9
Quantization of Gauge Fields

We will now turn to the problem of the quantization of gauge theories. We will begin with the simplest gauge theory, the free electromagnetic field. This is an *abelian* gauge theory. After that we will discuss at length the quantization of non-abelian gauge fields. Unlike abelian theories, such as the free electromagnetic field, even in the absence of matter fields non-abelian gauge theories are not free fields and have highly non-trivial dynamics.

9.1 Canonical quantization of the free electromagnetic field

The Maxwell theory was the first field theory to be quantized. The quantization procedure of a gauge theory, even for a free field, involves a number of subtleties not shared by the other problems that we have considered so far. The issue is the fact that this theory has a local gauge invariance. Unlike systems which only have global symmetries, not all the classical configurations of vector potentials represent physically distinct states. It could be argued that one should abandon the picture based on the vector potential and go back to a picture based on electric and magnetic fields instead. However, there is no local Lagrangian that can describe the time evolution of the system in that representation. Furthermore, is not clear which fields, $\boldsymbol{E}$ or $\boldsymbol{B}$ (or some other field) plays the role of coordinates and which can play the role of momentum. For that reason, and others, one sticks to the Lagrangian formulation with the vector potential A_μ as its independent coordinate-like variable.

The Lagrangian for the Maxwell theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{9.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, can be written in the form

$$\mathcal{L} = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) \tag{9.2}$$

where

$$E_j = -\partial_0 A_j - \partial_j A_0, \qquad B_j = -\epsilon_{jk\ell}\partial_k A_\ell \tag{9.3}$$

The electric field E_j and the space components of the vector potential A_j form a canonical pair since, by definition, the momentum Π_j conjugate to A_j is

$$\Pi_j(x) = \frac{\delta \mathcal{L}}{\delta \partial_0 A_j(x)} = \partial_0 A_j + \partial_j A_0 = -E_j \tag{9.4}$$

Notice that since $\mathcal{L}$ does not contain any terms which include $\partial_0 A_0$, the momentum Π_0, conjugate to A_0, vanishes

$$\Pi_0 = \frac{\delta \mathcal{L}}{\delta \partial_0 A_0} = 0 \tag{9.5}$$

A consequence of this result is that A_0 is essentially arbitrary and it plays the role of a Lagrange multiplier. Indeed, it is always possible to find a gauge transformation ϕ

$$A_0' = A_0 + \partial_0 \phi \qquad A_j' = A_j - \partial_j \phi \tag{9.6}$$

such that $A_0' = 0$. The solution is

$$\partial_0 \phi = -A_0 \tag{9.7}$$

which is consistent provided that A_0 vanishes both in the remote part and in the remote future, $x_0 \to \pm\infty$.

The canonical formalism can be applied to Maxwell electrodynamics if we notice that the fields $A_j(\boldsymbol{x})$ and $\Pi_{j'}(\boldsymbol{x}')$ obey the equal-time Poisson Brackets

$$\{A_j(\boldsymbol{x}), \Pi_{j'}(\boldsymbol{x}')\}_{PB} = \delta_{jj'}\delta^3(\boldsymbol{x} - \boldsymbol{x}') \tag{9.8}$$

or, in terms of the electric field $\boldsymbol{E}$,

$$\{A_j(\boldsymbol{x}), E_{j'}(\boldsymbol{x}')\}_{PB} = -\delta_{jj'}\delta^3(\boldsymbol{x} - \boldsymbol{x}') \tag{9.9}$$

Thus, the spatial components of the vector potential and the components of the electric field are canonical pairs. However, the time component of the vector field, A_0, does not have a canonical pair. Thus, the quantization procedure treats it separately, as a Lagrange multiplier field that is imposing a constraint, that we will see is the Gauss Law. However, at the operator level, the condition $\Pi_0 = 0$ must then be imposed as a constraint. This fact led Dirac to formulate the theory of quantization of systems with constraints (Dirac, 1966). There is, however, another approach, also initiated by Dirac, consisting in setting $A_0 = 0$ and to impose the Gauss Law as a constraint on the space of quantum states. As we will see, this amounts to fixing the gauge first (at the price of manifest Lorentz invariance).

The classical Hamiltonian density is defined in the usual manner

$$\mathcal{H} = \Pi_j \partial_0 A_j - \mathcal{L} \tag{9.10}$$

We find

$$\mathcal{H}(x) = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) - A_0(x) \nabla \cdot \boldsymbol{E}(x) \tag{9.11}$$

Except for the last term, this is the usual answer. It is easy to see that the last term is a *constant of motion*. Indeed the equal-time Poisson Bracket between the Hamiltonian *density* $\mathcal{H}(\boldsymbol{x})$ and $\nabla \cdot \boldsymbol{E}(\boldsymbol{y})$ is zero. By explicit calculation, we get

$$\{\mathcal{H}(\boldsymbol{x}), \nabla \cdot \boldsymbol{E}(\boldsymbol{y})\}_{PB} = \int d^3z \Big[- \frac{\delta \mathcal{H}(\boldsymbol{x})}{\delta A_j(\boldsymbol{z})} \frac{\delta \nabla \cdot \boldsymbol{E}(\boldsymbol{y})}{\delta E_j(\boldsymbol{z})} + \frac{\delta \mathcal{H}(\boldsymbol{x})}{\delta E_j(\boldsymbol{z})} \frac{\delta \nabla \cdot \boldsymbol{E}(\boldsymbol{y})}{\delta A_j(\boldsymbol{z})} \Big] \tag{9.12}$$

But

$$\begin{aligned} \frac{\delta \mathcal{H}(\boldsymbol{x})}{\delta A_j(\boldsymbol{z})} = \int d^3w \frac{\delta \mathcal{H}(\boldsymbol{x})}{\delta B_k(\boldsymbol{w})} \frac{\delta B_k(\boldsymbol{w})}{\delta A_j(\boldsymbol{z})} &= \int d^3w B_k(\boldsymbol{w}) \delta(\boldsymbol{x} - \boldsymbol{w}) \epsilon_{k\ell j} \nabla_\ell^w \, \delta(\boldsymbol{w} - \boldsymbol{z}) \\ &= -\epsilon_{k\ell j} \nabla_\ell^z \int d^3w B_k(\boldsymbol{w}) \delta(\boldsymbol{x} - \boldsymbol{w}) \delta(\boldsymbol{w} - \boldsymbol{z}) \end{aligned} \tag{9.13}$$

Hence

$$\frac{\delta \mathcal{H}(\boldsymbol{x})}{\delta A_j(\boldsymbol{z})} = \epsilon_{j\ell k} \nabla_\ell^z \left(B_k(\boldsymbol{x}) \delta(\boldsymbol{x} - \boldsymbol{z}) \right) = \epsilon_{j\ell k} B_k(\boldsymbol{x}) \nabla_\ell^x \, \delta(\boldsymbol{x} - \boldsymbol{z}) \tag{9.14}$$

Similarly, we get

$$\frac{\delta \nabla \cdot \boldsymbol{E}(\boldsymbol{y})}{\delta E_j(\boldsymbol{z})} = \nabla_j^y \delta(\boldsymbol{y} - \boldsymbol{z}), \qquad \frac{\delta \nabla \cdot \boldsymbol{E}(\boldsymbol{y})}{\delta A_j(\boldsymbol{z})} = 0 \tag{9.15}$$

Thus, the Poisson Bracket is

$$\begin{aligned} \{\mathcal{H}(\boldsymbol{x}), \nabla \cdot \boldsymbol{E}(\boldsymbol{y})\}_{PB} &= \int d^3z [-\epsilon_{j\ell k} B_k(\boldsymbol{x}) \nabla_\ell^x \, \delta(\boldsymbol{x} - \boldsymbol{z}) \nabla_j^y \, \delta(\boldsymbol{y} - \boldsymbol{z})] \\ &= -\epsilon_{j\ell k} B_k(\boldsymbol{x}) \nabla_\ell^x \nabla_j^y \delta(\boldsymbol{x} - \boldsymbol{y}) \\ &= \epsilon_{j\ell k} B_k(\boldsymbol{x}) \nabla_\ell^x \nabla_j^x \delta(\boldsymbol{x} - \boldsymbol{y}) = 0 \end{aligned} \tag{9.16}$$

provided that $\boldsymbol{B}(\boldsymbol{x})$ is non-singular. Thus, $\nabla \cdot \boldsymbol{E}(\boldsymbol{x})$ is a *constant of motion.* It is easy to check that $\nabla \cdot \boldsymbol{E}$ generates infinitesimal gauge transformations. We will prove this statement directly in the quantum theory.

Since $\nabla \cdot \boldsymbol{E}(\boldsymbol{x})$ is a constant of motion, if we pick a value for it at some initial time $x_0 = t_0$, it will remain constant in time. Thus we can write

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = \rho(\boldsymbol{x}) \tag{9.17}$$

which we recognize to be Gauss's Law. Naturally, an external charge distribution may be explicitly time dependent and then

$$\frac{d}{dx_0}(\nabla \cdot \boldsymbol{E}) = \frac{\partial}{\partial x_0}(\nabla \cdot \boldsymbol{E}) = \frac{\partial}{\partial x_0}\rho_{ext}(\boldsymbol{x}, x_0) \tag{9.18}$$

Before turning to the quantization of this theory, we must notice that A_0 plays the role of a Lagrange multiplier field whose variation yields the Gauss Law, $\nabla \cdot \boldsymbol{E} = 0$. Hence, the Gauss Law should be regarded as a *constraint* rather than an equation of motion. This issue becomes very important in the quantum theory. Indeed, without the constraint $\nabla \cdot \boldsymbol{E} = 0$, the theory is absolutely trivial, and wrong.

Constraints impose very severe restrictions on the allowed states of a quantum theory. Consider for instance a particle of mass m moving *freely* in three dimensional space. Its stationary states have plane wave wave functions $\Psi_{\boldsymbol{p}}(\boldsymbol{r}, x_0)$, with an energy $E(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m}$. If we constrain the particle to move only on the surface of a sphere of radius R, it becomes equivalent to a rigid rotor of moment of inertia $I = mR^2$ and energy eigenvalues $\epsilon_{\ell m} = \frac{\hbar^2}{2I}\ell(\ell+1)$ where $\ell = 0, 1, 2, \ldots$, and $|m| \leq \ell$. Thus, even the simple constraint $\boldsymbol{r}^2 = R^2$, does have non-trivial effects.

Unlike the case of a particle forced to move on the surface of a sphere, the constraints that we have to impose when quantizing Maxwell electrodynamics do not change the energy spectrum. This is so because we can reduce the number of degrees of freedom to be quantized by taking advantage of the gauge invariance of the classical theory. This procedure is called *gauge fixing.* For example, the classical equation of motion

$$\partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0 \tag{9.19}$$

in the Coulomb gauge, $A_0 = 0$ and $\nabla \cdot \boldsymbol{A} = 0$, becomes

$$\partial^2 A_j = 0 \tag{9.20}$$

However the Coulomb gauge is not compatible with the Poisson Bracket

$$\left\{A_j(\boldsymbol{x}), \Pi_j, (\boldsymbol{x}')\right\}_{PB} = \delta_{jj'}\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{9.21}$$

since the spatial divergence of the delta function does not vanish. It will follow that the quantization of the theory in the Coulomb gauge is achieved at the price of a modification of the commutation relations.

Since the classical theory is gauge-invariant, we can always fix the gauge without any loss of physical content. The procedure of gauge fixing has the attractive that the number of independent variables is greatly reduced. A standard approach to the quantization of a gauge theory is to fix the gauge first, at the classical level, and to quantize later.

However, a number of problems arise immediately. For instance, in most gauges, such as the Coulomb gauge, Lorentz invariance is lost, or at least it is manifestly so. Thus, although the Coulomb gauge, also known as the radiation or transverse gauge, spoils Lorentz invariance, it has the attractive feature that the nature of the physical states (the *photons*) is quite transparent. We will see below that the quantization of the theory in this gauge has some peculiarities.

Another standard choice is the Lorentz gauge

$$\partial_\mu A^\mu = 0 \tag{9.22}$$

whose main appeal is its manifest covariance. The quantization of the system is this gauge follows the method developed by and Gupta and Bleuer. While highly successful, it requires the introduction of states with negative norm (known as ghosts) which cancel all the gauge-dependent contributions to physical quantities. This approach is described in detail in the book by Itzykson and Zuber (Itzykson and Zuber, 1980).

