1 Vertex Functions, Effective Potential and Ward Identities

1. Looking at the anticommutation relation:

\[ \{ \psi_{\alpha, a}^\dagger (x), \psi_{\beta, b} (y) \} = \delta_{\alpha, \beta} \delta_{a, b} \delta(x - y) \]  

we can deduce the momentum operator from the equal time anticommutation relations:

\[ \{ \psi_{\alpha, a} (x), \Pi_{\beta, b} (y) \} = i \delta_{\alpha, \beta} \delta_{a, b} \delta(x - y) \Rightarrow \Pi_a (x) = i \psi_a^\dagger (x) \]  

The Lagrangian density is then given by:

\[ \mathcal{L} = \Pi_a \partial_0 \psi_a - \mathcal{H} = i \psi_a^\dagger \partial_0 \psi_a - i \psi_a^\dagger \gamma_5 \partial_1 \psi_a - \frac{g}{4} \left[ (\bar{\psi}_a \psi_a)^2 - (\bar{\psi}_a \gamma_5 \psi_a)^2 \right] \]  

Inserting the Identity matrix \( I = \gamma_0^2 \) in the first two terms of the previous expression and using that \( \gamma_0 \gamma_5 = \gamma_1 \) and \( \bar{\psi}_a = \psi_a^\dagger \gamma_0 \) we finally get:

\[ \mathcal{L} = i \bar{\psi}_a \dot{\psi}_a - \frac{g}{4} \left[ (\bar{\psi}_a \psi_a)^2 - (\bar{\psi}_a \gamma_5 \psi_a)^2 \right] \]  

where \( \dot{\psi} = \gamma^\mu \partial_\mu = \gamma_0 \partial_0 - \gamma_1 \partial_1 \) and sum over repeated indices is understood.

2. We can directly read off the propagator from the free part of the Lagrangian in eq. (4), \( S^{(F)} = (i\dot{\psi})^{-1} \), which in momentum space reads as:

\[ S^{(F)} (p) = \frac{\not{p}}{p^2 + i\epsilon} \]  

3. Let us start by writing the interaction term of the Lagrangian in eq. (4):

\[ \mathcal{L}_{int} = -\frac{g}{4} \left[ (\bar{\psi}_a \psi_a)^2 - (\bar{\psi}_a \gamma_5 \psi_a)^2 \right] \]  

\[ = -\frac{g}{4} \left[ (\bar{\psi}_{\alpha, a} \delta^{\alpha\beta} \psi_{\beta, a}) (\bar{\psi}_{\delta, b} \delta^{\delta\rho} \psi_{\rho, b}) - (\bar{\psi}_{\alpha, a} \gamma_5^\alpha \psi_{\beta, a}) (\bar{\psi}_{\delta, b} \gamma_5^\delta \psi_{\rho, b}) \right] \]  

\[ = -\bar{\lambda}^{\alpha\beta\delta\rho} (\bar{\psi}_{\alpha, a} \psi_{\beta, a}) (\bar{\psi}_{\delta, b} \psi_{\rho, b}) \]
where $\lambda^{\alpha\beta\rho} = \frac{g}{4}(\delta^{\alpha\beta}\delta^{\delta\rho} - \gamma_5^{\alpha\beta}\gamma_5^{\delta\rho})$. Using that $\gamma_5$ is diagonal ($\gamma_5^{11} = -1$, $\gamma_5^{22} = 1$, $\gamma_5^{12} = \gamma_5^{21} = 0$) we have that:

$$\mathcal{L}_{int} = -\sum_{\alpha,\beta} \lambda^{\alpha\beta}(\bar{\psi}_{\alpha,a}\psi_{\alpha,a})(\bar{\psi}_{\beta,b}\psi_{\beta,b})$$

(9)

where we defined $\lambda^{\alpha\beta} = \frac{g}{4}(1 - \gamma_5^{\alpha}\gamma_5^{\beta}) = \frac{g}{2}\lambda^{\alpha\beta}$. (Notice that $g$ is included in the definition of the $\lambda$ matrix.) The partition function (or the path integral) reads:

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left( i\int d^2x (\bar{\psi}_a i\partial_x \psi_a + \bar{\psi}_a \eta_a + \mathcal{L}_{int}) \right)$$

(10)

Since the interaction term is quartic in the fermionic fields, we cannot integrate out the fermionic fields. We can expand the exponential of the interaction action:

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left( i\int d^2x (\bar{\psi}_a i\partial_x \psi_a + \bar{\psi}_a \eta_a + \mathcal{L}_{int}) \right)$$

$$= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \sum_{n=0}^{\infty} \frac{1}{n!} \left[ -i\lambda^{\alpha\beta} \int dy (\bar{\psi}_{\alpha,a}\psi_{\alpha,a})(\bar{\psi}_{\beta,b}\psi_{\beta,b}) \right]^n \exp \left( i\int d^2x (\bar{\psi}_a i\partial_x \psi_a + \bar{\psi}_a \eta_a + \mathcal{L}_{int}) \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ -i\lambda^{\alpha\beta} \int dy \frac{\delta}{\delta\eta_{\alpha,a}} \frac{\delta}{\delta\eta_{\beta,b}} \right]^n \text{det}(i\partial_x) e^{-\frac{1}{2} d^2x d^2x' S^{(F)}(x-x')\eta(x') \eta(x')}$$

