1 Renormalization Group for the Ashkin-Teller Model

The Hamiltonian for the Ashkin-Teller model is given by:

\[ H = -K_2 \sum_{\langle \vec{r}, \vec{r}' \rangle} (\sigma(\vec{r})\sigma(\vec{r}') + \tau(\vec{r})\tau(\vec{r}')) - K_4 \sum_{\langle \vec{r}, \vec{r}' \rangle} \sigma(\vec{r})\sigma(\vec{r}')\tau(\vec{r})\tau(\vec{r}') \]  

(1)

where \( \langle \vec{r}, \vec{r}' \rangle \) denotes nearest neighboring sites and \( K_2 > 0 \) and \( K_4 > 0 \) are two coupling constants.

1. (a) For \( K_2 \to \infty \) the system tends to minimize the first term in Eq. (1). This is achieved when \( \sigma(\vec{r})\sigma(\vec{r}') = \tau(\vec{r})\tau(\vec{r}') = 1 \) which translates to \( \sigma(\vec{r}) = \sigma(\vec{r}') \) and \( \tau(\vec{r}) = \tau(\vec{r}') \) \( \forall \vec{r} \). The ground state is then four-fold degenerate. The four states are given by:

<table>
<thead>
<tr>
<th>( \sigma(\vec{r}) )</th>
<th>( \tau(\vec{r}) )</th>
</tr>
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<tbody>
<tr>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>+1</td>
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<td>-1</td>
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(b) For \( K_4 \to \infty \) the system tends to minimize the second term in Eq. (1). In this case \( \sigma(\vec{r})\sigma(\vec{r}')\tau(\vec{r})\tau(\vec{r}') = 1 \) which is the same as \( \sigma(\vec{r})\tau(\vec{r}) = \sigma(\vec{r}')\tau(\vec{r}') \implies \sigma(\vec{r})\tau(\vec{r}) = \pm 1 \) \( \forall \vec{r} \). In this case the Hamiltonian (1) is given by:

\[ H = -K_2 \sum_{\langle \vec{r}, \vec{r}' \rangle} (\sigma(\vec{r})\sigma(\vec{r}') + \tau(\vec{r})\tau(\vec{r}')) - \text{const.} \]  

(2)

\[ = -K_2 \sum_{\langle \vec{r}, \vec{r}' \rangle} (\sigma(\vec{r})\sigma(\vec{r}') + (\pm \sigma(\vec{r}))(\pm \sigma(\vec{r}'))) - \text{const.} \]  

(3)

\[ = -2K_2 \sum_{\langle \vec{r}, \vec{r}' \rangle} \sigma(\vec{r})\sigma(\vec{r}') - \text{const.} \]  

(4)

So the system is equivalent to a single Ising model with coupling constant \( K = 2K_2 \).
(c) In this case the the first term in Eq. (1) doesn’t play a role. The Hamiltonian is simple given by:

\[ H = -K_4 \sum_{\langle \vec{r}, \vec{r}' \rangle} \sigma(\vec{r}) \sigma(\vec{r}') \tau(\vec{r}) \tau(\vec{r}') \]

(5)

\[ = -K_4 \sum_{\langle \vec{r}, \vec{r}' \rangle} S(\vec{r}) S(\vec{r}') \]

(6)

where we have define the new variable \( S(\vec{r}) = \sigma(\vec{r}) \tau(\vec{r}) \) which also take values ±1. We see that the system is equivalent to a single Ising model with coupling constant \( K_{\text{eff}} = K_4 \).