More general covariant gauges can also be defined. A general approach consists not on imposing a rigid restriction on the degrees of freedom, but to add new terms to the Lagrangian which eliminate the gauge freedom. For instance, the modified Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A^\mu(x))^2 \tag{9.23}$$

is not gauge invariant because of the presence of the last term. We can easily see that this term weighs gauge equivalent configurations differently and the parameter $1/\alpha$ plays the role of a Lagrange multiplier field. In fact, in the limit $\alpha \to 0$ we recover the Lorentz gauge condition. In the path integral quantization of the Maxwell theory it is proven that this approach is equivalent to an average over gauges of the physical quantities. If $\alpha = 1$, the equations of motion become very simple, i.e. $\partial^2 A_\mu = 0$. This is the Feynman gauge. In this gauge the calculations are simplest although, here too, the quantization of the theory has subtleties (such as ghosts, etc.).

Still, within the Hamiltonian or canonical quantization procedure, a third approach has been developed. In this approach one fixes the gauge $A_0 = 0$. This condition is not enough to eliminate completely the gauge freedom. In this gauge a residual set of *time-independent* gauge transformations are still allowed. In this approach quantization is achieved by replacing the Poisson Brackets by commutators and Gauss Law condition becomes now a *constraint* on the space of physical quantum states. So, we quantize first and constrain later.

In general, it is a non-trivial task to prove that all the different quantizations yield a theory with the same physical properties. In practice what one has to prove is that these different gauge choices yield theories whose states differ from each other by, *at most*, a unitary transformation. Otherwise, the quantized theories would be physically inequivalent. In addition, the recovery of Lorentz invariance may be a bit tedious in some cases. There is however, an alternative, complementary, approach to the quantum theory in which most of these issues become very transparent. This is the path-integral approach. This method has the advantage that all the symmetries are taken care of from the outset. In addition, the canonical methods encounter very serious difficulties in the treatment of the non-abelian generalizations of Maxwell electrodynamics.

We will consider here two canonical approaches: 1) quantization in the Coulomb gauge and 2) canonical quantization in the $A_0 = 0$ gauge in the Schrödinger picture.

9.2 Coulomb gauge

Quantization in the Coulomb gauge follows the methods developed for the scalar field very closely. Indeed, the classical constraints $A_0 = 0$ and $\nabla \cdot \boldsymbol{A} = 0$ allow for a Fourier expansion of the vector potential $\boldsymbol{A}(\boldsymbol{x}, x_0)$. In Fourier space, we write

$$\boldsymbol{A}(\boldsymbol{x}, x_0) = \int \frac{d^3p}{(2\pi)^3 2p_0} \boldsymbol{A}(\boldsymbol{p}, x_0) \exp(i\boldsymbol{p} \cdot \boldsymbol{x}) \tag{9.24}$$

where $\boldsymbol{A}(\boldsymbol{p}, x_0) = \boldsymbol{A}^*(-\boldsymbol{p}, x_0)$. The Maxwell equations yield the classical equation of motion, the wave equation

$$\partial^2 \boldsymbol{A}(\boldsymbol{x}, x_0) = 0 \tag{9.25}$$

The Fourier expansion is consistent only if the amplitude $\boldsymbol{A}(\boldsymbol{p}, x_0)$ satisfies

$$\partial_0^2 \boldsymbol{A}(\boldsymbol{p}, x_0) + \boldsymbol{p}^2 \boldsymbol{A}(\boldsymbol{p}, x_0) = 0 \tag{9.26}$$

The constraint $\nabla \cdot \boldsymbol{A} = 0$ in turn becomes the transversality condition

$$\boldsymbol{p} \cdot \boldsymbol{A}(\boldsymbol{p}, x_0) = 0 \tag{9.27}$$

Hence, $\boldsymbol{A}(\boldsymbol{p}, x_0)$ has the time dependence

$$\boldsymbol{A}(\boldsymbol{p}, x_0) = \boldsymbol{A}(\boldsymbol{p})e^{ip_0x_0} + \boldsymbol{A}(-\boldsymbol{p})e^{-ip_0x_0} \tag{9.28}$$

where $p_0 = |\boldsymbol{p}|$. Then, the mode expansion takes the form

$$\boldsymbol{A}(\boldsymbol{x}, x_0) = \int \frac{d^3p}{(2\pi)^3 2p_0}\Big[\boldsymbol{A}^*(\boldsymbol{p})e^{ip\cdot x} + \boldsymbol{A}(\boldsymbol{p})e^{-ip\cdot x}\Big] \tag{9.29}$$

where $p \cdot x = p_\mu x^\mu$. The transversality condition, Eq.(9.27), is satisfied by introducing two polarization unit vectors $\boldsymbol{\epsilon}_1(\boldsymbol{p})$ and $\boldsymbol{\epsilon}_2(\boldsymbol{p})$, such that $\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}_1 \cdot \boldsymbol{p} = \boldsymbol{\epsilon}_2 \cdot \boldsymbol{p} = 0$, and $\boldsymbol{\epsilon}_1^2 = \boldsymbol{\epsilon}_2^2 = 1$. Hence, if the amplitude $\boldsymbol{A}$ has to be orthogonal to $\boldsymbol{p}$, it must be a linear combination of $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, i.e.

$$\boldsymbol{A}(\boldsymbol{p}) = \sum_{\alpha=1,2} \boldsymbol{\epsilon}_\alpha(\boldsymbol{p}) a_\alpha(\boldsymbol{p}) \tag{9.30}$$

where the factors $a_\alpha(\boldsymbol{p})$ are complex amplitudes. In terms of $a_\alpha(\boldsymbol{p})$ and $a_\alpha^*(\boldsymbol{p})$ the Hamiltonian looks like a sum of oscillators.

In the coulomb gauge, the passage to the quantum theory is achieved by assigning to each amplitude $a_\alpha(\boldsymbol{p})$ a Heisenberg annihilation operator $\hat{a}_\alpha(\boldsymbol{p})$. Similarly $a_\alpha^*(\boldsymbol{p})$ maps onto the adjoint operator, the creation operator $\hat{a}_\alpha^\dagger(\boldsymbol{p})$. The expansion of the vector potential in modes now is

$$\hat{\boldsymbol{A}}(x) = \int \frac{d^3p}{(2\pi)^3 2p_0} \sum_{\alpha=1,2} \boldsymbol{\epsilon}_\alpha(p)\Big[\hat{a}_\alpha(\boldsymbol{p})e^{-ip\cdot x} + \hat{a}_\alpha^\dagger(\boldsymbol{p})e^{ip\cdot x}\Big] \tag{9.31}$$

with $p^2 = 0$ and $p_0 = |\boldsymbol{p}|$. The operators $\hat{a}_\alpha(\boldsymbol{p})$ and $\hat{a}_\alpha^\dagger(\boldsymbol{p})$ satisfy canonical commutation relations

$$\begin{aligned}
[\hat{a}_\alpha(\boldsymbol{p}), \hat{a}_{\alpha'}^\dagger(\boldsymbol{p}')] =& 2p_0(2\pi)^3\delta(\boldsymbol{p} - \boldsymbol{p}') \\
[\hat{a}_\alpha(\boldsymbol{p}), \hat{a}_{\alpha'}(\boldsymbol{p}')] =& [\hat{a}_\alpha^\dagger(\boldsymbol{p}), \hat{a}_{\alpha'}^\dagger(\boldsymbol{p}')] = 0
\end{aligned} \tag{9.32}$$

It is straightforward to check that the vector potential $\boldsymbol{A}(\boldsymbol{x})$ and the electric field $\boldsymbol{E}(\boldsymbol{x})$ obey the (unconventional) equal-time commutation relation

$$[A_j(\boldsymbol{x}), E_{j'}(\boldsymbol{x}')] = -i\left(\delta_{jj'} - \frac{\nabla_j \nabla_{j'}}{\nabla^2}\right)\delta^3(\boldsymbol{x} - \boldsymbol{x}') \tag{9.33}$$

where the symbol $1/\nabla^2$ represents the inverse of the Laplacian, i.e. the

Laplacian Green function. In the derivation of this relation, the following identity was used

$$\sum_{\alpha=1,2} \epsilon_\alpha^j(\boldsymbol{p})\epsilon_\alpha^{j'}(\boldsymbol{p}) = \delta_{jj'} - \frac{p_j p_{j'}}{\boldsymbol{p}^2} \tag{9.34}$$

These commutation relations are an extension of the canonical commutation relation, and are a consequence of the transversality condition, $\nabla \cdot \boldsymbol{A} = 0$

In this gauge, the (normal-ordered) Hamiltonian is

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3 2p_0}\, p_0 \sum_{\alpha=1,2} \hat{a}_\alpha^\dagger(\boldsymbol{p})\hat{a}_\alpha(\boldsymbol{p}) \tag{9.35}$$

The ground state, the vacuum state $|0\rangle$, is annihilated by both polarizations $\hat{a}_\alpha(\boldsymbol{p})|0\rangle = 0$. The single-particle states are $\hat{a}_\alpha^\dagger(\boldsymbol{p})|0\rangle$ and represent transverse *photons* with momentum $\boldsymbol{p}$, energy $p_0 = |\boldsymbol{p}|$ and with the two possible linear polarizations labelled by $\alpha = 1, 2$. Circularly polarized photons can be constructed in the usual manner.

The Coulomb gauge has the advantage that, in this picture, the electromagnetic field can be regarded as a collection of linear harmonic oscillators which are then quantized. This, of course, is a simple reflection of the fact that Maxwell electrodynamics is a free field theory. It has, however, several problems. One is that Lorentz invariance is violated from the outset, and has to be recovered afterwards in the computation of observables. The other is that, as we will discuss below, in non-abelian theories the Coulomb gauge does not exist globally. For these reasons, its usefulness is essentially limited to the Maxwell theory.

9.3 The gauge $A_0 = 0$

In this gauge we will apply directly the canonical formalism. In what follows we will fix $A_0 = 0$ and associate to the three spatial components A_j of the vector potential an operator, $\hat{A}_j$ which acts on a Hilbert space of states. Similarly, to the canonical momentum $\Pi_j = -E_j$, we assign an operator $\widehat{\Pi}_j$. These operators obey the equal-time *commutation* relations

$$[\widehat{A}_j(\boldsymbol{x}), \widehat{\Pi}_{j'}(\boldsymbol{x}')] = i\delta(\boldsymbol{x} - \boldsymbol{x}')\delta_{jj'} \tag{9.36}$$

Hence, the vector potential $\boldsymbol{A}$ and the electric field $\boldsymbol{E}$ do not are canonically conjugate operators, and do not commute with each other,

$$[\hat{A}_j(\boldsymbol{x}), \hat{E}_{j'}(\boldsymbol{x}')] = -i\delta_{jj'}\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{9.37}$$

Let us now specify the Hilbert space to be the space of states $|\Psi\rangle$ with wave functions which, in the field representation, have the form $\Psi(\{A_j(\boldsymbol{x})\})$. When acting on these states, the electric field is the functional differential operator

$$\hat{E}_j(\boldsymbol{x}) \equiv i\frac{\delta}{\delta A_j(\boldsymbol{x})} \tag{9.38}$$

In this Hilbert space, the inner product is

$$\langle\{A_j(\boldsymbol{x})\}|\{A_j(\boldsymbol{x})\}\rangle \equiv \Pi_{\boldsymbol{x},j}\delta\left(A_j(\boldsymbol{x}) - A_j(\boldsymbol{x})\right) \tag{9.39}$$

This Hilbert space is actually much too large. Indeed states with wave functions that differ by time-independent gauge transformations

$$\Psi_\phi(\{A_j(\boldsymbol{x})\}) \equiv \Psi(\{A_j(\boldsymbol{x}) - \nabla_j\phi(\boldsymbol{x})\}) \tag{9.40}$$

are *physically equivalent* since the matrix elements of the electric field operator $\hat{E}_j(\boldsymbol{x})$ and magnetic field operator $\hat{B}_j(\boldsymbol{x}) = \epsilon_{jk\ell}\nabla_k \hat{A}_\ell(\boldsymbol{x})$ are the same for all gauge-equivalent states, i.e.

$$\begin{aligned}\langle\Psi'_{\phi'}(\{A_j(\boldsymbol{x})\})|\hat{E}_j(\boldsymbol{x})|\Psi_\phi(\{A_j(\boldsymbol{x})\})\rangle &= \langle\Psi'(\{A_j(\boldsymbol{x})\})|\hat{E}_j(\boldsymbol{x})|\Psi(\{A_j(\boldsymbol{x})\})\rangle\\ \langle\Psi'_{\phi'}(\{A_j(\boldsymbol{x})\})|\hat{B}_j(\boldsymbol{x})|\Psi_\phi(\{A_j(\boldsymbol{x})\})\rangle &= \langle\Psi'(\{A_j(\boldsymbol{x})\})|\hat{B}_j(\boldsymbol{x})|\Psi(\{A_j(\boldsymbol{x})\})\rangle\end{aligned} \tag{9.41}$$

The (local) operator $\hat{Q}(\boldsymbol{x})$

$$\hat{Q}(\boldsymbol{x}) = \nabla\cdot\hat{\boldsymbol{E}}(\boldsymbol{x}) \tag{9.42}$$

commutes locally with the Hamiltonian and with each other

$$[\hat{Q}(\boldsymbol{x}), \hat{H}] = 0, \qquad [\hat{Q}(\boldsymbol{x}), \hat{Q}(\boldsymbol{y})] = 0 \tag{9.43}$$

Hence, all the local operators $\hat{Q}(\boldsymbol{x})$ can be diagonalized simultaneously with $\hat{H}$.