So finally we write a formal series in the coupling constant in momentum space

$$Z[\eta, \eta] = Z[0,0] \sum_{n=0}^{\infty} \frac{1}{n!} \left[ -i\lambda^{\alpha\beta} \int \prod_{i=1}^{4} \frac{d^2k_i}{(2\pi)^2} \delta \frac{\delta}{\delta\eta_{\alpha,a}(k_i)} \delta \frac{\delta}{\delta\eta_{\beta,b}(k_i)} \delta \frac{\delta}{\delta\eta_{\beta,b}(k_i)} \delta \frac{\delta}{\delta\eta_{\alpha,a}(k_i)} \delta \frac{\delta}{\delta\eta_{\alpha,a}(k_i)} \right]^n$$

$$\times \exp \left( -\int \frac{d^2p}{(2\pi)^2} \frac{S^{(F)}(p)}{\eta(p)} \eta(-p) \right)$$

where $Z[0,0] = \text{det}(i\partial_x)$, $\lambda^{\alpha\beta} = \frac{g}{2}\gamma_5^{\alpha\beta}$ and $S^{(F)}(p) = \left( \frac{p^2 + i\epsilon}{2p^2 + i\epsilon} \right)_{\alpha,\beta} \delta_{a,b}$.

We can now use the previous expansion to derive the Feynman rules for the two point function for the $n^{th}$ order in $g$:

- Two external legs, labeled by a set of external momentum. $n$ vertices, with two momentums coming in and two coming out.
• Draw the topologically distinct graphs.
• Weight of each graph.
  (a) Every vertex preserves momentum and has a factor of \(-i\lambda_{\alpha,\beta}\).
  (b) Each line corresponds to the free Feynman propagator:

\[
\alpha, a \quad \overrightarrow{p} \quad \beta, b = iS_{a,b}^{(F)\alpha,\beta}(p)
\]

(c) Include multiplicity factor, integrate over internal momentum and sum over indices. Moreover, include a \(-1\) for every closed loop.

Using these rules we can compute the two point correlation function, which is given by:

\[
S_{a,b}^{\alpha,\beta}(p) = \frac{1}{Z[0, 0]} \frac{\delta}{\delta\bar{\eta}_{\alpha,a}(p) \delta\eta_{\beta,b}(-p)} \bigg|_{\eta=\bar{\eta}=0}
\]

(11)

we will compute the two point correlation function up to second order in \(g\) in the next subsection.

4. (a) Fermion two-point correlation function.

• \(O(g^0)\):

\[
\alpha, a \quad \overrightarrow{p} \quad \beta, b = iS_{a,b}^{(F)\alpha,\beta}(p)
\]

• \(O(g^1)\): Here we have the following two diagrams.

\[
\begin{align*}
\text{(12)} \\
\text{(13)} \\
\text{(14)}
\end{align*}
\]

Notice that the first diagram above gives zero contribution. This is so, since there is a trace over spinor indices and \(\text{tr}(\gamma^\mu) = 0\). The contribution from the second diagram is given by:

\[
2 \times iS_{a,c}^{(F)\alpha,\gamma}(p)(-i\lambda\gamma^\delta) \left( \int \frac{d^2q}{(2\pi)^2} iS_{c,d}^{(F)\gamma,\delta}(q) \right) S_{d,b}^{(F)\delta,\beta}(p) = 2 \times iS_{a,a}^{(F)\alpha,\gamma}(p)(-i\lambda\gamma^\delta) \left( \int \frac{d^2q}{(2\pi)^2} iS_{a,b}^{(F)\gamma,\delta}(q) \right) S_{b,b}^{(F)\delta,\beta}(p)
\]

\[
= -g \left( \frac{p}{p^2 + i\epsilon} \right)_{\alpha,\beta}^2 \left( \frac{q}{q^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{a,b}
\]

(14)

where there is no “sum” over repeated indices and we have used \(\lambda_{\alpha,\beta} = \frac{g}{2} \lambda_{\alpha,\beta}\).
• $O(g^2)$: Here we have contributions from the following three diagrams.

(i)

\[
\frac{2 \times 2}{2!} \left( \frac{-ig}{2} \right)^2 i^5 \left[ \left( \frac{\psi}{p^2 + i\epsilon} \right)_{\alpha,\beta} \right]^3 \int \frac{d^2 q_1}{(2\pi)^2} \left( \frac{\phi_1}{q_1^2 + i\epsilon} \right)_{\beta,\alpha} \int \frac{d^2 q_2}{(2\pi)^2} \left( \frac{\phi_2}{q_2^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{a,b} \]

\[= 2 \times 2 \left( \frac{-ig}{2} \right)^2 i^5 \left[ \left( \frac{\psi}{p^2 + i\epsilon} \right)_{\alpha,\beta} \right]^3 \int \frac{d^2 q_1}{(2\pi)^2} \left( \frac{\phi_1}{q_1^2 + i\epsilon} \right)_{\beta,\alpha} \int \frac{d^2 q_2}{(2\pi)^2} \left( \frac{\phi_2}{q_2^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{a,b} \]

(ii)

\[= (-1) \frac{2 \times 2 \times 2}{2!} \left( \frac{-ig}{2} \right)^2 i^5 \left[ \left( \frac{\psi}{p^2 + i\epsilon} \right)_{\alpha,\beta} \right]^2 \int \frac{d^2 q_1}{(2\pi)^2} \int \frac{d^2 q_2}{(2\pi)^2} \left( \frac{\phi_1}{q_1^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{c,d} \left( \frac{\phi_2}{q_2^2 + i\epsilon} \right)_{\alpha,\beta} \delta_{d,c} \left( \frac{\psi'}{p^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{a,b} \]

\[= N i^5 g^2 \left[ \left( \frac{\psi}{p^2 + i\epsilon} \right)_{\alpha,\beta} \right]^2 \int \frac{d^2 q_1}{(2\pi)^2} \int \frac{d^2 q_2}{(2\pi)^2} \left( \frac{\phi_1}{q_1^2 + i\epsilon} \right)_{\beta,\alpha} \left( \frac{\phi_2}{q_2^2 + i\epsilon} \right)_{\alpha,\beta} \left( \frac{\psi'}{p^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{a,b} \]

where $p' = p + q_1 - q_2$ and $N$ comes from $\delta_{c,d} \delta_{d,c} = \delta_{c,c} = N$