2. In 1-D, the Hamiltonian for the Ashkin-Teller model is given by:

\[ H = -K_2 \sum_i (\sigma_i \sigma_{i+1} + \tau_i \tau_{i+1}) - K_4 \sum_i \sigma_i \sigma_{i+1} \tau_i \tau_{i+1} \]

(7)

We need to integrate out every other spin (decimation). Using the notation \( \sigma_{2n} = \mu_n \) and \( \tau_{2n} = \eta_n \) Eq. (7) becomes:

\[ H = -K_2 \sum_i (\sigma_{2i+1}(\mu_i + \mu_{i+1}) + \tau_{2i+1}(\eta_i + \eta_{i+1})) - K_4 \sum_i \sigma_{2i+1} \tau_{2i+1}(\mu_i \eta_i + \mu_{i+1} \eta_{i+1}) \]

(8)

Defining \( J_2 = \beta K_2 \) and \( J_4 = \beta K_4 \) we get:

\[ \sum_{\sigma_{2i+1}=\pm 1 \tau_{2i+1}=\pm 1} \exp(-\beta H) = \exp [J_2(\mu_i + \mu_{i+1}) + J_2(\eta_i + \eta_{i+1}) + J_4(\mu_i \eta_i + \mu_{i+1} \eta_{i+1})] \]

\[ + \exp [J_2(\mu_i + \mu_{i+1}) - J_2(\eta_i + \eta_{i+1}) - J_4(\mu_i \eta_i + \mu_{i+1} \eta_{i+1})] \]

\[ + \exp [-J_2(\mu_i + \mu_{i+1}) + J_2(\eta_i + \eta_{i+1}) - J_4(\mu_i \eta_i + \mu_{i+1} \eta_{i+1})] \]

\[ + \exp [-J_2(\mu_i + \mu_{i+1}) - J_2(\eta_i + \eta_{i+1}) + J_4(\mu_i \eta_i + \mu_{i+1} \eta_{i+1})] \]

(9)

After some straightforward algebra the previous expression can be written as:

\[ \sum_{\sigma_{2i+1}=\pm 1 \tau_{2i+1}=\pm 1} \exp(-\beta H) = 4 \cosh^4 J_2 \cosh^2 J_4 \left[ (1 + A^2 B)^2 + (\mu_i \mu_{i+1} + \eta_i \eta_{i+1}) A^2 (1 + B)^2 \right. \]

\[ \left. + \mu_i \eta_i \mu_{i+1} \eta_{i+1} (A^2 + B)^2 \right] \]

(10)

where \( A = \tanh J_2 \) and \( B = \tanh J_4 \). We are looking for an expression for the effective Hamiltonian of the form:

\[ H' = \text{const.} - K_2' \sum_i (\mu_i \mu_{i+1} + \eta_i \eta_{i+1}) - K_4' \sum_i \mu_i \mu_{i+1} \eta_i \eta_{i+1} \]

(11)

The Boltzmann weight can then be writte as:

\[ \exp(-\beta H_i') = \exp [\alpha + J_2' (\mu_i \mu_{i+1} + \eta_i \eta_{i+1}) + J_4' \mu_i \mu_{i+1} \eta_i \eta_{i+1}] \]

(12)
where $\alpha$ is just some constant. The previous expression can be written as:

$$\exp(-\beta H_i) = e^{\alpha} \left[ \cosh^2 J'_2 \cosh J'_4 \left\{ 1 + (\mu_i \mu_{i+1} + \eta_i \eta_{i+1}) \left( A' + A'B' \right) + \mu_i \mu_{i+1} \eta_i \eta_{i+1} (A'^2 + B' + A'^2 B') \right\} \right]$$

Comparing each term in Eq. (10) and Eq. (13) we have:

$$e^{\alpha} \cosh^2 J'_2 \cosh J'_4 (1 + A'^2 B') = 4 \cosh^4 J'_2 \cosh^2 J'_4 (1 + A^2 B)^2 \quad (14)$$

$$e^{\alpha} \cosh^2 J'_2 \cosh J'_4 A' (1 + B') = 4 \cosh^4 J'_2 \cosh^2 J'_4 A' (1 + B)^2 \quad (15)$$

$$e^{\alpha} \cosh^2 J'_2 \cosh J'_4 (A'^2 + B') = 4 \cosh^4 J'_2 \cosh^2 J'_4 (A'^2 + B)^2 \quad (16)$$