Let us show now that $\hat{Q}(\boldsymbol{x})$ generates local infinitesimal time-independent gauge transformations. From the canonical commutation relation

$$[\hat{A}_j(\boldsymbol{x}), \hat{E}_{j'}(\boldsymbol{x}')] = -i\delta_{jj'}\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{9.44}$$

we get (by differentiation)

$$[\hat{A}_j(\boldsymbol{x}), \hat{Q}(\boldsymbol{x}')] = [\hat{A}_j(\boldsymbol{x}), \nabla_j\hat{E}_{j'}(\boldsymbol{x}')] = i\nabla_j^x\,\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{9.45}$$

Hence, we also find

$$[i\int d\boldsymbol{z}\phi(\boldsymbol{z})\hat{Q}(\boldsymbol{z}), \hat{A}_j(\boldsymbol{x})] = -\int d\boldsymbol{z}\phi(\boldsymbol{z})\nabla_j^z\,\delta(\boldsymbol{z}-\boldsymbol{x}) = \nabla_j\phi(\boldsymbol{x}) \tag{9.46}$$

and

$$
e^{i\int dz\phi(z)\hat{Q}(z)}\hat{A}_j(x)e^{-i\int dz\phi(z)\hat{Q}(z)} =
$$

$$
\begin{aligned}
&= e^{-i\int dz \bigtriangledown_k \phi(z)\hat{E}_k(z)}\hat{A}_j(x)\, e^{i\int dz) \bigtriangledown_k \phi(z)\hat{E}_k(z)} \\
&=\hat{A}_j(x) + \bigtriangledown_j\phi(x)
\end{aligned}
\tag{9.47}
$$

The physical requirement that states that differ by time-independent gauge transformations be equivalent to each other leads to the demand that we should *restrict* the Hilbert space to the *space of gauge-invariant states*. These states, which we will denote by $|\text{Phys}\rangle$, satisfy

$$
\hat{Q}(x)|\text{Phys}\rangle \equiv \bigtriangledown \cdot \hat{\boldsymbol{E}}(x)|\text{Phys}\rangle = 0 \tag{9.48}
$$

Thus, the constraint means that only the states which obey the Gauss Law are in the *physical Hilbert space*. Unlike the quantization in the Coulomb gauge, in the $A_0 = 0$ gauge the commutators are canonical and the states are constrained to obey the Gauss Law.

In the Schrödinger picture, the eigenstates of the system obey the Schrödinger equation

$$
\int dx\frac{1}{2}\Big[-\frac{\delta^2}{\delta A_j(x)^2} + B_j(x)^2\Big]\Psi[A] = \mathcal{E}\Psi[A] \tag{9.49}
$$

where $\Psi[A]$ is a shorthand for the wave functional $\Psi(\{A_j(x)\})$. In this notation, the constraint of Gauss law is

$$
\bigtriangledown_j^x \hat{E}_j(x)\Psi[A] \equiv i\bigtriangledown_j^x \frac{\delta}{\delta A_j(x)}\Psi[A] = 0 \tag{9.50}
$$

This constraint can be satisfied by separating the real field $A_j(x)$ into longitudinal $A_j^L(x)$ and transverse $A_j^T(x)$ parts

$$
A_j(x) = A_j^L(x) + A_j^T(x) = \int \frac{d^3p}{(2\pi)^3}\left(A_j^L(p) + A_j^T(p)\right)e^{i p\cdot x} \tag{9.51}
$$

where $A_j^L(x)$ and $A_j^T(x)$ satisfy

$$
\bigtriangledown_j A_j^T(x) = 0 \qquad A_j^L(x) = \bigtriangledown_j\phi(x) \tag{9.52}
$$

and $\phi(x)$ is, for the moment, arbitrary. In terms of A_j^L and A_j^T the constraint of Gauss law simply becomes

$$
\bigtriangledown_j^x\frac{\delta}{\delta A_j^L(x)}\Psi[A] = 0 \tag{9.53}
$$

and the Hamiltonian now is

$$\hat{H} = \int d^3p\, \frac{1}{2}\Big[- \frac{\delta^2}{\delta A_j^T(\boldsymbol{p})\delta A_j^T(-\boldsymbol{p})} - \frac{\delta^2}{\delta A_j^L(\boldsymbol{p})\delta A_j^L(-\boldsymbol{p})} + \boldsymbol{p}^2 A_j^T(\boldsymbol{p}) A_j^T(-\boldsymbol{p})\Big] \tag{9.54}$$

We satisfy the constraint by looking only at gauge-invariant states. Their wave functions do not depend on the longitudinal components of $\boldsymbol{A}(\boldsymbol{x})$. Hence, $\Psi[A] = \Psi[A^T]$. When acting on those states, the Hamiltonian is

$$H\Psi = \int d^3p \frac{1}{2}\Big[- \frac{\delta^2}{\delta A_j^T(\boldsymbol{p})\delta A_j^T(-\boldsymbol{p})} + \boldsymbol{p}^2 A_j^T(\boldsymbol{p}) A_j^T(-\boldsymbol{p})\Big]\Psi = \mathcal{E}\Psi \tag{9.55}$$

Let $\boldsymbol{\epsilon}_1(\boldsymbol{p})$ and $\boldsymbol{\epsilon}_2(\boldsymbol{p})$ be two vectors which together with the unit vector $\boldsymbol{n}_p = \boldsymbol{p}/|\boldsymbol{p}|$ form an orthonormal basis. Let us define the operators ($\alpha = 1, 2; j = 1, 2, 3$)

$$\begin{aligned} \hat{a}(\boldsymbol{p}, \alpha) &= \frac{1}{\sqrt{2|\boldsymbol{p}|}} \epsilon_j^\alpha(\boldsymbol{p})[\frac{\delta}{\delta A_j^T(-\boldsymbol{p})} + |\boldsymbol{p}| A_j^T(\boldsymbol{p})] \\ \hat{a}^\dagger(\boldsymbol{p}, \alpha) &= \frac{1}{\sqrt{2|\boldsymbol{p}|}} \epsilon_j^\alpha(\boldsymbol{p})[-\frac{\delta}{\delta A_j^T(\boldsymbol{p})} + |\boldsymbol{p}| A_j^T(-\boldsymbol{p}) \end{aligned} \tag{9.56}$$

These operators satisfy the commutation relations

$$[\hat{a}(\boldsymbol{p}, \alpha), \hat{a}^\dagger(\boldsymbol{p}', \alpha')] = \delta_{\alpha\alpha'} \delta^3(\boldsymbol{p} - \boldsymbol{p}') \tag{9.57}$$

In terms of these operators, the Hamiltonian $\hat{H}$ and the expansion of the transverse part of the vector potential are

$$\begin{aligned} \hat{H} &= \int d^3p \frac{|\boldsymbol{p}|}{2} \sum_{\alpha=1,2} [\hat{a}^\dagger(\boldsymbol{p}, \alpha)\hat{a}(\boldsymbol{p}, \alpha) + \hat{a}(\boldsymbol{p}, \alpha)\hat{a}^\dagger(\boldsymbol{p}, \alpha)] \\ A_j^T(\boldsymbol{x}) &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2|\boldsymbol{p}|}} \sum_{\alpha=1,2} \epsilon_\alpha^j(\boldsymbol{p})[\hat{a}(\boldsymbol{p}, \alpha)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger(\boldsymbol{p}, \alpha)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}] \end{aligned} \tag{9.58}$$

We recognize these expressions to be the same ones that we obtained before in the Coulomb gauge (except for the normalization factors).

It is instructive to derive the wave functional for the ground state. The ground state $|0\rangle$ is the state annihilated by all the oscillators $\hat{a}(\boldsymbol{p}, \alpha)$. Hence its wave function $\Psi_0[A]$ satisfies

$$\langle\{A_j(\boldsymbol{x})\}|\hat{a}(\boldsymbol{p}, \alpha)|0\rangle = 0 \tag{9.59}$$

This equation is the functional differential equation

$$\sum_{\alpha}^{j}(\boldsymbol{p})[\frac{\delta}{\delta A_j^T(-\boldsymbol{p})} + |\boldsymbol{p}|A_j^T(\boldsymbol{p})]\Psi_0(\{A_j^T(\boldsymbol{p})\}) = 0 \tag{9.60}$$

It is easy to check that the unique solution of this equation is

$$\Psi_0[A] = N\exp[-\frac{1}{2}\int d^3p|\boldsymbol{p}|A_j^T(\boldsymbol{p})A_j^T(-\boldsymbol{p})] \tag{9.61}$$

Since the transverse components of $A_j(\boldsymbol{p})$ satisfy

$$A_j^T(\boldsymbol{p}) = \epsilon_{jk\ell}\frac{p_k A_\ell(\boldsymbol{p})}{|\boldsymbol{p}|} = \left(\frac{\boldsymbol{p}\times\boldsymbol{A}(\boldsymbol{p})}{|\boldsymbol{p}|}\right)_j \tag{9.62}$$

we can write $\Psi_0[A]$ in the form

$$\Psi_0[A] = \mathcal{N}\exp[-\frac{1}{2}\int\frac{d^3p}{|\boldsymbol{p}|}(\boldsymbol{p}\times\boldsymbol{A}(\boldsymbol{p}))\cdot(\boldsymbol{p}\times\boldsymbol{A}(-\boldsymbol{p}))] \tag{9.63}$$

It is instructive to write this wave function in position space, i.e. as a functional of the configuration of magnetic fields $\{\boldsymbol{B}(\boldsymbol{x})\}$. Clearly, we have

$$\begin{aligned}\boldsymbol{p}\times\boldsymbol{A}(\boldsymbol{p}) &= -i\int\frac{d^3x}{(2\pi)^{3/2}}(\nabla_x\times\boldsymbol{A}(\boldsymbol{x}))\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\\ \boldsymbol{p}\times\boldsymbol{A}(-\boldsymbol{p}) &= i\int\frac{d^3x}{(2\pi)^{3/2}}(\nabla_x\times\boldsymbol{A}(\boldsymbol{x}))\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\end{aligned} \tag{9.64}$$

By substitution of these identities back into the exponent of the wave function, we get

$$\Psi_0[A] = \mathcal{N}\exp\Big(-\frac{1}{2}\int d^3x\int d^3x'\boldsymbol{B}(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x}')G(\boldsymbol{x}-\boldsymbol{x}')\Big) \tag{9.65}$$

where $G(\boldsymbol{x},\boldsymbol{x}')$ is given by

$$G(\boldsymbol{x}-\boldsymbol{x}') = \int\frac{d^3p}{(2\pi)^3}\frac{e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{|\boldsymbol{p}|} \tag{9.66}$$

This function has a singular behavior at large values of $|\boldsymbol{p}|$. We will define a *smoothed* version $G_\Lambda(\boldsymbol{x}-\boldsymbol{x}')$ to be

$$G_\Lambda(\boldsymbol{x}-\boldsymbol{x}') = \int\frac{d^3p}{(2\pi)^3}\frac{e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{|\boldsymbol{p}|}e^{-|\boldsymbol{p}|/\Lambda} \tag{9.67}$$

which cuts off the contributions with $|\boldsymbol{p}| \gg \Lambda$. Also, $G_\Lambda(\boldsymbol{x},\boldsymbol{x}')$ formally goes back to $G(\boldsymbol{x}-\boldsymbol{x}')$ as $\Lambda \to \infty$. $G_\Lambda(\boldsymbol{x}-\boldsymbol{x}')$ can be evaluated explicitly to give

$$G_\Lambda(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{2\pi^2|\boldsymbol{x}-\boldsymbol{x}'|^2}\int_0^\infty dt\ \sin t\ e^{-t/\Lambda|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$= \frac{1}{2\pi^2|\boldsymbol{x}-\boldsymbol{x}'|^2}\mathrm{Im}\left[\frac{1}{\frac{1}{\Lambda|\boldsymbol{x}-\boldsymbol{x}'|}-i}\right] \tag{9.68}$$

Thus,

$$\lim_{\Lambda\to\infty} G_\Lambda(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{2\pi^2|\boldsymbol{x}-\boldsymbol{x}'|^2} \tag{9.69}$$

Hence, the ground state wave functional $\Psi_0[A]$ is

$$\Psi_0[A] = \mathcal{N}\exp\left(-\frac{1}{4\pi^2}\int d^3x\int d^3x\frac{\boldsymbol{B}(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\right) \tag{9.70}$$

which is only a functional of the configuration of magnetic fields.