(iii)

\[= 2 \times 2 \left( \frac{-ig}{2} \right)^2 i^5 \left[ \left( \frac{\psi}{p^2 + i\epsilon} \right)_{\alpha,\beta} \right]^2 \int \frac{d^2 q_1}{(2\pi)^2} \left( \frac{\phi_1}{q_1^2 + i\epsilon} \right)_{\beta,\alpha} \int \frac{d^2 q_2}{(2\pi)^2} \left( \frac{\phi_2}{q_2^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{a,b} \]

\[= 2 \times 2 \left( \frac{-ig}{2} \right)^2 i^5 \left[ \left( \frac{\psi}{p^2 + i\epsilon} \right)_{\alpha,\beta} \right]^2 \int \frac{d^2 q_1}{(2\pi)^2} \left( \frac{\phi_1}{q_1^2 + i\epsilon} \right)_{\beta,\alpha} \int \frac{d^2 q_2}{(2\pi)^2} \left( \frac{\phi_2}{q_2^2 + i\epsilon} \right)_{\beta,\alpha} \delta_{a,b} \]
(b) For the effective coupling $g$ we need to consider the four-point vertex function.

- $O(g^1)$: Is simply given by the tree level vertex $= 2 \times \frac{-ig}{2} = -ig$.
- $O(g^2)$: At this order we need to consider the one loop diagrams. There are two such diagrams that contribute to the four-point vertex function.

\[
\frac{2 \times 2 \times 2}{2!} \left( \frac{-ig}{2} \right)^2 i^2 \int \frac{d^2 q}{(2\pi)^2} \left( \frac{q}{q^2 + i\epsilon} \right)_{\beta, \alpha} \left( \frac{p'}{p'^2 + i\epsilon} \right)_{\alpha, \beta} \delta_{\gamma, \alpha} \delta_{\delta, \beta} \delta_{\delta, \alpha} \delta_{\alpha, c} \delta_{\beta, d} \tag{20}
\]

\[
= -N \frac{2 \times 2 \times 2}{2!} \left( \frac{-ig}{2} \right)^2 i^2 \int \frac{d^2 q}{(2\pi)^2} \left( \frac{q}{q^2 + i\epsilon} \right)_{\beta, \alpha} \left( \frac{p'}{p'^2 + i\epsilon} \right)_{\alpha, \beta} \delta_{\gamma, \alpha} \delta_{\delta, \beta} \delta_{\delta, \alpha} \delta_{\alpha, c} \delta_{\beta, d} \tag{21}
\]

Now that we have all the contributions to the four-point vertex function we can compute the new effective coupling constant up to $O(g^2)$. This is simply given by evaluating the four-point vertex function at zero external momentum.

\[-ig_{\text{new}} = \Gamma^{(4)}(0, 0, 0, 0) \tag{22}\]

5. (a) We already know from last semester how the Dirac fields transform:

\[
\psi_a \rightarrow e^{i\theta_{\gamma_5}} \psi_a \quad \bar{\psi}_a \rightarrow \bar{\psi}_a e^{i\theta_{\gamma_5}} \tag{23}
\]

Let us look first at the free part of the Lagrangian.

\[
i\bar{\psi}_a \gamma_{\mu} \partial_{\mu} \psi_a \rightarrow \bar{\psi}_a e^{i\theta_{\gamma_5}} \gamma_{\mu} \partial_{\mu} e^{i\theta_{\gamma_5}} \psi_a = \bar{\psi}_a \gamma_{\mu} e^{-i\theta_{\gamma_5}} \partial_{\mu} e^{i\theta_{\gamma_5}} \psi_a = i\bar{\psi}_a \gamma_{\mu} \partial_{\mu} \psi_a \tag{24}
\]
where we have used that \( \{\gamma_\mu, \gamma_5\} = 0 \). Let us now see the interaction part of the Lagrangian. Let us first notice that we can rewrite the interaction as:

\[
L_{\text{int}} = -\frac{g}{4} \left[ (\bar{\psi}_a \psi_a)^2 - (\bar{\psi}_a \gamma_5 \psi_a)^2 \right] = -\frac{g}{4} [\Delta_0^2 + \Delta_5^2]
\]  

(25)

where

\[
\Delta_0 = \bar{\psi}_a \psi_a \quad \text{and} \quad \Delta_5 = i \bar{\psi}_a \gamma_5 \psi_a
\]  

(26)

Let us investigate how \( \Delta_0 \) and \( \Delta_5 \) transform under a chiral transformation.

\[
\Delta_0 = \bar{\psi}_a \psi_a \rightarrow \Delta_0' = \bar{\psi}_a e^{2i\theta\gamma_5} \psi_a = \cos(2\theta) \bar{\psi}_a \psi_a + i \sin(2\theta) \bar{\psi}_a \gamma_5 \psi_a
\]

\[
\Delta_0' = \cos(2\theta) \Delta_0 + \sin(2\theta) \Delta_5
\]