Taking (15)/(14) and (16)/(14) we have the following RG transformations:

$$\begin{align*}
\frac{A'(1 + B')}{1 + (A'^2 B')} &= \frac{A^2(1 + B)^2}{(1 + A^2 B)^2} \\
\frac{(A'^2 + B')}{1 + (A'^2 B')} &= \frac{(A^2 + B)^2}{(1 + A^2 B)^2}
\end{align*} \quad (17)$$

We now look for the FP solutions of the system (17) where $A = A' = A^*$. The solutions are:

$$(A^*, B^*) = \{(0, 0); (1, x); (0, 1)\}, \quad 0 \leq x \leq 1 \quad (18)$$

Now we have to study the stability of the FP’s. We write then $A = A^* + \delta A$ and $B = B^* + \delta B$. The system (17) becomes:

$$\frac{(A^* + \delta A')(1 + (B^* + \delta B'))}{(1 + (A^* + \delta A)^2(B^* + \delta B'))} = \frac{(A^* + \delta A)^2(1 + (B^* + \delta B)^2)}{(1 + (A^* + \delta A)^2(B^* + \delta B)^2)} \quad (19)$$

$$\frac{(A^* + \delta A)^2 + (B^* + \delta B')}{(1 + (A^* + \delta A)^2(B^* + \delta B'))} = \frac{(A^* + \delta A)^2 + (B^* + \delta B')}{(1 + (A^* + \delta A)^2(B^* + \delta B'))} \quad (20)$$

- $(A^*, B^*) = (0, 0)$: eq. (20) becomes:

$$\frac{(\delta A'^2 + \delta B')}{(1 + \delta A'^2 \delta B')} = \frac{(\delta A^2 + \delta B)^2}{(1 + \delta A^2 \delta B)^2} \quad (21)$$

So we have that: $\delta A' = \delta A^2$ and $\delta B' = \delta B^2$, so to lowest order in $\delta A$ and $\delta B$ we have:

$$\beta(A) = \frac{\delta A' - \delta A}{\delta A} = -1 < 0 \quad \text{and} \quad \beta(B) = \frac{\delta B' - \delta B}{\delta B} = -1 < 0 \quad (22)$$

so the FP $(A^*, B^*) = (0, 0)$ is stable.

- $(A^*, B^*) = (1, x)$: eq. (20) becomes:

$$\frac{((1 + \delta A')^2 + (x + \delta B'))}{(1 + (1 + \delta A')^2(x + \delta B'))} = \frac{((1 + \delta A')^2 + (x + \delta B')^2}{(1 + (1 + \delta A')^2(x + \delta B')^2} \quad (23)$$

Expanding to lowest order on both sides we get that: $\delta A' = 2\delta A$ and $\delta B' = \delta B$, so:

$$\beta(A) = \frac{\delta A' - \delta A}{\delta A} = +1 > 0 \quad \text{and} \quad \beta(B) = \frac{\delta B' - \delta B}{\delta B} = 0. \quad (24)$$

so the FP $(A^*, B^*) = (1, x)$ is unstable in the $A$-direction.
• \((A^*, B^*) = (0, 1)\): Expanding to lowest order in eq. (19) and in eq. (20) we get that: 
\[
\delta A' = 2\delta A^2 \quad \text{and} \quad \delta B' = 2\delta B,
\]
so:
\[
\beta(A) = \frac{\delta A' - \delta A}{\delta A} = -1 < 0 \quad \text{and} \quad \beta(B) = \frac{\delta B' - \delta B}{\delta B} = +1 > 0.
\]
(25)
so the FP \((A^*, B^*) = (0, 1)\) is stable in the \(A\)-direction while unstable in the \(B\)-direction.

We can now draw the RG flow diagram (see Fig. (1)):

\[\begin{align*}
K_1 & \quad \infty \\
\infty & \quad K_2
\end{align*}\]

Figure 1: RG flow for the Ashkin-Teller Model in 1D

3. \(D = 1 + \epsilon\), where \(0 < \epsilon \ll 1\).