9.4 Path integral quantization of gauge theories

We have discussed at length the quantization of the abelian gauge theory, i.e. Maxwell electromagnetism, within canonical quantization in the $A_0 = 0$ gauge and a modified canonical formalism in the Coulomb gauge. Although conceptually what we have done is correct, it poses a number of questions.

The canonical formalism is natural in the gauge $A_0 = 0$, and can be generalized to other gauge theories. However, this gauge is highly non-covariant and it is necessary to prove covariance of physical observables at the end. In addition, the gauge field propagator in this gauge is very complicated.

The particle spectrum is most transparent in the transverse (or Coulomb) gauge. However, in addition of being non-covariant, it is not possible to generalize this gauge to non-abelian gauge theories (or even to abelian gauge theories on a compact gauge group) due to subtle topological problems known as Gribov ambiguities (or Gribov "copies"). The propagator is equally awful in this gauge. The commutation relations in real space look quite different from those in scalar field theory. In addition, for non-abelian gauge groups, even in the absence of matter fields, the theory is already non-linear and needs to be regularized in a manner in which gauge invariance is preserved. Although it is possible to use covariant gauges, such as the Lorentz gauge $\partial_\mu A^\mu = 0$, the quantization of the theory is these gauges requires an approach, known as Gupta-Bleuer quantization, of difficult generalization.

At the root of this problems is the issue of quantizing a theory which has a local (or gauge) invariance in a manner in which both Lorentz and gauge invariance are kept explicitly. It turns out that path-integral quantization is the most direct approach to deal with these problems.

We will now construct the path integral for the free electromagnetic field. However, formally the procedure that we will present will hold, at least formally, for any gauge theory.

We will begin with the theory quantized canonically in the gauge $A_0 = 0$ (Dirac, 1966). We saw above that, in the gauge $A_0 = 0$, the electric field $\boldsymbol{E}$ is (minus) the momentum canonically conjugate to the vector potential $\boldsymbol{A}$, the spatial components of the gauge field, and both fields obey equal-time canonical commutation relations

$$\left[E_j(\boldsymbol{x}), A_k(\boldsymbol{x}')\right] = i\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{9.71}$$

In addition, in this gauge, the Gauss Law becomes a constraint on the space of states, i.e.

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x})|\text{Phys}\rangle = J_0(\boldsymbol{x})|\text{Phys}\rangle \tag{9.72}$$

which defines the physical Hilbert space. Here $J_0(x)$ is a charge density distribution. In the presence of a set of conserved sources $J_\mu(x)$, that satisfy $\partial_\mu J^\mu = 0$, the Hamiltonian of the free field theory is

$$\hat{H} = \int d^3x\, \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \int d^3x\, \boldsymbol{J}\cdot\boldsymbol{A} \tag{9.73}$$

We will construct the path-integral in the Hilbert space of gauge-invariant states defined by the condition of Eq.(9.72).

Let su denote by $Z[J_\mu]$ the partition function

$$Z[J] = \text{tr}' T e^{-i\int dx_0 \hat{H}} \equiv \text{tr}\left(T e^{-i\int dx_0 \hat{H}}\hat{P}\right) \tag{9.74}$$

where tr' means a trace (or sum) over the space of states that satisfy the Gauss Law, Eq.(9.72). We implement this constraint by means of the operator $\hat{P}$ that projects onto the gauge-invariant states,

$$\hat{P} = \prod_{\boldsymbol{x}} \delta\left(\nabla\cdot\hat{\boldsymbol{E}}(\boldsymbol{x}) - J_0(\boldsymbol{x})\right) \tag{9.75}$$

We will now follow the standard construction of the path integral, while making sure that we only sum over histories that are consistent with the constraint. In principle all we need to do is to insert complete sets of states which are eigenstates of the field operator $\hat{\boldsymbol{A}}(x)$ at all intermediate times. These states, denoted by $|\{\boldsymbol{A}(\boldsymbol{x},x_0)\}\rangle$, are not gauge invariant, and do not satisfy the constraint. However, the projection operator $\hat{P}$ projects-out the unphysical components of these states.

Hence, if the projection operator is included in the evolution operator, the inserted states actually are gauge-invariant. Thus, to insert at every intermediate time x_0^k ($k = 1, \ldots, N$ with $N \to \infty$ and $\Delta x_0 \to 0$) a complete set of gauge-invariant states, amounts to writing $Z[J]$ as

$$Z[J] = \prod_{k=1}^{N} \int \mathcal{D}A_j(\boldsymbol{x}, x_0^k)$$
$$\langle \{A_j(\boldsymbol{x}, x_0^k)\}| \left(1 - i\Delta x_0 \hat{H}\right) \prod_{\boldsymbol{x}} \delta\left(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, x_0^k) - J_0(\boldsymbol{x}, x_0^k)\right) |\{A_j(\boldsymbol{x}, x_0^{k+1})\}\rangle \tag{9.76}$$

As an operator, the projection operator $\hat{P}$ is naturally spanned by the eigenstates of the electric field operator $|\{\boldsymbol{E}(\boldsymbol{x}, x_0)\}\rangle$, i.e.

$$\prod_{\boldsymbol{x}} \delta\left(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, x_0) - J_0(\boldsymbol{x}, x_0)\right) \equiv$$
$$\int \mathcal{D}\boldsymbol{E}(\boldsymbol{x}, x_0)\ |\{\boldsymbol{E}(\boldsymbol{x}, x_0)\}\rangle\langle\{\boldsymbol{E}(\boldsymbol{x}, x_0)\}| \prod_{\boldsymbol{x}} \delta\left(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, x_0) - J_0(\boldsymbol{x}, x_0)\right) \tag{9.77}$$

The delta function has the integral representation

$$\prod_{\boldsymbol{x}} \delta\left(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, x_0) - J_0(\boldsymbol{x}, x_0)\right) =$$
$$= \mathcal{N} \int \mathcal{D}A_0(\boldsymbol{x}, x_0) e^{i\Delta x_0 \int d^3x\ A_0(\boldsymbol{x}, x_0)\left(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, x_0) - J_0(\boldsymbol{x}, x_0)\right)} \tag{9.78}$$

Hence, the matrix elements of interest become

$$\int \mathcal{D}\boldsymbol{A} \prod_{x_0} \langle\{\boldsymbol{A}(\boldsymbol{x}, x_0)\}| \left(1 - i\Delta x_0 \hat{H}\right) \prod_{\boldsymbol{x}} \delta\left(\nabla_j \hat{E}_j - J_0\right) |\{\boldsymbol{A}(\boldsymbol{x}, x_0 + \Delta x_0)\}\rangle$$
$$= \int \mathcal{D}A_0 \mathcal{D}\boldsymbol{A} \mathcal{D}\boldsymbol{E} \prod_{x_0} \langle\{\boldsymbol{A}(\boldsymbol{x}, x_0)\}|\{\boldsymbol{E}(\boldsymbol{x}, x_0)\}\rangle\langle\{\boldsymbol{E}(\boldsymbol{x}, x_0)\}|\{\boldsymbol{A}(\boldsymbol{x}, x_0 + \Delta x_0)\}\rangle$$
$$\times\ \exp\left[i\Delta x_0 \int d^3x\ A_0(\boldsymbol{x}, x_0)\left(\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, x_0) - J_0(\boldsymbol{x}, x_0)\right)\right]$$
$$\times \exp\left[-\frac{\langle\{\boldsymbol{A}(\boldsymbol{x}, x_0)\}|\hat{H}|\{\boldsymbol{E}(\boldsymbol{x}, x_0)\}\rangle}{\langle\{\boldsymbol{A}(\boldsymbol{x}, x_0)\}|\{\boldsymbol{E}(\boldsymbol{x}, x_0)\}\rangle}\right] \tag{9.79}$$

The overlaps are equal to

$$\langle\{\boldsymbol{A}(\boldsymbol{x},x_0)\}|\{\boldsymbol{E}(\boldsymbol{x},x_0)\}\rangle = e^{i\int d^3x\ \boldsymbol{A}(\boldsymbol{x},x_0)\cdot\boldsymbol{E}(\boldsymbol{x},x_0)} \tag{9.80}$$

Hence, we find that the product of the overlaps is given by

$$\prod_{x_0}\ \langle\{\boldsymbol{A}(\boldsymbol{x},x_0)\}|\{\boldsymbol{E}(\boldsymbol{x},x_0)\}\rangle\langle\{\boldsymbol{E}(\boldsymbol{x},x_0)\}|\{\boldsymbol{A}(\boldsymbol{x},x_0+\Delta x_0)\}\rangle = \\ = e^{-i\int dx_0\ \int d^3x\ \boldsymbol{E}(\boldsymbol{x},x_0)\cdot\partial_0\boldsymbol{A}(\boldsymbol{x},x_0)} \tag{9.81}$$

The matrix elements of the Hamiltonian are

$$\frac{\langle\{\boldsymbol{A}(\boldsymbol{x},x_0)\}|\hat{H}|\{\boldsymbol{E}(\boldsymbol{x},x_0)\}\rangle}{\langle\{\boldsymbol{A}(\boldsymbol{x},x_0)\}|\{\boldsymbol{E}(\boldsymbol{x},x_0)\}\rangle} = \int d^3x\ \left[\frac{1}{2}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)+\boldsymbol{J}\cdot\boldsymbol{A}\right] \tag{9.82}$$

Putting everything together we find that the path integral expression for $Z[J]$ has the form

$$Z[J] = \int \mathcal{D}A_\mu\mathcal{D}\boldsymbol{E}\ e^{iS[A_\mu,\boldsymbol{E}]} \tag{9.83}$$

where

$$\mathcal{D}A_\mu = \mathcal{D}\boldsymbol{A}\mathcal{D}A_0 \tag{9.84}$$

and the action $S[A_\mu,\boldsymbol{E}]$ is given by

$$S[A_\mu,\boldsymbol{E}] = \int d^4x\left[-\boldsymbol{E}\cdot\partial_0\boldsymbol{A}-\frac{1}{2}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)-\boldsymbol{J}\cdot\boldsymbol{A}+A_0\left(\nabla\cdot\boldsymbol{E}-J_0\right)\right] \tag{9.85}$$

Notice that the Lagrange multiplier field A_0, which appeared when we introduced the integral representation of the delta function, has become the time component of the vector potential.

Since the action is quadratic in the electric fields, we can integrate them out explicitly to find

$$\int \mathcal{D}\boldsymbol{E}\ e^{i\int d^4x\left(-\frac{1}{2}\boldsymbol{E}^2-\boldsymbol{E}\cdot(\partial_0\boldsymbol{A}+\nabla A_0)\right)} = \\ = \text{const.}\ e^{i\int d^4x\frac{1}{2}\left(-\partial_0\boldsymbol{A}-\nabla A_0\right)^2} \tag{9.86}$$

We now collect everything and find that the path integral is

$$Z[J] = \int \mathcal{D}A_\mu \; e^{i\int d^4x\mathcal{L}} \tag{9.87}$$

where the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu \tag{9.88}$$

which is what we should have expected. We should note here that this formal argument is valid for *all* gauge theories, abelian or non-abelian. In other words, the path integral is *always* the sum over the histories of the field A_μ with a weight factor which is the exponential of $i/\hbar$ times the action S of the gauge theory.

Therefore, we found that, at least formally, we can write a functional integral which will play the role of the generating functional of the N-point functions of this theories,

$$\langle 0|TA_{\mu_1}(x_1)\ldots A_{\mu_N}(x_N)|0\rangle \tag{9.89}$$

9.5 Path integrals and gauge fixing

The expression for the path integral in Eq.(9.87) is formal because we are summing over *all* histories of the field without restriction. In fact, since the action S and the integration measure $\mathcal{D}A_\mu$ are both gauge invariant, histories that differ by gauge transformations have the same weight in the path integral, and the partition function has an apparent divergence of the form $v(G)^V$, where $v(G)$ is the volume of the gauge group G and V is the (infinite) volume of space-time.

In order to avoid this problem we must implement a procedure that restricts the sum over configurations in such a way that configurations which differ by local gauge transformations are counted only once. This procedure is known as gauge fixing. We will follow that approach introduced by L. Faddeev and V. Popov (Faddeev and Popov, 1967; Faddeev, 1976). Although the method works for all gauge theories, the non-abelian theories have subtleties and technical issues that we will discuss below. We will begin with a general discussion of the method, and then we will specialize it first for the case of Maxwell theory, the $U(1)$ gauge theory without matter fields, and later to the case of a general compact gauge group.