(27)

\[
\Delta_5 = i \bar{\psi}_a \gamma_5 \psi_a \rightarrow \Delta_5' = i \bar{\psi}_a \gamma_5 \psi_a = \cos(2\theta) \bar{\psi}_a \gamma_5 \psi_a - \sin(2\theta) \bar{\psi}_a \psi_a
\]

\[
\Delta_5' = \cos(2\theta) \Delta_5 - \sin(2\theta) \Delta_0
\]

(28)

We can see then that \( \Delta_0^2 + \Delta_5^2 \) is invariant under a chiral transformation. Therefore, the Lagrangian is invariant under a chiral transformation. (b) As we discussed above:

\[
\begin{pmatrix}
\Delta_0' \\
\Delta_5'
\end{pmatrix} =
\begin{pmatrix}
\cos(2\theta) & \sin(2\theta) \\
-sin(2\theta) & \cos(2\theta)
\end{pmatrix}
\begin{pmatrix}
\Delta_0 \\
\Delta_5
\end{pmatrix}
\]

(29)

(c) Looking how the \( \psi_{a,a} \) transforms, we can see that \( \psi_{R,a}^\dagger \psi_{R,a} \) and \( \psi_{L,a}^\dagger \psi_{L,a} \) are invariant under the chiral transformation. This means that symmetry conserves the number of left and right movers independently of each other.

6.

\[
L_{\text{chiral}} = H_0(x) \Delta_0(x) + H_5(x) \Delta_5(x)
\]

(30)

If we want to maintain \( L_{\text{chiral}} \) under the chiral transformation, we need to transform \( H_0 \) and \( H_5 \) accordingly. Using the transformation given in eq. (29), we have that \( H_0 \) and \( H_5 \) has to transform as:

\[
\begin{pmatrix}
H_0' \\
H_5'
\end{pmatrix} =
\begin{pmatrix}
\cos(2\theta) & \sin(2\theta) \\
-sin(2\theta) & \cos(2\theta)
\end{pmatrix}
\begin{pmatrix}
H_0 \\
H_5
\end{pmatrix}
\]

(31)

7. We can read off the Effective Potential directly from the lecture notes, for instance eq. (1.124) (or Amit chapter 5.).

\[
U[\Delta_0, \Delta_5] = \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{i_1 \cdots i_N} \Gamma^{(N)}_{i_1 \cdots i_N} (0, \ldots, 0) \tilde{\Delta}_{i_1} \cdots \tilde{\Delta}_{i_N}
\]

(32)

where \( i_j = 0, 5 \). This can be rewriting as:

\[
U[\bar{\Delta}_0, \bar{\Delta}_5] = \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{M=2}^{\infty} \frac{1}{M!} \Gamma^{(N,M)} (0, \ldots, 0) \bar{\Delta}_0^N \bar{\Delta}_5^M
\]

(33)
8. At the tree level \( U[\Delta_0, \Delta_5] = \frac{g}{2} (\Delta_0^2 + \Delta_5^2) \) Now we move on to the one-loop corrections. First of all, notice that we have two types of vertices: \( \Delta_0 : -ig/4 \) and \( \Delta_5 : -ig\gamma_5/4 \).

The Feynman diagram for the effective potential is then the following:

\[
\begin{array}{c}
\text{q} \\
\text{1} \\
\text{2n} \\
\text{2n + 2m}
\end{array}
\]

where to each vertex there is attached a \( \Delta_0 \) or \( \Delta_5 \). The diagram corresponds to \( 2n \) vertices for \( \Delta_0 \) and \( 2m \) vertices for \( \Delta_5 \) (odd number of vertices of \( \Delta_0 \) and/or \( \Delta_5 \) have zero contribution).

Proceeding as in the lecture notes we have:

\[
\Gamma^{(n,m)}(0,\ldots,0) = (-1)^{2n+2m} N \left( \frac{-ig}{4} \right)^{2n+2m} \frac{S_{n,m}}{(2n+2m)!} \int \frac{d^2q}{(2\pi)^2} \text{tr} \left[ \left( \frac{i\gamma_\mu\gamma_\nu}{q^2 + i\epsilon} \right) \left( \frac{i\gamma_\alpha\gamma_\beta}{q^2 + i\epsilon} \right)^{2m} \right] \times (2n + 2m - 1)!
\]

where \( S_{n,m} = (2n+2m)! \) being the number of ways of reordering the vertices and \( (2n+2m-1)! \) is the number of ways of attaching \( p \)'s to the vertices. We have then:

\[
\Gamma^{(n,m)}(0) = -N \left( \frac{g}{2} \right)^{2n+2m} (2n + 2m - 1)! \int \frac{d^2q}{(2\pi)^2} \text{tr} \left[ \left( \frac{i\gamma_\mu\gamma_\nu}{q^2 + i\epsilon} \right)^{2n} \left( \frac{i\gamma_\alpha\gamma_\beta}{q^2 + i\epsilon} \right)^{2m} \right]
\]

Using the properties of the gamma matrices \( \text{tr}(\gamma_\mu\gamma_\nu) = 2g_{\mu\nu} \) and \( \text{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda) = 2\epsilon_{\mu\nu\lambda} \) we get:

\[
\text{tr} \left[ \left( \frac{i\gamma_\mu\gamma_\nu}{q^2 + i\epsilon} \right)^{2n} \left( \frac{i\gamma_\alpha\gamma_\beta}{q^2 + i\epsilon} \right)^{2m} \right] = 2 \frac{1}{(q^2 + i\epsilon)^{n+m}}
\]