(a) We need to rescale our variables. As it was shown in class we have:
\[
J_i = 2^{D-1} J_i = 2^\epsilon J_i \approx (1 + \epsilon \ln 2) J_i
\]
we have then:
\[
\bar{J}_2 = (1 + \epsilon \ln 2) J_2 \\
\bar{J}_4 = (1 + \epsilon \ln 2) J_4
\]
(27) (28)

Now that we have rescaled our coupling constants we need to follow the procedure from the previous sections. Then we will get the same RG equations (17) with \(J_i \to \bar{J}_i\).

\[
\frac{A'(1 + B')}{(1 + A'^2 B')} = \frac{\bar{A}^2 (1 + \bar{B})^2}{(1 + \bar{A}^2\bar{B})^2}
\]
(29)
\[
\frac{(A^2 + B')}{(1 + A'^2 B')} = \frac{(\bar{A}^2 + \bar{B})^2}{(1 + \bar{A}^2\bar{B})^2}
\]
(30)

where \(\bar{A} = \tanh \bar{J}_2\) and \(\bar{B} = \tanh \bar{J}_4\). We can read off the FPs from the previous section. Those are given by:
\[
(\bar{A}^*, \bar{B}^*) = \{(0, 0); (1, x); (0, 1)\}, \quad 0 \leq x \leq 1
\]
(31)

Now, as in the previous section, we have to compute the FPs and their stabilities.
• $(\bar{A}^*, \bar{B}^*) = (0, 0)$: We have:
\[
\tanh \bar{J}_2 = 0 \implies J_2 = 0 \tanh \bar{J}_4 = 0 \implies J_4 = 0
\]
The FP is $(J_2, J_4) = (0, 0)$, the trivial FP. As we saw in the previous section this FP is stable.

• $(\bar{A}^*, \bar{B}^*) = (1, x)$. As we saw in the previous section $\delta A' = 2 \delta \bar{A}$ and $\delta B' = \delta \bar{B}$. Close to the FP we have:
\[
\bar{A}^* + \delta \bar{A} = \tanh \bar{J}_2 = \tanh(1 + \epsilon \ln 2) J_2
\]
\[
\implies \tanh^{-1}(1 + \delta \bar{A}) = (1 + \epsilon \ln 2) J_2 = \frac{1}{2} \left[ \ln(1 + 1 + \delta \bar{A}) - \ln(1 - 1 - \delta \bar{A}) \right]
\]
Combining eqs. (35) and (36) we get:
\[
J_2 \approx (1 - \epsilon \ln 2) \left( \ln 2 + \frac{J_2^*}{2} \right)
\]
Now we have to solve for $x$, with $0 \leq x \leq 1$. Taking $x = 1$ we get the same expressions for $J_4^*$ as we did for $J_2^*$. For $x = 0$ we have the trivial point $J_4^* = 0$. We can interpolate now between these two points ($x = 0$ and $x = 1$).

• $(\bar{A}^*, \bar{B}^*) = (0, 1)$. As in the previous part, we have that $J_2^* = 0$ and $J_4^* = 1/2\epsilon$. We now draw the RG flow (see Fig. (2)) using the stability of the FPs from the previous section. The phase diagram is shown in Fig. (3).
Figure 2: RG flow for the Ashkin-Teller Model in $D = 1 + \epsilon$

\[
\langle \sigma \rangle = \langle \tau \rangle = 0 \\
\langle S \rangle = \langle \sigma \tau \rangle \neq 0 \\
\text{Ising in } S
\]

\[
\langle \sigma \rangle = \langle \tau \rangle = 0
\]

\[
\langle \sigma \rangle \neq 0 \\
\langle \tau \rangle \neq 0 \\
\text{Ising in } \sigma \text{ and } \tau
\]