Let the vector potential A_μ be a field that takes values in the *algebra* of a gauge group G, i.e. A_μ is a linear combination of the group generators. Let $U(x)$ be an unitary-matrix field that takes values on a representation of the group G (please recall our earlier discussion on this subject in section 3.6). For the abelian group $U(1)$, we have

$$U(x) = e^{i\phi(x)} \tag{9.90}$$

where $\phi(x)$ is a real (scalar) field. A gauge transformation is, for a group G

$$A_\mu^U = U A_\mu U^\dagger - iU\partial_\mu U^\dagger \tag{9.91}$$

For the abelian group $U(1)$ we have

$$A_\mu^U = A_\mu + \partial_\mu \phi \tag{9.92}$$

In order to avoid infinities in $Z[J]$, we must impose restrictions on the sum over histories such that histories that are related via a gauge transformation are *counted exactly once*. In order to do that we must find a way to classify the configurations of the vector field A_μ into *classes*. We will do this by defining gauge fixing conditions. Each class is labelled by a *representative* configuration and other elements in the class are related to it by smooth gauge transformations. Hence, all configurations in a given class are characterized by a set of gauge invariant data, such as field strengths in the case of the abelian theory. The set of configurations that differ from each other by a local gauge transformation belong to the same class. We can think of the class as a set obtained by the action on some reference configuration by the gauge group, and the elements of a class constitute an orbit of the gauge group. Mathematically, the elements of the gauge class form a vector bundle.

We must choose gauge conditions such that the theory remains local and, if possible, Lorentz covariant. It is essential that, whatever gauge condition we use, that *each class is counted exactly once* by the gauge condition. It turns out that for the Maxwell gauge theory this is always (and trivially) the case. However, in non-abelian theories, and in gauge theories with an abelian compact gauge group, there are many gauges in which a class may be counted more than once. The origin of this problem is a topological obstruction first shown by I. Singer. This question is known as the Gribov problem. The Coulomb gauge is well known to always have this problem, except for the trivial case of the Maxwell theory.

Finally we must also keep in mind that we are only fixing the local gauge invariance, but we should not alter the boundary conditions since they represent physical degrees of freedom. In particular, if the theory is defined on a closed manifold, e.g. a sphere, tori, etc., large gauge transformations, which wrap around the manifold, represent global degrees of freedom (or states). Large gauge transformations play a key role on gauge theories at finite temperature, where the transformation wrap around the (finite and periodic) imaginary time direction. Also, there is a class of gauge theories, known as topological field theories, whose only physical degrees of freedom are represented by large gauge transformations on closed manifolds. We will discuss these theories in chapter 22.

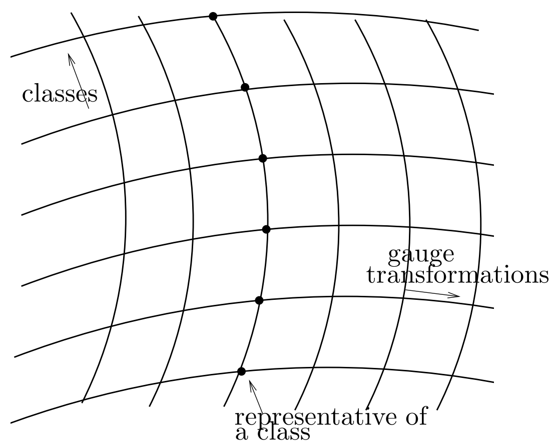

Figure 9.1 The gauge fixing condition selects a manifold of configurations.

How do we impose a gauge condition consistently? We will do it in the following way. Let us denote the gauge condition by that we wish to impose by

$$g(A_\mu) = 0 \tag{9.93}$$

where $g(A_\mu)$ is a local differentiable function of the gauge fields and/or of their derivatives. Examples of such local conditions are $g(A_\mu) = \partial_\mu A^\mu$ for the Lorentz gauge, or $g(A_\mu) = n_\mu A^\mu$ for an axial gauge.

The discussion that follows is valid for all compact Lie groups G of volume $v(G)$. For the special case of the Maxwell gauge theory, the gauge group is $U(1)$. Up to topological considerations, the group $U(1)$ is isomorphic to the real numbers $\mathbb{R}$, even though the volume of the compact $U(1)$ group is finite, $v(U(1)) = 2\pi$, while for the non-compact case the group are the real numbers $\mathbb{R}$ whose "volume" is infinite, $v(\mathbb{R}) = \infty$.

Naively, to impose a gauge condition would mean to restrict the path integral by inserting Eq. (9.93) as a delta function in the integrand,

$$Z[J] \sim \int \mathcal{D}A_\mu \, \delta\big(g(A_\mu)\big) \, e^{iS[A,J]} \tag{9.94}$$

We will now see that in general this is an inconsistent (and wrong) prescription. Following Faddeev and Popov, we begin by considering the expression defined by following integral

$$\Delta_g^{-1}[A_\mu] \equiv \int \mathcal{D}U \, \delta\Big(g(A_\mu^U)\Big) \tag{9.95}$$

where $A_\mu^U(x)$ are the configurations of gauge fields related by the gauge transformation $U(x)$ to the configuration $A_\mu(x)$, i.e. we move *inside one class*. In other words, the integral of Eq.(9.95) is a sum over the orbit of the gauge group. Thus, by construction, $\Delta_g[A_\mu]$ depends only on the class defined by the gauge-fixing condition g or, what is the same, it is gauge-invariant.

Let us show that $\Delta_g^{-1}[A_\mu]$ is gauge invariant. We first observe that the integration measure $\mathcal{D}U$, called the Haar measure, is invariant under the composition rule $U \to UU'$,

$$\mathcal{D}U = \mathcal{D}(UU') \tag{9.96}$$

where U' is and arbitrary but fixed element of G. For the case of $G = U(1)$, $U = \exp(i\phi)$ and $\mathcal{D}U \equiv \mathcal{D}\phi$.

Using the invariance of the measure, Eq. (9.96), we can write

$$\Delta_g^{-1}[A_\mu^{U'}] = \int \mathcal{D}U \, \delta\Big(g(A_\mu^{U'U})\Big) = \int \mathcal{D}U'' \, \delta\Big(g(A_\mu^{U''})\Big) = \Delta_g^{-1}[A_\mu] \tag{9.97}$$

where we have set $U'U = U''$. Therefore $\Delta_g^{-1}[A_\mu]$ is gauge invariant, i.e. it is a function of the *class* and not of the configuration A_μ itself. Obviously we can also write Eq. (9.95) in the form

$$1 = \Delta_g[A_\mu] \int \mathcal{D}U \, \delta\Big(g(A_\mu^U)\Big) \tag{9.98}$$

We will now insert the number 1, as given by Eq. (9.98), in the path integral for a general gauge theory to find

$$\begin{aligned} Z[J] &= \int \mathcal{D}A_\mu \times 1 \times e^{iS[A,J]} \\ &= \int \mathcal{D}A_\mu \, \Delta_g[A_\mu] \int \mathcal{D}U \, \delta\Big(g(A_\mu^U)\Big) \, e^{iS[A,J]} \end{aligned} \tag{9.99}$$

We now make the change of variables

$$A_\mu \to A_\mu^{U'} \tag{9.100}$$

where $U' = U'(x)$ is an arbitrary gauge transformation, and find

$$Z[J] = \int \mathcal{D}U \int \mathcal{D}A_\mu^{U'} \; e^{iS[A^{U'}, J]} \; \Delta_g[A_\mu^{U'}] \, \delta\left(g(A_\mu^{U'U})\right) \tag{9.101}$$

(Notice that we have changed the order of integration.) We now choose $U' = U^{-1}$, and use the gauge invariance of the action $S[A, J]$, of the measure $\mathcal{D}A_\mu$ and of $\Delta_g[A]$ to write the partition as

$$Z[J] = \left[\int \mathcal{D}U\right] \int \mathcal{D}A_\mu \; \Delta_g[A_\mu] \, \delta\left(g(A_\mu)\right) \; e^{iS[A, J]} \tag{9.102}$$

The factor in brackets in Eq. (9.102) is the infinite constant

$$\int \mathcal{D}U = v(G)^V \tag{9.103}$$

where $v(G)$ is the volume of the gauge group and V is the (infinite) volume of space-time. This infinite constant is nothing but the result of summing over gauge-equivalent states inside each class.

Thus, provided the quantity $\Delta_g[A_\mu]$ is finite, and that it does not vanish identically, we find that the consistent rule for fixing the gauge consists in dividing-out the (infinite) factor of the volume of the gauge group but, more importantly, to insert together with the constraint $\delta\left(g(A_\mu)\right)$ the factor $\Delta_g[A_\mu]$ in the integrand of $Z[J]$,

$$Z[J] \sim \int \mathcal{D}A_\mu \; \Delta_g[A_\mu] \, \delta\left(g(A_\mu)\right) \; e^{iS[A, J]} \tag{9.104}$$

Therefore the measure $\mathcal{D}A_\mu$ has to be understood as a sums over classes of configurations of the gauge fields and not over all possible configurations.

We are only left to compute $\Delta_g[A_\mu]$. We will show now that $\Delta_g[A_\mu]$ is a determinant of a certain operator, and is known as the Faddeev-Popov determinant. We will only compute first this determinant for the case of the abelian theory $U(1)$. Below we will also discuss the non-abelian case, relevant for Yang-Mills gauge theories.

We will compute $\Delta_g[A_\mu]$ by using the fact that $g[A_\mu^U]$ can be regarded as a function of $U(x)$ (for $A_\mu(x)$ fixed). We will now change variables from U to g. The price we pay is a Jacobian factor since

$$\mathcal{D}U = \mathcal{D}g \; \text{Det} \left|\frac{\delta U}{\delta g}\right| \tag{9.105}$$

where the determinant is the Jacobian of the change of variables. Since this is a non-linear change of variables, we expect a non-trivial Jacobian. Therefore we can write

$$\Delta_g^{-1}[A_\mu] = \int \mathcal{D}U \, \delta\left(g(A_\mu^U)\right) = \int \mathcal{D}g \,\text{Det}\left|\frac{\delta U}{\delta g}\right| \, \delta(g) \tag{9.106}$$

and we find

$$\Delta_g^{-1}[A_\mu] = \text{Det}\left|\frac{\delta U}{\delta g}\right|_{g=0} \tag{9.107}$$

or, conversely

$$\Delta_g[A_\mu] = \text{Det}\left|\frac{\delta g}{\delta U}\right|_{g=0} \tag{9.108}$$

All we have done thus far holds for all gauge theories with a compact gauge group. We will specialize our discussion first to the case of the $U(1)$ gauge theory, Maxwell electromagnetism. We will discuss how this applies to non-abelian Yang-Mill gauge theories shortly below.

For example, for the particular case of the abelian $U(1)$ gauge theory, the Lorentz gauge condition is obtained by the choice $g(A_\mu) = \partial_\mu A^\mu$. Then, for a general $U(1)$ gauge transformation $U(x) = \exp(i\phi(x))$, we get

$$g(A_\mu^U) = \partial_\mu \left(A^\mu + \partial^\mu \phi\right) = \partial_\mu A^\mu + \partial^2 \phi \tag{9.109}$$

Hence,

$$\frac{\delta g(x)}{\delta \phi(y)} = \partial^2 \delta(x-y) \tag{9.110}$$

Thus, for the Lorentz gauge of the abelian theory, the Faddeev-Popov determinant is given by

$$\Delta_g[A_\mu] = \text{Det}\partial^2 \tag{9.111}$$

which is a constant *independent* of A_μ. This is a peculiarity of the abelian theory and, as we will see below, it is not true in the non-abelian case.

Let us return momentarily to the general case of Eq. (9.104), and modify the gauge condition from $g(A_\mu) = 0$ to $g(A_\mu) = c(x)$, where $c(x)$ is some arbitrary function of x. The partition function now reads

$$Z[J] \sim \int \mathcal{D}A_\mu \, \Delta_g[A_\mu] \, \delta\left(g(A_\mu) - c(x)\right) \, e^{iS[A,J]} \tag{9.112}$$

We will now average over the arbitrary functions with a Gaussian weight

(properly normalized to unity)

$$Z_\alpha[J] = \mathcal{N} \int \mathcal{D}A_\mu \, \mathcal{D}c \, e^{-i \int d^4x \, \frac{c(x)^2}{2\alpha}} \, \Delta_g[A_\mu] \, \delta\left(g(A_\mu) - c(x)\right) \, e^{iS[A,J]}$$
$$= \mathcal{N} \int \mathcal{D}A_\mu \, \Delta_g(A_\mu) \, e^{i \int d^4x \left[\mathcal{L}[A,J] - \frac{1}{2\alpha}\left(g(A_\mu)\right)^2\right]} \tag{9.113}$$

From now on we will restrict our discussion to the $U(1)$ abelian gauge theory (the electromagnetic field) and $g(A_\mu) = \partial_\mu A^\mu$. From Eq. (9.113) we find that in this gauge the Lagrangian is

$$\mathcal{L}_\alpha = -\frac{1}{4}F_{\mu\nu}^2 + J_\mu A^\mu - \frac{1}{2\alpha}\left(\partial_\mu A^\mu\right)^2 \tag{9.114}$$

The parameter α labels a family of gauge fixing conditions known as the *Feynman-'t Hooft gauges*. For $\alpha \to 0$ we recover the strong constraint $\partial_\mu A^\mu = 0$, the Lorentz gauge. From the point of view of doing calculations the simplest is the gauge $\alpha = 1$, the Feynman gauge, as we will see now.