So at one loop we have:

\[
U^{(1)}[\Delta_0, \Delta_5] = -2N \sum_{n,m=1}^{\infty} \frac{1}{(2n)! (2m)!} \frac{1}{2^{n+m}} \frac{(2n + 2m)!}{2m + 2n} \int \frac{d^2q}{(2\pi)^2} \left( \frac{g^2\Delta_0^2 + g^2\Delta_5^2}{4(q^2 + i\epsilon)} \right)^n \left( \frac{g^2\Delta_5^2}{4(q^2 + i\epsilon)} \right)^m
\]

\[
= -N \sum_{j=1}^{\infty} \frac{1}{j} \int \frac{d^2q}{(2\pi)^2} \left( \frac{g^2(\Delta_0^2 + \Delta_5^2)}{4(q^2 + i\epsilon)} \right)^j
\]

\[
= -N \sum_{j=1}^{\infty} \frac{1}{j} \int \frac{d^2q}{(2\pi)^2} \left( \frac{g^2(\Delta_0^2 + \Delta_5^2)}{4(q^2 + i\epsilon)} \right)^j
\]

\[
= -N \int \frac{d^2q}{(2\pi)^2} \ln \left( 1 - \frac{g^2(\Delta_0^2 + \Delta_5^2)}{4(q^2 + i\epsilon)} \right)
\]
So we finally up to one loop:

\[ U[\bar{\Delta}_0, \bar{\Delta}_5] = \frac{g}{2} (\bar{\Delta}_0^2 + \bar{\Delta}_5^2) + N \int \frac{d^2 q}{(2\pi)^2} \ln \left( 1 - \frac{g^2(\bar{\Delta}_0^2 + \bar{\Delta}_5^2)}{4(q^2 + i\epsilon)} \right) \] (38)

9. Using:

\[
\begin{pmatrix}
H'_0 \\
H'_5
\end{pmatrix} = \begin{pmatrix}
\cos(2\theta) & \sin(2\theta) \\
-\sin(2\theta) & \cos(2\theta)
\end{pmatrix}
\begin{pmatrix}
H_0 \\
H_5
\end{pmatrix}
\] (39)

and taking the limit of infinitesimal transformation \(2\theta = \epsilon\) we have:

\[ \bar{H}' = T\bar{H} \] (40)

where

\[ T = I + \epsilon \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \] (41)

The transformations are then: \(\delta H_0 = \epsilon H_5\) and \(\delta H_5 = -\epsilon H_0\). Using that:

\[ \bar{\Delta}_i = \frac{\delta \mathcal{F}[H]}{\delta H_i}, \quad \Gamma[\bar{\Delta}] = \bar{\Delta}_0 H_0 + \bar{\Delta}_5 H_5 - \mathcal{F}[H] \quad \text{and} \quad H_i = \frac{\delta \Gamma[\bar{\Delta}]}{\delta \bar{\Delta}_i} \] (42)

we get:

\[ \int d^2 x \left[ \frac{\delta \mathcal{F}[H]}{\delta H_0} \delta H_0 + \frac{\delta \mathcal{F}[H]}{\delta H_5} \delta H_5 \right] = 0 \]
\[ \int d^2 x \left[ \bar{\Delta}_0(x) H_5(x) - \bar{\Delta}_5(x) H_0(x) \right] = 0 \]
\[ \int d^2 x \left[ \Delta_0(x) \frac{\delta \Gamma[\bar{\Delta}]}{\delta \bar{\Delta}_5(x)} - \bar{\Delta}_5(x) \frac{\delta \Gamma[\bar{\Delta}]}{\delta \bar{\Delta}_0(x)} \right] = 0 \] (43)

10. Let us start from the Ward identity:

\[ 0 = \int d^2 x \frac{\delta}{\delta \Delta_5(y)} \left[ \Delta_0(x) \frac{\delta \Gamma[\bar{\Delta}]}{\delta \Delta_5(x)} - \Delta_5(x) \frac{\delta \Gamma[\bar{\Delta}]}{\delta \bar{\Delta}_0(x)} \right] \]
\[ 0 = \int d^2 x \frac{\delta}{\delta \Delta_0(z)} \left[ \Delta_0(x) \frac{\delta^2 \Gamma[\bar{\Delta}]}{\delta \Delta_5(y) \delta \Delta_5(x)} - \Delta_5(x) \frac{\delta^2 \Gamma[\bar{\Delta}]}{\delta \Delta_5(y) \delta \bar{\Delta}_0(x)} - \frac{\delta \Gamma[\bar{\Delta}]}{\delta \bar{\Delta}_0(x)} \delta^2(x - y) \right] \] (44)
\[ 0 = \int d^2 x \left[ \Delta_0(x) \Gamma^{(3)}_{055}(x, y, z) + \Gamma^{(2)}_{55}(x, y) \delta^2(x - z) - \Gamma^{(3)}_{005}(x, y, z) \Delta_5(x) - \Gamma^{(2)}_{00}(x, z) \delta^2(x - y) \right] \] (45)

Taking \(\Delta_5(x)\) and \(\Delta_0(x) = \bar{\Delta}_0\) the previous expression simplifies to:

\[ \Gamma^{(2)}_{00}(y, z) - \Gamma^{(2)}_{55}(z, y) = \bar{\Delta}_0 \int d^2 x \Gamma^{(3)}_{055}(x, y, z) \] (46)
We have from eq. (44) that (taking $\Delta_5(x)$ and $\Delta_0(x) = \Delta_0$):

$$\Delta_0 \int d^2 x \Gamma_{55}^{(2)}(x, y) = \frac{\delta \Gamma[\Delta]}{\delta \Delta_0(y)} = H_0$$

(47)