Figure 3: Phase diagram
2 Momentum-Shell RG for the Non-Linear $\sigma$-Model

The Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2g} (\partial_{\mu} \vec{n}(\vec{x}))^2 + \frac{1}{2g} \vec{H}(\vec{x}) \cdot \vec{n}(\vec{x})$$

and the partition function is given by:

$$Z = \int \mathcal{D} \vec{n}(\vec{x}) \prod_{\vec{x}} \delta(\vec{n}^2(\vec{x}) - 1) e^{-\int d^Dx \mathcal{L}}$$

1. Using dimensional analysis we have that:

$$[\mathcal{L}] = L^{-D}, \quad [\vec{n}] = 1, \quad [g] = L^{D-2} \quad \text{and} \quad \left[ \vec{H} \right] = L^{-2}$$

In $D = 2 + \epsilon$ we have that:

$$[g] = L^\epsilon$$

$$\left[ \vec{H} \right] = L^{-2}$$

We can define then:

$$g = u \Lambda^{-\epsilon}$$

$$\vec{H} = \vec{h} \Lambda^2$$

The Lagrangian can be written then as:

$$\mathcal{L} = \frac{1}{2u} \Lambda^\epsilon (\partial_{\mu} \vec{n}(\vec{x}))^2 + \frac{1}{2u} \Lambda^{2+\epsilon} \vec{h}(\vec{x}) \cdot \vec{n}(\vec{x})$$

2. Using eq. (5) from the problem set we have:

$$\partial_{\mu} n^a(\vec{x}) = \partial_{\mu} \left[ \sqrt{1 - \phi^2(\vec{x})} n^a_0(\vec{x}) + \sum_{i=1}^{N-1} \phi_i(\vec{x}) e_i^a(\vec{x}) \right]$$

$$= \frac{-1}{\sqrt{1 - \phi^2(\vec{x})}} \vec{\phi} \cdot \partial_{\mu} \vec{\phi} + \sqrt{1 - \phi^2(\vec{x})} \partial_{\mu} n^a_0 + \sum_{i=1}^{N-1} (\partial_{\mu} \phi_i) e_i^a + \sum_{i=1}^{N-1} \phi_i \partial_{\mu} e_i^a$$

Using eq. (6) and eq. (7) from the problem set we have:

$$\partial_{\mu} n^a(\vec{x}) = \frac{-1}{\sqrt{1 - \phi^2(\vec{x})}} \vec{\phi} \cdot \partial_{\mu} \vec{\phi} + \sqrt{1 - \phi^2(\vec{x})} \partial_{\mu} \left[ \sum_{i=1}^{N-1} B_i^a e_i^a \right]$$

$$+ \sum_{i=1}^{N-1} (\partial_{\mu} \phi_i) e_i^a + \sum_{i=1}^{N-1} \phi_i \left[ \sum_{j=1}^{N-1} A_{ij}^a e_j^a - B_i^a n_0^a \right]$$
Using that \( n^a_0 c^a_i = 0 \) and \( c^a_j c^a_i = \delta_{ij} \) we have that:

\[
(\partial_n n^a(x))(\partial_n n^a(x)) = \frac{(\vec{\phi} \cdot \partial_n \vec{\phi})^2}{1 - \vec{\phi}^2} + 2 \sqrt{1 - \vec{\phi}^2} \frac{\phi_i B^i}{1 - \vec{\phi}^2} + \frac{1 - \vec{\phi}^2}{1 - \vec{\phi}^2} (\phi_i B^i)^2
\]

Expanding up to quadratic order in \( \phi_i(x) \) we get:

\[
S \approx \int d^D x \left\{ \frac{\Lambda^c}{2u} \left[ (1 - \vec{\phi}^2)(B^i)^2 + 2 B^i \partial_i \phi_i + 2 B^i A^k_i \phi_k + (\phi_i)^2 + 2 (\partial_i \phi_i) A^k_i \phi_k \right] + \phi_i \phi_k (A^j_i A^k_j + B^i B^k) \right\} \]