After some algebra is straightforward to see that, up to surface terms, in this family of gauges parametrized by α the Lagrangian is equal to

$$\mathcal{L}_\alpha = \frac{1}{2}A_\mu \left[g^{\mu\nu} \, \partial^2 - \frac{\alpha - 1}{\alpha} \, \partial^\mu \partial^\nu\right] A_\nu + J_\mu A^\mu \tag{9.115}$$

and the partition function reduces to

$$Z[J] = \mathcal{N} \, \text{Det}\left[\partial^2\right] \int \mathcal{D}A_\mu \, e^{i \int d^4x \, \mathcal{L}_\alpha[A,J]} \tag{9.116}$$

Hence, in a general gauge labelled by α, we get

$$Z[J] = \mathcal{N} \, \text{Det}\left[\partial^2\right] \, \text{Det}\left[g^{\mu\nu} \, \partial^2 - \frac{\alpha - 1}{\alpha} \, \partial^\mu \partial^\nu\right]^{-1/2}$$
$$\times \, \exp\left(\frac{i}{2} \int d^4x \int d^4y \, J_\mu(x) \, G^{\mu\nu}(x - y) \, J_\nu(y)\right) \tag{9.117}$$

where

$$G^{\mu\nu}(x - y) = -\langle x| \left(g^{\mu\nu} \, \partial^2 - \frac{\alpha - 1}{\alpha} \, \partial^\mu \partial^\nu\right)^{-1} |y\rangle \tag{9.118}$$

is the propagator in this gauge, parametrized by α. By inspection we see that the propagator of the gauge field $G_{\mu\nu}(x - y)$, in the Feynman-'t Hooft gauge parametrized by α, is related to the vacuum expectation value of the gauge fields by

$$G_{\mu\nu}(x - y) = i\langle 0|T A_\mu(x) A_\nu(y)|0\rangle \tag{9.119}$$

The form of Eq. (9.117) may seem to imply that $Z[J]$ depends on the choice of gauge. However, this cannot be correct since the path integral is, by construction, gauge-invariant. We will show in the next subsection that gauge invariance is indeed protected. This result comes about because J_μ is a conserved current, and as such it satisfies the continuity equation $\partial_\mu J^\mu = 0$.

For the Feynman-'t Hooft family of gauges, the propagator takes the form

$$G_{\mu\nu}(x-y) = -\left[g^{\mu\nu} + (\alpha - 1)\frac{\partial^\mu \partial^\nu}{\partial^2}\right] G^{(0)}(x-y) \tag{9.120}$$

where $G^{(0)}(x-y)$ is the propagator of the free *massless* scalar field and hence satisfies the Green function equation

$$\partial^2 G^{(0)}(x-y) = \delta^4(x-y) \tag{9.121}$$

where we set the mass of the scalar field to zero.

Thus, as expected for a free field theory, $Z[J]$ is a product of two factors: a functional (or fluctuation) determinant, and a factor that depends solely on the sources J_μ which contains all the information on the correlation functions. For the case of a single scalar field we also found a contribution in the form of a determinant factor but its power was $-1/2$. Here there are two such factors. The first one is the Faddeev-Popov determinant. The second one is the determinant of the fluctuation operator for the gauge field. However, in the Feynman gauge, $\alpha = 1$, this operator is just $g^{\mu\nu}\partial^2$, and its determinant has the same form as the Faddeev-Popov determinant except that it has a power $-4/2$. This is what one would have expected for a theory with *four* independent fields (one for each component of A_μ). The Faddeev-Popov determinant has power $+1$. Thus the total power is just $1-4/2 = -1$, which is the correct answer for a theory with only two independent (real) fields.

9.6 The propagator

For general α, $G_{\mu\nu}(x-y)$ is the solution of the Green function equation

$$-\left[g^{\mu\nu}\,\partial^2 - \frac{\alpha - 1}{\alpha}\partial^\mu \partial^\nu\right] G_{\nu\lambda}(x-y) = g^\mu_\lambda \delta^4(x-y) \tag{9.122}$$

Notice that in the special case of the Feynman gauge, $\alpha = 1$, this equation becomes

$$-\partial^2 G^{\mu\nu}(x-y) = g^{\mu\nu}\delta^4(x-y) \tag{9.123}$$

Hence, in the Feynman gauge, $G_{\mu\nu}(x-y)$ takes the form

$$G^{\mu\nu}(x-y) = -g^{\mu\nu}\, G^{(0)}(x-y) \tag{9.124}$$

where $G^{(0)}(x-y)$ is just the propagator of a free massless scalar field, i.e.

$$\partial^2 G^{(0)}(x-y) = \delta^4(x-y) \tag{9.125}$$

However, in a general gauge, the propagator of the gauge fields

$$G_{\mu\nu}(x-y) = i\langle 0|TA_\mu(x)A_\nu(y)|0\rangle \tag{9.126}$$

does not coincide with the propagator of a scalar field. Therefore, $G_{\mu\nu}(x-y)$, as expected, is a *gauge-dependent* quantity.

In spite of being gauge-dependent, the propagator does contain physical information. Let us examine this issue by calculating the propagator in a general gauge α. The Fourier transform of $G_{\mu\nu}(x-y)$ in D space-time dimensions is

$$G_{\mu\nu}(x-y) = \int \frac{d^D p}{(2\pi)^D}\, \widetilde{G}_{\mu\nu}(p)\, e^{-ip\cdot(x-y)} \tag{9.127}$$

This a solution of Eq. (9.122) provided $\widetilde{G}_{\mu\nu}(p)$ satisfies

$$\left[g^{\mu\nu}p^2 - \frac{\alpha-1}{\alpha}p^\mu p^\nu\right]\widetilde{G}_{\nu\lambda}(p) = g^\mu_\lambda \tag{9.128}$$

The formal solution is

$$\widetilde{G}_{\mu\nu}(p) = \frac{1}{p^2}\left[g^{\mu\nu} + (\alpha-1)\frac{p^\mu p^\nu}{p^2}\right] \tag{9.129}$$

In space-time the form of this (still formal) solution is given by Eq. (9.120).

In particular, in the Feynman gauge $\alpha = 1$, (formally) we get

$$\widetilde{G}^F_{\mu\nu}(p) = \frac{g^{\mu\nu}}{p^2} \tag{9.130}$$

whereas in the Lorentz gauge we find instead

$$\widetilde{G}^L_{\mu\nu}(p) = \frac{1}{p^2}\left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right] \tag{9.131}$$

Hence, in all cases there is a pole in p^2 in front of the propagator and a matrix structure that depends on the gauge choice. Notice that the matrix in brackets in the Lorentz gauge, $\alpha \to 0$, becomes the transverse projection operator which satisfies

$$p_\mu \left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right] = 0 \tag{9.132}$$

which follows from the gauge condition $\partial_\mu A^\mu = 0$.

The physical information of this propagator is contained in its analytic structure. It has a pole at $p^2 = 0$ which implies that $p_0 = \sqrt{\boldsymbol{p}^2} = |\boldsymbol{p}|$ is the singularity of $\widetilde{G}_{\mu\nu}(p)$. Hence the pole in the propagator tells us that this theory has a massless particle, the *photon*.

To actually compute the propagator in space-time from $\widetilde{G}_{\mu\nu}(p)$ requires that we define the integrals in momentum space carefully. As it stands, the Fourier integral Eq. (9.127) is ill-defined due to the pole in $\widetilde{G}_{\mu\nu}(p)$ at $p^2 = 0$. A proper definition requires that we move the pole into the complex plane by shifting $p^2 \to p^2 + i\epsilon$, where ϵ is real and $\epsilon \to 0^+$. This prescription yields the *Feynman propagator*. We will see in the next chapter that this rule applies to any theory and that it always yields the vacuum expectation value of the *time ordered* product of fields. For the rest of this section we will use the propagator in the Feynman gauge which reduces to the propagator of a scalar field. This is a quantity we know quite well, both in Euclidean and Minkowski space-times.

9.7 Physical meaning of $Z[J]$ and the Wilson loop operator

We discussed before that a general property of the path integral of any theory is that, in Euclidean space-time, $Z[0]$ is just

$$Z[0] = \langle 0|0\rangle \sim e^{-TE_0} \tag{9.133}$$

where T is the time span, which in general will be such that $T \to \infty$ (beware that here T is not the temperature!), and E_0 is the vacuum energy. Thus, if the sources J_μ are static (or quasi-static) we get instead

$$\frac{Z[J]}{Z[0]} \sim e^{-T\,[E_0(J) - E_0]} \tag{9.134}$$

Thus, the *change* in the vacuum energy due to the presence of the sources is

$$U(J) = E_0(J) - E_0 = -\lim_{T\to\infty} \frac{1}{T} \ln \frac{Z[J]}{Z[0]} \tag{9.135}$$

As we will see, the behavior of this quantity has a lot of information about the physical properties of the vacuum (i.e. the ground state) of a theory.

Quite generally, if the quasi-static sources J_μ are well separated from each other, $U(J)$ can be split into two terms: a self-energy of the sources, and an interaction energy,

$$U(J) = E_{\text{self-energy}}[J] + V_{\text{int}}[J] \tag{9.136}$$

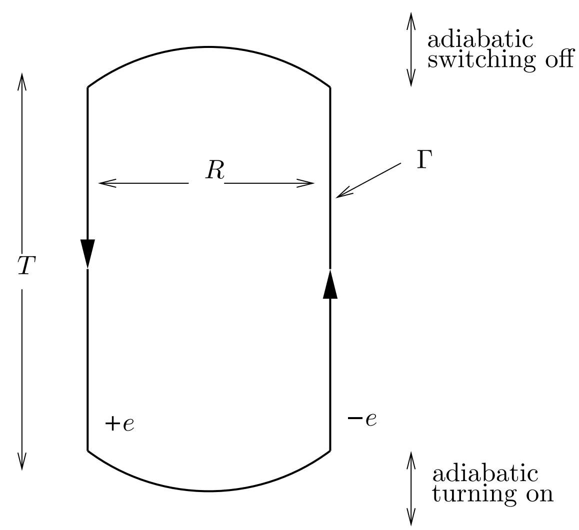

Figure 9.2 The Wilson loop operator can be viewed as representing a pair of quasi-static sources of charge $\pm e$ separated a distance R from each other.

As an example, we will now compute the expectation value of the Wilson loop operator,

$$W_\Gamma = \langle 0 | P e^{ie \oint_\Gamma dx_\mu A^\mu} | 0 \rangle \tag{9.137}$$

where Γ is the closed path in space-time shown in Fig.9.2, and P is the path-ordering symbol. Physically, what we are doing is looking at the electromagnetic field created by the current

$$J_\mu(x) = e\delta(x_\mu - s_\mu)\ \hat{s}_\mu \tag{9.138}$$

where s_μ is the set of points of space-time on the loop Γ , and $\hat{s}_\mu$ is a unit vector field tangent to Γ. The loop Γ has time span T and spatial size R.

We will be interested in loops such that $T \gg R$ so that the sources are turned on adiabatically in the remote past and switched off also adiabatically in the remote future. By current conservation the loop must be oriented. Thus, at a fixed time x_0 the loop looks like a pair of static sources with charges $\pm e$ at $\pm R/2$. In other words, we are looking of the effects of a particle-antiparticle pair which is created at rest in the remote past, the members of the pair are slowly separated (to avoid bremsstrahlung radiation) and live happily apart from each other, at a prudent distance R, for a long time T, and finally, are adiabatically annihilated in the remote future. Thus, we are in the quasi-static regime described above and $Z[J]/Z[0]$ should tell us what is the effective interaction between this pair of sources (or "electrodes").

What are the possible behaviors of the Wilson loop operator in general, that is for any gauge theory? The answer to this depends on the nature of the vacuum state. In later chapters we will see that a given theory may have different *vacua* or *phases* (as in thermodynamic phases), and that the behavior of the physical observables is different in different vacua (or phases). Here we will do an explicit computation for the case of the simple Maxwell $U(1)$ gauge theory. However the behavior that will find only holds for a free field and it is not generic.