So we have that

$$\lim_{p \to 0} \Gamma_{55}^{(2)}(p) = \frac{H_0}{\Delta_0} \Rightarrow \lim_{p \to 0} \Gamma_{55}^{(2)}(p) \xrightarrow{H_0 \to \Delta_0 \neq 0} 0$$

(48)

11. From eq. (48) we have that when $H_0 \to 0$ and $\Delta_0 \neq 0$, $\lim_{p \to 0} \Gamma_{55}^{(2)}(p) \to 0$. This means that $G_{55}^{(2)}(p)$ has a pole at zero momentum in the spontaneously broken phase, so there is a massless excitation, GOLDSTONE BOSON!
2 Perturbative $SU(2)$ Yang-Mills Gauge Theory

1. Let us start by rewriting the Lagrangian in a more suggestive way:

$$
\mathcal{L} = -\frac{1}{2} \text{tr} \left( F^a_{\mu\nu} t^a F^b_{\mu\nu} t^b \right) + \frac{\lambda}{2} \text{tr} \left( \partial_{\mu} A^\mu_a \partial^\nu A^\nu_a - \tilde{\eta}_a \partial_{\mu} D^\mu_{ab} \eta_b \right) - \frac{1}{4} \left( \partial_{\mu} A^\nu_a - \partial^\nu A^\mu_a + g f_{abc} A^b_{\mu} A^c_{\nu} \right) \left( \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + g f_{ade} A^d_{\mu} A^e_{\nu} \right)
$$

Using that $f_{abc} = \epsilon_{abc}$, $\epsilon_{abc} \epsilon_{ade} = \delta_{bd} \delta_{ce} - \delta_{be} \delta_{cd}$ (notice that we make no difference between upper and lower “latin” indices $v_a = v^a$) and integrating by parts after some straightforward algebra we get:

$$
\mathcal{L} = \frac{1}{2} A^a_{\mu} \left[ g^{\mu\nu} \partial^2 - \left( 1 + \frac{\lambda}{2} \right) \partial^\mu \partial^\nu \right] \delta_{ab} A^b_{\mu} - \tilde{\eta}_a \delta_{\mu}^2 \delta_{ab} \eta_b - \frac{g}{2} f_{abc} A^b_{\mu} A^c_{\nu} \left( \partial^\mu A^\nu_a - \partial^\nu A^\mu_a \right) - \frac{g^2}{4} f_{abc} f_{ade} A^b_{\mu} A^c_{\nu} A^d_{\mu} A^e_{\nu} - g \tilde{\eta}_a \partial_{\mu} f_{abc} A^\mu_a \eta_b
$$

Now it is straightforward to read the propagators from $\mathcal{L}_0$:

- **Gluon propagator**: the gluon propagator satisfies the following equation in momentum space:

$$
\begin{align*}
\mathcal{L}_0 &= \frac{1}{2} A^a_{\mu} \left[ g^{\mu\nu} \partial^2 - \left( 1 + \frac{\lambda}{2} \right) \partial^\mu \partial^\nu \right] \delta_{ab} A^b_{\mu} - \tilde{\eta}_a \delta_{\mu}^2 \delta_{ab} \eta_b \\
\mathcal{L}_\text{int} &= \frac{1}{2} f_{abc} A^b_{\mu} A^c_{\nu} \left( \partial^\mu A^\nu_a - \partial^\nu A^\mu_a \right) - \frac{g^2}{4} f_{abc} f_{ade} A^b_{\mu} A^c_{\nu} A^d_{\mu} A^e_{\nu} - g \tilde{\eta}_a \partial_{\mu} f_{abc} A^\mu_a \eta_b
\end{align*}
$$

Inverting the previous equation we get the gluon propagator in momentum space:

$$
i G^{ab}_{\mu\nu}(p) = -\frac{i}{p^2} \left[ g_{\mu\nu} - \left( 1 + \frac{2}{\lambda} \right) \frac{p_{\mu} p_{\nu}}{p^2} \right] \delta_{ab}
$$

- **Ghost propagator**:

$$
i S^{ab}_{\mu\nu}(p) = \frac{i}{p^2} \delta_{ab}
$$

2. Let us recall $\mathcal{L}_\text{int}$ again:

$$
\mathcal{L}_\text{int} = -\frac{g}{2} f_{abc} \left[ \left( g^{\mu\lambda} \partial^\mu - g^{\mu\lambda} \partial^\nu \right) A^a_{\lambda} \right] A^b_{\mu} A^c_{\nu} - \frac{g^2}{4} f_{abc} f_{ade} g^{\mu\lambda} g^{\nu\rho} A^d_{\lambda} A^e_{\rho} A^b_{\mu} A^c_{\nu} - g f_{abc} \tilde{\eta}_a \partial_{\mu} A^\mu_a \eta_b
$$

From the previous expression we can read off the different types of vertices. We draw them in momentum space for simplicity, although the expression in real space can be read immediately from there:
We have three types of vertices, represented by $V_1$, $V_2$ and $V_3$. 

- Two types of propagators (two different types of lines):
  \[
  a \quad \overset{p}{\longrightarrow} \quad b = iS_{ab}(p)
  \]
  \[
  a, \mu \quad \overset{p}{\longrightarrow} \quad b, \nu = iG_{\mu\nu}^{ab}(p)
  \]

Now that we have defined the different types of vertices, the Feynman rules are the usual one, with the following distinction:

- We have three types of vertices, represented by $V_1$, $V_2$ and $V_3$. 