3. Using the Euler-Lagrange equation for linear order in \( \phi_i \) in Eq. (55) we have:

\[
\frac{\delta L}{\delta \phi_i} - \partial_i \left( \frac{\delta L}{\delta \partial_i \phi_i} \right) = 0 \quad \Rightarrow \quad (\partial_i \delta^{ij} - A^j_i) B^i(x) = 0
\]

4. Using the previous equation, the action can be written as:

\[
S \approx \int d^D x \left\{ \frac{\Lambda^c}{2u} \left[ \phi_i \phi_j \left( B^i + (\partial_i \phi_i) B^j + B^j \right) + \phi^2 \right] + \frac{\Lambda^D}{2u} \frac{\tilde{h}}{\phi_i + \phi_i \tilde{e}_i} \right\}
\]

Now we compute the contribution to the effective action due to fluctuations inside the momentum shell \( b\Lambda \leq |\vec{p}| \leq \Lambda \). The leading order correction is:

\[
\langle L^{(2)} \rangle = \frac{\Lambda^c}{2u} \left[ \tilde{B}^2 + \phi \delta^{ij} \right] + \frac{\Lambda^D}{2u} \frac{\tilde{h}^2}{\phi_i + \phi_i \tilde{e}_i}
\]

Defining \( \phi'_i = \phi_i / \sqrt{g} \) we have:

\[
\langle L^{(2)} \rangle = \frac{\Lambda^{D-2}}{2u} \tilde{B}^2 + \frac{1}{2} \phi' \phi'_j - \phi^2 \delta_{ij} \right\} \right] + \phi^2 \left[ \frac{\Lambda^D}{2u} \frac{\tilde{h}^2}{\phi'_i + \phi'_i \tilde{e}'_i} \right]
\]

Using that \( \langle \phi_i \phi_j \rangle_{\text{Shell}} = \langle \phi_i \rangle_{\text{Shell}} \langle \phi_j \rangle_{\text{Shell}} = 0 \) for \( i \neq j \) we have:

\[
\langle \phi'_i \phi'_j - \phi^2 \delta_{ij} \rangle_{\text{Shell}} = \delta_{ij} (1 - (N - 1)) \int_{\text{Shell}} \frac{1}{B^2} \]

\[
= \delta_{ij} (2 - N) \frac{S^D}{(2\pi)^D} \int_{b\Lambda} \frac{dpp^{D-1}}{p^2} \frac{1}{\Lambda^{D-2} \delta l}
\]

\[
= \delta_{ij} (2 - N) \frac{S_D}{(2\pi)^D} \Lambda^{D-2} \delta l
\]

8
where $\delta l = 1 - b$. Using this result we get:

$$\langle \mathcal{L}^{(2)} \rangle = \frac{\Lambda^{D-2}}{2u} \left[ 1 + (2 - N) \frac{S_D}{(2\pi)^D} u \delta l \right] \vec{B}_\mu^2 + \frac{\Lambda^D}{2u} h^a \left[ n^a \left( 1 - \frac{N - 1}{2} \frac{S_D}{(2\pi)^D} u \delta l \right) \right]$$

(61)

Recalling that $\vec{B}_\mu^2 = (\partial_\mu n^a)^2$ we can read off $u'$ and $h'$:

$$\frac{(b\Lambda)^{D-2}}{2u'} = \frac{\Lambda^{D-2}}{2u} \left[ 1 + (2 - N) \frac{S_D}{(2\pi)^D} u \delta l \right]$$

(62)

$$\frac{(b\Lambda)^D}{2u'} h'^a = \frac{\Lambda^D}{2u} h^a \left[ 1 - \frac{N - 1}{2} \frac{S_D}{(2\pi)^D} u \delta l \right]$$

(63)

Using $\delta l \approx 1 - \delta l \approx e^{-\delta l}$ we have that Eq. (62) can be written as:

$$\frac{1 - (D - 2)\delta l}{2u'} = \frac{\Lambda^{D-2}}{2u} \left[ 1 + (2 - N) \frac{S_D}{(2\pi)^D} u \delta l \right]$$