What are the possible behaviors, then? A loop is an extended object. In contrast to a local operator, the Wilson loop expectation value is characterized by its geometric properties: its area, perimeter, aspect ratio, and so on. We will show later on that these geometric properties of the loop to characterize the behavior of the Wilson loop operator. Here are the generic cases (Wilson, 1974; Kogut and Susskind, 1975a):

1. *Area Law*: Let $A = RT$ be the minimal area of a surface bounded by the loop. One possible behavior of the Wilson loop operator is the *area law*

$$W_\Gamma \sim e^{-\sigma RT} \tag{9.139}$$

We will show later on that this is the *fastest possible decay* of the Wilson loop operator as a function of size. If the area law is obeyed the effective potential for R large, but still small compared to the time span T, behaves as

$$V_{\text{int}}(R) = \lim_{T\to\infty} \frac{-1}{T} \ln W_\Gamma = \sigma R \tag{9.140}$$

Hence, in this case the energy to separate a pair of sources grows linearly with distance, and the sources are *confined*. We will say that in this case the the theory is in the *confined phase*. The quantity σ is known as the *string tension*.

2. *Perimeter Law*: Another possible decay behavior, weaker than the area

law, is a perimeter law

$$W_\Gamma \sim e^{-\rho(R+T)} + O\left(e^{-R/\xi}\right) \tag{9.141}$$

where ρ is a constant with units of energy, and ξ is a length scale. This decay law implies that in this case

$$V_{\text{int}} \sim \rho + \text{const. } e^{-R/\xi} \tag{9.142}$$

Thus, in this case the energy to separate two sources to infinite distance is *finite*. This is a *deconfined* phase. However since it is massive, with a mass scale $m \sim \xi^{-1}$, there are no long range gauge bosons. We will later see that this phase can also be regarded as a *Higgs* phase. Since the gauge bosons are massive, this phase bears a close analogy with a *superconductor*.

3. *Scale Invariant*: Yet another possibility is that the Wilson loop behavior is determined by the aspect ratio R/T or T/R, e.g.

$$W_\Gamma \sim e^{-\alpha\left(\frac{R}{T}+\frac{T}{R}\right)} \tag{9.143}$$

where α is a *dimensionless* constant. This behavior leads to an interaction

$$V_{\text{int}} \sim -\frac{\alpha}{R} \tag{9.144}$$

which coincides with the Coulomb law in four dimensions. We will see that this is a *deconfined* phase with massless gauge bosons (photons).

We will now compute the expectation value of the Wilson loop operator in the Maxwell $U(1)$ gauge theory. We will return to the general problem when we discuss the strong coupling behavior of gauge theories. We begin by using the analytic continuation of Eq. (9.117) to imaginary time,

$$\begin{aligned} Z[J] &= \mathcal{N}\, \text{Det}\left[\partial^2\right]^{-1}\, e^{-\frac{1}{2}\int d^4x \int d^4y\, J_\mu(x)\, \langle A_\mu(x)A_\nu(y)\rangle\, J_\nu(y)} \\ &= \mathcal{N}\, \text{Det}\left[\partial^2\right]^{-1}\, e^{-\frac{e^2}{2}\oint_\Gamma dx_\mu \oint_\Gamma dy_\nu\, \langle A_\mu(x)A_\nu(y)\rangle} \end{aligned} \tag{9.145}$$

where $\langle A_\mu(x)A_\nu(y)\rangle$ is the Euclidean propagator of the gauge fields in the family of gauges labelled by α. Here we have also analytically continued the temporal component of the gauge field $A_0 \to iA_D$ so that the inner products, such as $A_\mu A^\mu \to -A_\mu^2$ (where now $\mu = 1, \ldots, D$), behave as they should in D-dimensional Euclidean space-time.

In the Feynman gauge $\alpha = 1$ the propagator is given by the expression

$$\langle A_\mu(x) A_\nu(y) \rangle = \delta_{\mu\nu} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} e^{ip_\mu \cdot (x_\mu - y_\mu)} \tag{9.146}$$

where $\mu = 1, \ldots, D$. After doing the integral we find that the Euclidean propagator (the correlation function) in the Feynman gauge is

$$\langle A_\mu(x) A_\nu(y) \rangle = \delta_{\mu\nu} \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2} |x - y|^{D-2}} \tag{9.147}$$

Notice that the propagator has a short-distance singularity $\sim R^{-(D-2)}$, where R is a length scale. This singularity can be easily understood from dimensional analysis. Indeed, since the Lagrangian density must have units of inverse space-time volume, $[\mathcal{L}] = L^{-D}$, it follows that the gauge field has units of $[A_\mu] = L^{-(D-2)/2}$, just as in the case of the scalar field. Thus, the circulation of the gauge field has units of $L^{(D-4)/2}$, and the electric charge has units of $[e] = L^{(D-4)/2}$. We will see below that this scaling is consistent.

In order to carry out this calculation we will make the assumption that the time span T of the Wilson loop is much larger than it spatial extent R, as shown in Fig.9.2. We will further assume that the loop is everywhere smooth and that both at long times in the past and in the future the loop was turned on and off arbitrarily slowly (adiabatically). These assumptions are needed in order to avoid singularities that have the physical interpretation of the production of a large amount of soft photons in the form of Bremsstrahlung radiation, as we noted above. Within these assumptions, the contributions to the expectation value of the Wilson loop operator from the top and bottom of the loop of Fig.9.2 can be neglected. Therefore, $E[J] - E_0$ is equal to

$$\begin{aligned} E[J] - E_0 &= \lim_{T \to \infty} \frac{e^2}{2T} \oint_\Gamma \oint_\Gamma d\boldsymbol{x} \cdot d\boldsymbol{y} \; \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2} |x - y|^{D-2}} \\ &= 2 \times \text{self} - \text{energy} - \frac{e^2}{2T} \, 2 \int_{-T/2}^{+T/2} dx_D \int_{-T/2}^{+T/2} dy_D \; \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2} |x - y|^{D-2}} \end{aligned} \tag{9.148}$$

where $|x-y|^2 = (x_D - y_D)^2 + R^2$. The integral in Eq. (9.148) is equal to

$$\int_{-T/2}^{+T/2} dx_D \int_{-T/2}^{+T/2} dy_D \; \frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}\,|x-y|^{D-2}} =$$

$$= \int_{-T/2}^{+T/2} ds \int_{-(T/2+s)/R}^{+(T/2-s)/R} \frac{dt}{(t^2+1)^{(D-2)/2}} \frac{1}{R^{D-3}} \times \frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{D/2}}$$

$$\simeq \frac{1}{R^{D-3}} \int_{-T/2}^{+T/2} ds \int_{-\infty}^{+\infty} \frac{dt}{(t^2+1)^{(D-2)/2}} \times \frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{D/2}}$$

$$= \frac{T\sqrt{\pi}}{R^{D-3}} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\Gamma\left(\frac{D-2}{2}\right)} \times \frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{D/2}} \tag{9.149}$$

where

$$\Gamma(\nu) = \int_0^\infty dt\, t^{\nu-1} e^{-t} \tag{9.150}$$

is the Euler Gamma function. In Eq. (9.149) we already took the limit $T/R \to \infty$. Putting it all together we find that the *interaction energy* of a pair of static sources of charges $\pm e$ separated a distance R in D dimensional space-time is given by

$$V_{\text{int}}(R) = -\frac{\Gamma(\frac{D-1}{2})}{2\pi^{(D-1)/2}(D-3)} \frac{e^2}{R^{D-3}} \tag{9.151}$$

This is the Coulomb potential in D space-time dimensions. It is straightforward to see that this result is consistent (as it should) with our dimensional analysis. In the particular case of $D = 4$ dimensions we find

$$V_{\text{int}}(R) = -\left(\frac{e^2}{4\pi}\right) \frac{1}{R} \tag{9.152}$$

where the (dimensionless) quantity $\alpha = e^2/4\pi$ is the *fine structure constant*. Notice that in $D = 4$ space-time dimensions the charge e is dimensionless. This fact plays a key role in the perturbative analysis of quantum electrodynamics. On the other hand, in $D = 1+1$ dimensions this result implies that the Coulomb interaction is a *linear* function of the separation R between the sources, i.e. the charged sources are *confined*.

Therefore we find that, even at the quantum level, the effective interaction between a pair of static sources is the Coulomb interaction. This is true because the Maxwell theory is a free field theory. It is also true in Quantum Electrodynamics (QED), the quantum field theory of electrons and photons, at distances R much greater than the Compton wavelength of the electron. However it is not true at short distances where the effective charge is screened by fluctuations of the Dirac field, and the potential becomes exponentially suppressed. In contrast, in Quantum Chromodynamics (QCD) the situation is quite different: even in the absence of a matter field, for R large compared with a scale ξ determined by the dynamics of Yang-Mills theory, the effective potential $V(R)$ grows linearly with R. This long distance behavior is known as *confinement*. The existence of the not trivial scale ξ, known as the *confinement scale*, cannot be obtained in perturbation theory. Conversely, the potential is Coulomb-like at short distances, a behavior known as *asymptotic freedom*.

9.8 Path integral quantization of non-abelian gauge theories

In this section we will discuss the general properties of the path-integral quantization of non-abelian gauge theories. Most of what we did for the abelian case, carries over to non-abelian gauge theories where, as we will see, it plays a much more central role. However, here we will not deal with the non-linearities which, ultimately, require the use of the ideas of the Renormalization Group, and a non-perturbative treatment. We will do this in chapter 15. A more detailed presentation can be found in the classic books by Itzykson and Zuber (Itzykson and Zuber, 1980), and by Peskin and Schroeder (Peskin and Schroeder, 1995).

The path integral $Z[J]$ for a non-abelian gauge field A_μ with gauge condition(s) $g^a[A]$ is

$$Z[J] = \int \mathcal{D}A_\mu^a \, e^{iS[A,J]} \, \delta(g[A]) \, \Delta_{\text{FP}}[A] \tag{9.153}$$

where $A_\mu = A_\mu^a \lambda^a$ is in the *algebra* of a simply connected compact Lie group G, whose generators are the Hermitian matrices λ^a. We will use the family of covariant gauge conditions

$$g^a[A] = \partial^\mu A_\mu^a(x) + c^a(x) = 0 \tag{9.154}$$

and where $\Delta_{\text{FP}}[A]$ is the Faddeev-Popov determinant. Notice that we impose one gauge condition for each direction in the algebra of the gauge group G. We will proceed as we did in the abelian case and consider an average over gauges. In other words we will work in the manifestly covariant Feynman-'t Hooft gauges. Notice that in the partition function of Eq. (9.153) we have dropped the overall divergent factor $v(G)^V$ or, rather, that we defined the integration measure $\mathcal{D}A_\mu$ so that this factor is explicitly cancelled.

Let us work out the structure of the Faddeev-Popov determinant for a general gauge fixing condition $g^a[A]$. Let U be an infinitesimal gauge transformation,

$$U \simeq 1 + i\epsilon^a(x)\lambda^a + \dots \tag{9.155}$$

Under a gauge transformation, the vector field A_μ transforms as

$$A_\mu^U = UA_\mu U^{-1} + i\left(\partial_\mu U\right)U^{-1} \equiv A_\mu + \delta A_\mu \tag{9.156}$$

For an infinitesimal transformation, the change of A_μ is

$$\delta A_\mu = i\epsilon^a\left[\lambda^a, A_\mu\right] - \partial_\mu\epsilon^a\lambda^a + O(\epsilon^2) \tag{9.157}$$

where λ^a are the generators of the algebra of the gauge group G.

In components, we can also write

$$\begin{aligned}\delta A_\mu^c =& 2i\epsilon^b \operatorname{tr}\left(\lambda^c\left[\lambda^b, A_\mu\right]\right) - 2\,\partial_\mu\epsilon^b \operatorname{tr}\left(\lambda^c\lambda^b\right) + O(\epsilon^2)\\ =& i\epsilon^b\operatorname{tr}\left(\lambda^c\left[\lambda^b,\lambda^d\right]\right)A_\mu^d - \partial_\mu\epsilon^b\delta_{bc} + O(\epsilon^2)\\ =& -2\,f^{bde}\,\epsilon^b\operatorname{tr}\left(\lambda^c\lambda^e\right)A_\mu^d - \partial_\mu\epsilon^b\,\delta_{bc} + O(\epsilon^2)\\ =& -f^{bdc}\,\epsilon^b\,A_\mu^d - \partial_\mu\epsilon^b\,\delta_{bc} + O(\epsilon^2)\end{aligned} \tag{9.158}$$

where f^{abc} are the structure constants of the Lie group G.

