boldsymbol{n} \times (\boldsymbol{p} - \boldsymbol{A}) \tag{8.247}$$

satisfies the algebra

$$[\Lambda_a, \Lambda_b] = i\hbar \epsilon_{abc} (\Lambda_c - \hbar S n_c) \tag{8.248}$$

where $a, b, c = 1, 2, 3$, ϵ_{abc} is the (third rank) Levi-Civita tensor, and with

$$\boldsymbol{\Lambda} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\Lambda} = 0 \tag{8.249}$$

the generators of rotations for this system are

$$\boldsymbol{L} = \boldsymbol{\Lambda} + \hbar S \boldsymbol{n} \tag{8.250}$$

The operators $\boldsymbol{L}$ and $\boldsymbol{\Lambda}$ satisfy the (joint) algebra

$$\begin{aligned} [L_a, L_b] &= -i\hbar\epsilon_{abc}L_s \qquad \left[L_a, \boldsymbol{L}^2\right] = 0 \\ [L_a, n_b] &= i\hbar\epsilon_{abc}n_c \quad [L_a, \Lambda_b] = i\hbar\epsilon_{abc}\Lambda_c \end{aligned} \tag{8.251}$$

Hence

$$\left[L_a, \boldsymbol{\Lambda}^2\right] = 0 \;\Rightarrow\; [L_a, H] = 0 \tag{8.252}$$

since the operators L_a satisfy the angular momentum algebra, we can diagonalize $\boldsymbol{L}^2$ and L_3 simultaneously. Let $|m,\ell\rangle$ be the simultaneous eigenstates of $\boldsymbol{L}^2$ and L_3,

$$\boldsymbol{L}^2|m,\ell\rangle = \hbar^2\ell(\ell+1)|m,\ell\rangle \tag{8.253}$$

$$L_3|m,\ell\rangle = \hbar m|m,\ell\rangle \tag{8.254}$$

$$H_0|m,\ell\rangle = \frac{\hbar^2}{2MR^2}\left(\frac{\ell(\ell+1)-S}{2S}\right)|m,\ell\rangle \tag{8.255}$$

where $R = 1$ is the radius of the sphere. The eigenvalues ℓ are of the form $\ell = S + n$, $|m| \leq \ell$, with $n \in \mathbb{Z}^+ \cup \{0\}$ and $2S \in \mathbb{Z}^+ \cup \{0\}$. Hence each level is $2\ell+1$-fold degenerate, or what is equivalent, $2n+1+2S$-fold degenerate. Then, we get

$$\boldsymbol{\Lambda}^2 = \boldsymbol{L}^2 - \boldsymbol{n}^2\hbar^2S^2 = \boldsymbol{L}^2 - \hbar^2S^2 \tag{8.256}$$

Since $M = S\delta t \to 0$, the *lowest* energy in the spectrum of H_0 are those with the *smallest* value of ℓ, *i. e.* states with $n = 0$ and $\ell = S$. The degeneracy of this "Landau" level is $2S+1$, and the gap to the next excited states diverges as $M \to 0$. Thus, in the $M \to 0$ limit, the *lowest* energy states have the same degeneracy as the spin-S representation. Moreover, the operators $\boldsymbol{L}^2$ and L_3 become the corresponding spin operators. Thus, the equivalency found is indeed correct.

Thus, we have shown that the quantum states of a scalar (non-relativistic) particle bound to a magnetic monopole of magnetic charge $2S$, obeying the Dirac quantization condition, are identical to those of those of a spinning particle! (Wu and Yang, 1976)

We close this section with some observations on the semi-classical motion. From the (real time) action (already in the $M \to 0$ limit)

$$\mathcal{S} = -\int_0^T dt\,\mu S\,\boldsymbol{n}\cdot\boldsymbol{B} + S\int_0^T dt\int_0^1 ds\,\boldsymbol{n}\cdot\partial_t\boldsymbol{n}\times\partial_s\boldsymbol{n} \tag{8.257}$$

we can derive a Classical Equation of Motion by looking at the stationary configurations. The variation of the second term in Eq. (8.257) is

$$\delta\mathcal{S} = S\,\delta \int_0^T dt \int_0^1 ds\; \boldsymbol{n}\cdot\partial_t\boldsymbol{n}\times\partial_s\boldsymbol{n} = S\int_0^T dt\delta\boldsymbol{n}(t)\cdot\boldsymbol{n}(t)\times\partial_t\boldsymbol{n}(t) \tag{8.258}$$

the variation of the first term in Eq. (8.257) is

$$\delta\int_0^T dt\; \mu S\boldsymbol{n}(t)\cdot\boldsymbol{B} = \int_0^T dt\; \delta n(t)\cdot\mu S\boldsymbol{B} \tag{8.259}$$

Hence,

$$\delta\mathcal{S} = \int_0^T dt\; \delta\boldsymbol{n}(t)\cdot\Big(-\mu S\boldsymbol{B} + S\boldsymbol{n}(t)\times\partial_t\boldsymbol{n}(t)\Big) \tag{8.260}$$

which implies that the classical trajectories must satisfy the equation of motion

$$\mu\boldsymbol{B} = \boldsymbol{n}\times\partial_t\boldsymbol{n} \tag{8.261}$$

If we now use the vector identity

$$\boldsymbol{n}\times\boldsymbol{n}\times\partial_t\boldsymbol{n} = (\boldsymbol{n}\cdot\partial_t\boldsymbol{n})\,\boldsymbol{n} - \boldsymbol{n}^2\partial_t\boldsymbol{n} \tag{8.262}$$

and

$$\boldsymbol{n}\cdot\partial_t\boldsymbol{n} = 0, \qquad \text{and} \qquad \boldsymbol{n}^2 = 1 \tag{8.263}$$

we get the classical equation of motion

$$\partial_t\boldsymbol{n} = \mu\boldsymbol{B}\times\boldsymbol{n} \tag{8.264}$$

Therefore, the classical motion is *precessional* with an angular velocity $\boldsymbol{\Omega}_{\text{pr}} = \mu\boldsymbol{B}$.