(64)

$$\implies u' = \frac{u(1 - (D - 2)\delta l)}{1 + (2 - N) \frac{S_D}{(2\pi)^D} u \delta l} \approx u - u(D - 2)\delta l - u^2(2 - N) \frac{S_D}{(2\pi)^D} \delta l + O(\delta l^2)$$

(65)

$$\implies \frac{u' - u}{\delta l} \approx - \left[ (D - 2) + (2 - N) \frac{S_D}{(2\pi)^D} u \right] u$$

(66)

Following the same procedure for $h'$ we get:

$$\beta_u(u, h) = \frac{du}{dl} = -u \left[ (D - 2) + (2 - N) \frac{S_D}{(2\pi)^D} u \right]$$

$$\beta_h(u, h) = \frac{dh}{dl} = h \left[ 2 + \frac{(N - 3) S_D}{(2\pi)^D} u \right]$$

(67)

5. The FPs are given by: $\beta_u(u^*, h^*) = \beta_h(u^*, h^*) = 0$.

$$\beta_u(u^*, h^*) = 0 \implies u^* \left[ (D - 2) + (2 - N) \frac{S_D}{(2\pi)^D} u^* \right] = 0 \implies u^* = \left\{ 0, \frac{D - 2 (2\pi)^D}{N - 2 S_D} \right\}$$

(68)

$$\beta_h(u^*, h^*) = 0 \implies h^* \left[ 2 + \frac{(N - 3) S_D}{(2\pi)^D} u^* \right] = 0 \implies h^* = 0$$

(69)

Using that $D = 2 + \epsilon$ and that $\frac{S_D}{(2\pi)^D} = \frac{1}{\Gamma(D/2) (2\pi)^{D/2}} \approx \frac{1}{2 \pi} \approx \frac{1}{2 \pi}$ we get:

$$u^* = \left\{ 0, \frac{2\pi \epsilon}{N - 2} \right\} \quad \text{and} \quad h^* = 0$$

(70)

Let us rewrite Eq. (67) as:

$$\beta_u(u, h) = \frac{du}{dl} = -\epsilon u + \frac{N - 2}{2\pi} u^2$$

(71)

$$\beta_h(u, h) = \frac{dh}{dl} = \left( 2 + \frac{(N - 3) u}{4\pi} \right) h$$

(72)
Now we compute the derivatives of the $\beta$ functions

$$\frac{d\beta_u(u, h)}{du} = -\epsilon + \frac{N - 2}{\pi} u \quad \frac{d\beta_h(u, h)}{du} = \frac{(N - 3)}{4\pi}$$

$$\frac{d\beta_u(u, h)}{dh} = 0 \quad \frac{d\beta_h(u, h)}{dh} = 2 + \frac{(N - 3)}{4\pi} u$$ (73)

Evaluating the previous expressions at the FPs we have:

- $(u^*, h^*) = (0, 0) \implies \frac{d\beta_u(u, h)}{du} = -\epsilon < 0$ and $\frac{d\beta_h(u, h)}{dh} = 2 > 0$.

- $(u^*, h^*) = \left(\frac{2\pi\epsilon}{N - 2}, 0\right) \implies \frac{d\beta_u(u, h)}{du} = \epsilon > 0$ and $\frac{d\beta_h(u, h)}{dh} = 2 + \left(\frac{N - 3}{N - 2}\right) \frac{\epsilon}{2} > 0$.