Therefore, we find

$$\frac{\delta A_\mu^c(x)}{\delta\epsilon^b(y)} = -\left[\partial_\mu\delta_{bc} + f^{bcd}\,A_\mu^d\right]\delta(x-y) \equiv -D_\mu^{cd}[A]\delta(x-y) \tag{9.159}$$

where we have denoted by $D_\mu[A]$ the covariant derivative in the *adjoint* representation, which in components is given by

$$D_\mu^{ab}[A] = \delta_{ab}\,\partial_\mu - f^{abc}\,A_\mu^c \tag{9.160}$$

Using these results we can put the Faddeev-Popov determinant (or Jacobian) in the form

$$\Delta_{FP}[A] = \operatorname{Det}\left(\frac{\delta g}{\delta\epsilon}\right) = \operatorname{Det}\left(\frac{\partial g^a}{\partial A_\mu^c}\frac{\delta A_\mu^c}{\delta\epsilon^b}\right) \tag{9.161}$$

where we used that

$$\frac{\delta g^a}{\delta\epsilon^b} = \frac{\partial g^a}{\partial A_\mu^c}\,\frac{\delta A_\mu^c}{\delta\epsilon^b} \tag{9.162}$$

We will now define an operator M_{FP} whose matrix elements are

$$\begin{aligned}\langle x,a|M_{FP}|y,b\rangle =&\langle x,a|\frac{\partial g}{\partial A_\mu^c}\frac{\delta A_\mu^c}{\delta\epsilon}|y,b\rangle\\ =&\int_z \frac{\partial g^a(x)}{\partial A_\mu^c(z)}\frac{\delta A_\mu^c(z)}{\delta\epsilon^b(y)}\\ =&-\int_z \frac{\partial g^a(x)}{\partial A_\mu^c(z)}\, D_\mu^{cb}\delta(z-y)\end{aligned} \tag{9.163}$$

For the case of $g^a[A]=\partial^\mu A_\mu^a(x)-c^a(c)$, appropriate for the Feynman-'t Hooft gauges, we have

$$\frac{\partial g^a(x)}{\partial A_\mu^c(z)}=\delta_{ac}\,\partial^\mu\delta(x-z) \tag{9.164}$$

and also

$$\begin{aligned}\langle x,a|M_{FP}|y,b\rangle =&-\int_z \delta_{ac}\partial_x^\mu\delta(x-z)D_\mu^{cb}[A]\delta(z-y)\\ =&-\int_z \delta_{ac}\delta(x-z)\,\partial_z^\mu D_\mu^{cb}\delta(x-y)\\ =&-\partial^\mu D_\mu^{ab}\delta(x-y)\end{aligned} \tag{9.165}$$

Thus, the Faddeev-Popov determinant now is

$$\Delta_{FP}=\text{Det}\left(\partial^\mu D_\mu[A]\right) \tag{9.166}$$

Notice that in the non-abelian case this determinant is an explicit function of the gauge field A_μ.

Since $\Delta_{FP}[A]$ is a determinant, it can be written as a path integral over a set of fermionic fields, denoted by $\eta_a(x)$ and $\bar\eta_a(x)$ (known as *ghosts*), one per gauge condition (i.e. one per generator):

$$\text{Det}\left[\partial^\mu D_\mu\right]=\int \mathcal{D}\eta_a\mathcal{D}\bar\eta_a\; e^{i\int d^Dx\;\bar\eta_a(x)\,\partial^\mu D_\mu^{ab}[A]\,\eta_b(x)} \tag{9.167}$$

Notice that these "ghost" fields are not spinors, and hence are quantized with the "wrong" statistics. In other words, these "particles" do not satisfy the general conditions for causality and unitarity to be obeyed. Hence, ghosts cannot create physical states (thereby their ghostly character).

The full form of the path integral of a Yang-Mills gauge theory with coupling constant g, in the Feynman-'t Hooft covariant gauges with gauge parameter λ, is given by

$$Z = \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}\bar{\eta}\; e^{i\int d^Dx\, \mathcal{L}_{YM}[A,\eta,\bar{\eta}]} \tag{9.168}$$

where $\mathcal{L}_{YM}$ is the effective Lagrangian density (defined in section 3.7.2)

$$\mathcal{L}_{YM}[A,\eta,\bar{\eta}] = -\frac{1}{4g^2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{\lambda}{2g^2}\left(\partial_\mu A^\mu\right)^2 - \bar{\eta}\,\partial_\mu D^\mu[A]\,\eta \tag{9.169}$$

Thus the pure gauge theory, even in the absence of matter fields, is non-linear. We will return to this problem later on when we look at both the perturbative and non-perturbative aspects of Yang-Mills gauge theories.

9.9 BRST invariance

In the previous section we developed in detail the path-integral quantization of non-abelian Yang-Mills gauge theories. We payed close attention to the role of gauge invariance and how to consistently fix the gauge in order to define the path-integral. Here we will show that the effective Lagrangian of a Yang-Mills gauge field, Eq. (9.169), has an extended symmetry, closely related to supersymmetry. This extended symmetry plays a crucial role in proving the renormalizability of non-abelian gauge theories.

Let us consider the QCD Lagrangian in the Feynman-'t Hooft covariant gauges (with gauge parameter λ and coupling constant g). The Lagrangian density $\mathcal{L}$ of this theory is

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} - \frac{1}{2\lambda}B_aB_a + B_a\partial^\mu A^a_\mu - \bar{\eta}^a\partial^\mu D^{ab}_\mu\eta^b \tag{9.170}$$

Here ψ is a Dirac Fermi field that represents quarks and transforms under the fundamental representation of the gauge group G. The "Hubbard-Stratonovich" field B_a is an auxiliary field which has no dynamics of its own, and transforms as a vector in the adjoint representation of G.

Becchi, Rouet, and Stora (Becchi et al., 1974, 1976) and Tyutin (Tyutin, 2008) realized that this gauge-fixed Lagrangian has the following ("BRST") symmetry, where ϵ is an infinitesimal anti-commuting parameter:

$$\delta A^a_\mu = \epsilon D^{ab}_\mu \eta_b \tag{9.171}$$

$$\delta\psi = ig\epsilon\eta^a t^a\psi \tag{9.172}$$

$$\delta\eta^a = -\frac{1}{2}\, g\epsilon f^{abc}\eta_b\eta_c \tag{9.173}$$

$$\delta\bar{\eta}^a = \epsilon B^a \tag{9.174}$$

$$\delta B^a = 0 \tag{9.175}$$

Eq. (9.171) and Eq. (9.172) are local gauge transformations, and as such, leave invariant the first two terms of the effective Lagrangian $\mathcal{L}$ of Eq. (9.170). The third term of Eq. (9.170) is trivial. The invariance of the fourth and fifth terms holds because the change of δA in the fourth term cancels against the change of $\bar{\eta}$ in the fifth term. Finally, it remains to see that the changes of the fields A_μ and η in the fifth term of Eq. (9.170) cancel out. To see that this is the case, we check that

$$\begin{aligned}\delta\left(D_\mu^{ab}\eta^b\right) &= D_\mu^{ab}\delta\eta^b + gf^{abc}\delta A_\mu^b\eta^c \\ &= -\frac{1}{2}g^2\epsilon f^{abc}f^{cde}\left(A_\mu^b\eta^d\eta^e + A_\mu^d\eta^e\eta^b + A_\mu^e\eta^b\eta^d\right)\end{aligned} \tag{9.176}$$

which vanishes due to the Jacobi identity for the structure constants

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0 \tag{9.177}$$

or, equivalently, from the nested commutators of the generators t^a:

$$\left[t^a,\left[t^b,t^c\right]\right] + \left[t^b,\left[t^c,t^a\right]\right] + \left[t^c,\left[t^a,t^b\right]\right] = 0 \tag{9.178}$$

Hence, BRST is at least a global symmetry of the gauge-fixed action with gauge fixing parameter λ.

This symmetry has a remarkable property which follows from its fermionic nature. Let ϕ be any of the fields of the Lagrangian and $Q\phi$ be the BRST transformation of the field,

$$\delta\phi = \epsilon Q\phi \tag{9.179}$$

For instance,

$$Q^a A_\mu^a = D_\mu^{ab}\eta^b \tag{9.180}$$

and so on. It follows that for *any* field ϕ

$$Q^2\phi = 0 \tag{9.181}$$

i.e. the BRST transformation of $Q\phi$ vanishes. This rule works for the field A_μ due to the transformation property of $\delta(D_\mu^{ab}\eta^b)$. It also holds for the ghosts since

$$Q^2\eta^a = \frac{1}{2}g^2 f^{abc}f^{bde}\eta^c\eta^d\eta^e = 0 \tag{9.182}$$

which holds due to the Jacobi identity.

What are the implications of the existence of BRST as a continuous symmetry? To begin with, it implies that there is a conserved self-adjoint charge Q that must necessarily commute with the Hamiltonian H of the Yang-Mills gauge theory. Above we saw how Q acts on the fields, $Q^2\phi = 0$, for all the fields in the Lagrangian. Hence, as an operator $Q^2 = 0$, that is, the BRST charge Q is nilpotent, and it commutes with H.

Let us now show that Q divides the Hilbert space of the eigenstates of H into three sectors

1. For $Q^2 = 0$ to hold, many eigenstates of H must be annihilated by Q. Let $\mathcal{H}_1$ be the set of eigenstates of H which *are not* annihilated by Q. Hence, if $|\psi_1\rangle \in \mathcal{H}_1$, then $Q|\psi_1\rangle \neq 0$. Thus, the states in $\mathcal{H}_1$ are not BRST invariant.
2. Let us consider the subspace of states $\mathcal{H}_2$ of the form $|\psi_2\rangle = Q|\psi_1\rangle$, i.e. $\mathcal{H}_2 = Q\mathcal{H}_1$. Then, for these states $Q|\psi_2\rangle = Q^2|\psi_1\rangle = 0$. Hence, the states in $\mathcal{H}_2$ are BRST invariant but are the BRST transform of states in $\mathcal{H}_1$.
3. Finally, let $\mathcal{H}_0$ be the set of eigenstates of H that are annihilated by Q, $Q|\psi_0\rangle = 0$, but which *are not* in $\mathcal{H}_2$, i.e. $|\psi_0\rangle \neq Q|\psi_1\rangle$. Hence, the states in $\mathcal{H}_0$ are BRST invariant, and are not the BRST transform of any other state. This is the *physical space of states*.

It follows from the above classification that *any* pair of states in $\mathcal{H}_2$, $|\psi_2\rangle$ and $|\psi_2'\rangle$, has a zero inner product:

$$\langle\psi_2|\psi_2'\rangle = \langle\psi_1|Q|\psi_2'\rangle = 0 \tag{9.183}$$

where we used that $|\psi_2\rangle$ is the BRST transform of a state in $\mathcal{H}_1$, $|\psi_1\rangle$. Similarly, one can show that if $|\psi_0\rangle \in \mathcal{H}_0$, then $\langle\psi_2|\psi_0\rangle = 0$.

What is the physical meaning of BRST and of this classification? Peskin and Schroeder give a simple argument (Peskin and Schroeder, 1995). Consider the weak coupling limit of the theory, $g \to 0$. In this limit we can find out what BRST does by looking at the transformation properties of the fields that appear in the Lagrangian of Eq. (9.170). In particular, Q transforms a forward polarized (i.e. longitudinal) component of A_μ into a ghost. At $g = 0$, we see that $Q\eta = 0$ and that the anti-ghost $\bar{\eta}$ transforms into the auxiliary field B. Also, at the classical level, $B = \lambda\partial^\mu A_\mu$. Hence, the auxiliary fields B are backward (longitudinally) polarized quanta of A_μ. Thus, forward polarized gauge bosons and anti-ghosts are in $\mathcal{H}_1$, since they are not the BRST transform of states created by other fields. Ghosts and backward polarized gauge bosons are in $\mathcal{H}_2$ since they are the BRST transform of the former. Finally, transverse gauge bosons are in $\mathcal{H}_0$. Hence, in general, states with ghosts, anti-ghosts, and gauge bosons with unphysical polarization belong to either $\mathcal{H}_1$ or $\mathcal{H}_2$. Only the physical states belong to $\mathcal{H}_0$. It turns out that the S-matrix, when restricted to the physical space $\mathcal{H}_0$, is unitary (as it should).

Finally, we note that BRST symmetry appears in all theories with constraints (Henneaux and Teitelboim, 1992). For example, it plays a key role in the study of critical dynamics in the path integral approach to the Langevin equation description of systems out of equilibrium, and in the statistical mechanics of disordered systems (Martin et al., 1973; Parisi, 1988; Hertz et al., 2016).