1 Finite temperature propagator: Erratum

The propagator for a scalar field at finite temperature T and in imaginary time is

$$G_T(\boldsymbol{x},\tau) = \frac{1}{\beta}\int \frac{d^dp}{(2\pi)^d}\sum_{n=-\infty}^{\infty}\frac{e^{i\omega_n\tau+i\boldsymbol{p}\cdot\boldsymbol{x}}}{\omega_n^2+\boldsymbol{p}^2+m^2} \tag{1}$$

where $\beta = 1/T$ and $\omega_n = 2\pi T n$. We then write the propagator as

$$G_T(\boldsymbol{x},\tau) = \frac{\beta}{4\pi^2}\int \frac{d^dp}{(2\pi)^d}\sum_{n=-\infty}^{\infty}\frac{e^{in(2\pi T\tau)+i\boldsymbol{p}\cdot\boldsymbol{x}}}{n^2+\frac{\boldsymbol{p}^2+m^2}{4\pi^2T^2}} \tag{2}$$

We can now use the identity

$$\sum_{n=-\infty}^{\infty}\frac{e^{inz}}{n^2+a^2} = \frac{\pi}{a}\Big[\coth(\pi a)\cosh(az)-\sinh(az)\Big] \tag{3}$$

which can be derived using a contour integral with the auxiliary function

$$F(z) = \frac{\pi e^{i\pi z}}{\sin(\pi z)} \tag{4}$$

which has simple poles at $z = n \in \mathbb{Z}$ with unit residue. For $z \in \mathbb{R}$, the sum is an even function of z with period 2π. Hence, it is defined outside the interval $[0, 2\pi)$ by its periodic extension.

Using the identity, the imaginary time propagator becomes

$$\begin{aligned} G_T(\boldsymbol{x},\tau) &= \int \frac{d^dp}{(2\pi)^d}\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{2\sqrt{\boldsymbol{p}^2+m^2}}\Big[\coth\left(\frac{\sqrt{\boldsymbol{p}^2+m^2}}{2T}\right)\cosh\left(\sqrt{\boldsymbol{p}^2+m^2}\tau\right)-\sinh\left(\sqrt{\boldsymbol{p}^2+m^2}\tau\right)\Big] \\ &= \int \frac{d^dp}{(2\pi)^d}\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}-\sqrt{\boldsymbol{p}^2+m^2}\tau}}{2\sqrt{\boldsymbol{p}^2+m^2}} + \\ &\quad + \int \frac{d^dp}{(2\pi)^d}\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{2\sqrt{\boldsymbol{p}^2+m^2}}\left(\coth\left(\frac{\sqrt{\boldsymbol{p}^2+m^2}}{2T}\right)-1\right)\cosh\left(\sqrt{\boldsymbol{p}^2+m^2}\tau\right) \end{aligned} \tag{5}$$

which can be brought to the form

$$\begin{aligned} G_T(\boldsymbol{x},\tau) &= \int \frac{d^dp}{(2\pi)^d}\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}-\sqrt{\boldsymbol{p}^2+m^2}\tau}}{2\sqrt{\boldsymbol{p}^2+m^2}} + \\ &\quad + \int \frac{d^dp}{(2\pi)^d}\frac{1}{e^{\frac{\sqrt{\boldsymbol{p}^2+m^2}}{T}}-1}\,\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{\boldsymbol{p}^2+m^2}}\cosh\left(\sqrt{\boldsymbol{p}^2+m^2}\tau\right) \end{aligned} \tag{6}$$

where we recognize that the first term of the r.h.s. is the imaginary time propagator at $T = 0$ and that

$$n_B(\boldsymbol{p}) = \frac{1}{e^{\frac{\sqrt{\boldsymbol{p}^2+m^2}}{T}}-1} \tag{7}$$

is the Bose-Einstein distribution for a bosons with energy $\sqrt{\boldsymbol{p}^2+m^2}$.

The (time-ordered) propagator in *real time* x_0 at finite temperature T is the obtained by the analytic continuation $\tau \to ix_0$ and it is given by

$$G_T(\boldsymbol{x},x_0) = \int \frac{d^dp}{(2\pi)^d}\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}-i\sqrt{\boldsymbol{p}^2+m^2}x_0}}{2\sqrt{\boldsymbol{p}^2+m^2}}+ \\ + \int \frac{d^dp}{(2\pi)^d}\frac{1}{e^{\frac{\sqrt{\boldsymbol{p}^2+m^2}}{T}}-1}\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{\sqrt{\boldsymbol{p}^2+m^2}}\cos\left(\sqrt{\boldsymbol{p}^2+m^2}x_0\right) \qquad (8)$$