The Feynman rules are the usual one, with the following distinction:

\[
V_1 = g f_{abc} [(r_\mu - q_\mu)g_{\nu\rho} + (p_\nu - q_\nu)g_{\mu\rho} + (q_\rho - p_\rho)g_{\mu\nu}] \delta(p + q + r) \tag{55}
\]

\[
V_2 = -ig^2 [f_{abc}f_{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + f_{ace}f_{bed}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})
+ f_{ade}f_{bce}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})] \delta(p + q + r + s) \tag{56}
\]

\[
V_3 = g f_{abc} p_\mu \delta(k + p - q) \tag{57}
\]
• The ghost fields only appear as internal lines. In addition, since they are fermionic fields, we have to include a factor of $(-1)$ for each closed loop.

3. In order to simplify the equations, I will use the following notation (see for instance Amit.)

• $1 = a, \mu, x$, $2 = b, \nu, y$, etc. For example $\langle 0| T A^a_\mu(x) A^b_\nu(y) |0 \rangle = \langle 0| T A(1) A(2) |0 \rangle$.

• $\int d3d4 = \sum_{\text{int. indices}} \int dz \int dz'$

Let us also notice that in the Feynman gauge $\lambda = -2$, $iG_{\mu\nu}^{ab}(p) = -\frac{i}{p^2 + i\epsilon} g_{\mu\nu} \delta_{ab}$. With the above definitions, we will compute the one-loop correction for the gauge propagator and the three and four gauge vertices. I will not include symmetry and topological factors and I will only list the diagrams that give nonzero contributions.

(a) The gauge field propagator:

\[
\begin{align*}
\int d3d4iG(1-3)[iG(3-4)]^2iG(4-2)V_1(3)V_1(4) &= \int d3d4iG(1-3)[iS(3-4)]^2iG(4-2)V_3(3)V_3(4) \\
\int d3d4iG(1-3)iG(3-2)V_1(3) &= \int d3d4iG(1-3)iG(3-3)iG(3-2)V_1(3)
\end{align*}
\]

(b) Three and four gauge field vertices:

(i) $\Gamma^{(3)}(1, 2, 3)$:

\[
\begin{align*}
\int d4d5[iG(4-5)]^2V_1(4)V_1(5) &= \int d4d5d6iG(4-5)iG(5-6)iG(6-4)V_1(4)V_1(5)V_1(6)
\end{align*}
\]
(ii) $\Gamma^{(4)}(1, 2, 3, 4)$:

\[
\int d5d6d7d8iG(5 - 7)iG(7 - 8)iG(8 - 6)V_1(5) V_1(6) V_1(7) V_1(8)
\]

\[
\int d5d6d7iS(5 - 7)iS(7 - 8)iS(8 - 6)V_3(5) V_3(6) V_3(7) V_3(8)
\]

\[
\int d5d6d7iG(5 - 7)iG(7 - 6)V_1(5) V_1(6) V_2(7)
\]
### 3 Renormalization of the $O(N)$ scalar $\phi^4$ theory

The Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2} (\nabla_\mu \phi_a)^2 + \frac{m_0^2}{2} \phi_a^2 + \frac{\lambda}{4!} (\phi_a^2)^2$$  \hspace{1cm} (58)

Using that:

$$S_{abcd} \phi_a \phi_b \phi_c \phi_d = \frac{1}{3} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \phi_a \phi_b \phi_c \phi_d$$

$$= \frac{1}{3} \left[ \sum_{a,d} \phi_a^2 \phi_d^2 + \sum_{a,d} \phi_a^2 \phi_d^2 + \sum_{a,b} \phi_a^2 \phi_b^2 \right]$$

$$= \left( \sum_a \phi_a^2 \right)^2$$  \hspace{1cm} (59)

We have that:

$$\mathcal{L}_{\text{int}} = \frac{\lambda}{4!} S_{abcd} \phi_a \phi_b \phi_c \phi_d$$  \hspace{1cm} (60)

1. $\Gamma^{(2)}_{ab}(p)$
   
   (i)

   \[ a \xrightarrow{p} b = (p^2 - m_0^2) \delta_{ab} \]

   (ii)

   \[ a \xrightarrow{p} c \xrightarrow{q} b = \frac{\lambda}{4!} S_{abcc} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m_0^2} \times M \]

   where $M = 4 \times 3$ is the multiplicity factor and $S_{abcc} = \frac{1}{3} (\delta_{ab} \delta_{cc} + \delta_{ac} \delta_{bc} + \delta_{ad} \delta_{bc}) = \frac{1}{3} (N + 2) \delta_{ab}$. Combining these two results we have that up to one loop:

$$\Gamma^{(2)}_{ab}(p) = \left[ p^2 + m_0^2 + \frac{(N + 2)\lambda}{6} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m_0^2} \right] \delta_{ab}$$  \hspace{1cm} (61)
\( \Gamma^{(4)}_{abcd}(p_1, p_2, p_3, p_4) \)

(i) Tree level:

\[
= \frac{\lambda}{4!} S_{abcd} \times M = S_{abcd} \lambda
\]

(ii) One loop:

\[
= -M \left( \frac{\lambda}{4!} \right) \frac{1}{2!} S_{abef} S_{efcd} \\
\int \frac{d^d q}{(2\pi)^d} \left\{ \frac{1}{q^2 + m_0^2 (p_1 + p_2 - q)^2 + m_0^2} + 2\text{perm.} \right\}
\]

where \( M = 4^2 \times 3^2 \times 2 \times 2 \) is the multiplicity factor and \( S_{abef} S_{efcd} = \frac{2}{3} S_{abcd} + \frac{1}{9} (N+2) \delta_{ab} \delta_{cd} \).

Combining these two results we get:

\[
\Gamma^{(4)}_{abcd}(p_1, p_2, p_3, p_4) = \lambda S_{abcd} \frac{\lambda^2}{2} \left[ \frac{2}{3} S_{abcd} + \frac{1}{9} (N+2) \delta_{ab} \delta_{cd} \right] \\
\times \int \frac{d^d q}{(2\pi)^d} \left\{ \frac{1}{q^2 + m_0^2 (p_1 + p_2 - q)^2 + m_0^2} + (p_1 \leftrightarrow p_3, p_1 \leftrightarrow p_4) \right\}
\]

(62)

2. Renormalization conditions:

\[
\mu^2 = \Gamma^{(2)}(\mu = 0)
\]

(63)

\[
g = \Gamma^{(4)}(\mu = 0)
\]

(64)

3. Applying the renormalization conditions we have for the bare mass:

\[
\mu^2 = m_0^2 + \frac{(N+2)\lambda}{6} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m_0^2}
\]

\[
\Rightarrow m_0^2 = \mu^2 - \frac{(N+2)\lambda}{6} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m_0^2} = \mu^2 - \frac{(N+2)g}{6} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu^2} + O(2\text{-loops})
\]

Introducing a cut-off we get:

\[
m_0^2 = \mu^2 - \frac{(N+2)g}{6} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu^2} = \mu^2 - \frac{(N+2)g}{6} D_1(\mu^2, \Lambda)
\]

(65)
Now for the coupling constant we have:

\[ g = \lambda - \frac{\lambda^2 N + 8}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + m_0^2)^2} \quad (66) \]

Introducing a cut-off we get:

\[ \lambda = g + \frac{g^2}{6} (N + 8) \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + \mu^2)^2} = g + \frac{g^2}{6} (N + 8) I(0, \mu^2, \Lambda) \quad (67) \]

4. The two vertex function is given by:

\[
\Gamma^{(2)}_{ab}(p) = \left[ p^2 + m_0^2 + \frac{(N + 2)\lambda}{6} D_1(m_0^2, \Lambda) \right] \delta_{ab}
\]

\[ \Gamma^{(2)}_{ab}(p) = \left[ p^2 + \mu^2 - \frac{(N + 2)\lambda}{6} D_1(\mu^2, \Lambda) + \frac{(N + 2)\lambda}{6} D_1(\mu^2, \Lambda) + O(2\text{-loops}) \right] \delta_{ab} \]

\Rightarrow \Gamma^{(2)}_{ab}(p) = [p^2 + \mu^2] \delta_{ab} \quad (68) \]

For the four vertex function we have:

\[
\Gamma^{(4)}_{abcd}(p_1, p_2, p_3, p_4) = \lambda S_{abcd} - \frac{\lambda^2}{2} \left[ \left( \frac{2}{3} S_{abcd} + \frac{1}{9} (N + 2) \delta_{ab} \delta_{cd} \right) I(p_1 + p_2, m_0^2, \Lambda) \right. \\
+ \left( \frac{2}{3} S_{abcd} + \frac{1}{9} (N + 2) \delta_{ac} \delta_{bd} \right) I(p_1 + p_3, m_0^2, \Lambda) \\
+ \left( \frac{2}{3} S_{abcd} + \frac{1}{9} (N + 2) \delta_{ad} \delta_{bc} \right) I(p_1 + p_4, m_0^2, \Lambda) \right]
\]

which can be written in terms of the renormalized mass and coupling constant as:

\[
\Gamma^{(4)}_{abcd}(p_1, p_2, p_3, p_4) = \left( g + \frac{g^2}{6} (N + 8) I(0, \mu^2, \Lambda) \right) S_{abcd}
\]

\[ - \frac{g^2}{2} \left[ \left( \frac{2}{3} S_{abcd} + \frac{1}{9} (N + 2) \delta_{ab} \delta_{cd} \right) I(p_1 + p_2, \mu^2, \Lambda) \\
+ \left( \frac{2}{3} S_{abcd} + \frac{1}{9} (N + 2) \delta_{ac} \delta_{bd} \right) I(p_1 + p_3, \mu^2, \Lambda) \\
+ \left( \frac{2}{3} S_{abcd} + \frac{1}{9} (N + 2) \delta_{ad} \delta_{bc} \right) I(p_1 + p_4, \mu^2, \Lambda) \right] + O(2\text{-loops}) \quad (70) \]

We write the previous expression in a more suggestive form as:

\[
\Gamma^{(4)}_{abcd}(p_1, p_2, p_3, p_4) = g S_{abcd} + \frac{g^2}{2} \left[ \frac{2S_{abcd}}{3} + \frac{N + 2}{9} \delta_{ab} \delta_{cd} \right] (I(0, \mu^2, \Lambda) - I(p_1 + p_2, \mu^2, \Lambda)) \\
+ \frac{g^2}{2} \left[ \frac{2S_{abcd}}{3} + \frac{N + 2}{9} \delta_{ac} \delta_{bd} \right] (I(0, \mu^2, \Lambda) - I(p_1 + p_3, \mu^2, \Lambda)) \\
+ \frac{g^2}{2} \left[ \frac{2S_{abcd}}{3} + \frac{N + 2}{9} \delta_{ad} \delta_{bc} \right] (I(0, \mu^2, \Lambda) - I(p_1 + p_4, \mu^2, \Lambda)) \quad (71) \]
Notice that the terms \( I(0, \mu^2, \Lambda) - I(\kappa, \mu^2, \Lambda) \) are finite for \( d < 6 \), so in \( d = 4 \) we have that both \( \Gamma_{abcd}(p) \) and \( \Gamma_{abcd}(p_1, p_2, p_3, p_4) \) are finite to one loop order in perturbation theory.