We have then that the eigenvalues are given by:

- $(u^*, h^*) = (0, 0)$

  $$\implies \gamma_u = -\epsilon < 0 \quad \text{(irrelevant), marginal if } D = 2 \ (\epsilon = 0)$$

  $$\gamma_h = 2 > 0 \quad \text{(relevant)}$$ (75)

- $(u^*, h^*) = \left(\frac{2\pi\epsilon}{N - 2}, 0\right)$

  $$\implies \gamma_u = \epsilon > 0 \quad \text{(relevant), marginal if } D = 2 \ (\epsilon = 0)$$

  $$\gamma_h = 2 + \left(\frac{N - 3}{N - 2}\right) \frac{\epsilon}{2} > 0 \quad \text{(relevant)}$$ (77)

Notice that at $D = 2$ there is a marginal operator (the one associated with $u$). With the previous results we are ready to compute the RG flow (see Fig. 4).

6. From the lecture notes we know that:

$$\xi = \frac{1}{\Lambda} f(u, h)$$

$$\implies \frac{\partial \xi}{\partial \Lambda} = 0 = -\frac{1}{\Lambda^2} f + \frac{1}{\Lambda} \frac{\partial f}{\partial \Lambda} = -\frac{1}{\Lambda^2} f + \frac{1}{\Lambda} \left( \frac{\partial f}{\partial u} \frac{\partial u}{\partial \Lambda} + \frac{\partial f}{\partial h} \frac{\partial h}{\partial \Lambda} \right)$$

Using that $d \ln \Lambda = \frac{N' - \Lambda}{\Lambda} = b - 1 = -\delta l$, so $\frac{\partial g}{\partial l} = -\Lambda \frac{\partial g}{\partial \Lambda}$ we have:

$$f + \frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial h} \beta_h = 0$$ (82)

Fixing $h = 0$, we have up to linear order:

$$f + \epsilon \frac{\partial f}{\partial u} (u - u^*) = 0$$ (83)
Figure 4: RG flow for the Non-linear $\sigma$ model in $D = 2 + \epsilon$. The black dots correspond to the two fixed points $(u^*, h^*) = (0, 0)$ and $(u^*, h^*) = \left(\frac{2\pi\epsilon}{N - 2}, 0\right)$

Solving the previous differential equation we get:

$$f = f_0 \left(\frac{u - u^*}{u_0 - u^*}\right)^{-1/\epsilon}$$

(84)

So the correlation length is given by:

$$\xi \sim |u - u^*|^{-\nu}, \quad \text{with} \quad \nu = \frac{1}{\epsilon}$$

(85)

7. As we did in the previous section, we look at a physical quantity so $\frac{\partial P_Q}{\partial \Lambda} = 0$. Let us look at the free energy:

$$F(\tilde{u}, h) = L^D f(\tilde{u}, h) = \Lambda^{-D} f(\tilde{u}, h) = (b\Lambda)^{-D} f(\tilde{u}', h')$$

(86)

$$\implies f(\tilde{u}, h) = b^{-D} f(\tilde{u}b^{y_u}, h^{y_h})$$

(87)

Now:

$$M(\tilde{u}, h) = \left. \frac{\partial f(\tilde{u}, h)}{\partial h} \right|_{h=0} = b^{-D} \frac{\partial}{\partial h} f(\tilde{u}', h') = \frac{b^{-D}}{b^{-y_u} \partial h'} f(\tilde{u}', h')$$

(88)

$$\implies M(\tilde{u}, h) = b^{y_h - D} M(\tilde{u}', h')$$

At $h = 0$ and choosing $b = \tilde{u}^{-1/y_u}$ we get that:

$$M(\tilde{u}, 0) = \tilde{u}^{D- y_h/y_u} M(1, 0) \implies M \sim (u - u_c)^\beta, \quad \text{with} \quad \beta = \frac{D - y_h}{y_u}$$

(89)
From section 5 we know \( y_u \) and \( y_h \), so we have that:

\[
\beta = \frac{(N-1)}{2(N-2)}
\]  

(90)

Now taking \( \bar{u} = u - u_c = 0 \) in Eq. (88) and choosing \( b = h^{-1/y_h} \) we get:

\[
\mathcal{M}(0,h) = h^{(D-y_h)/y_h} \mathcal{M}(0,1) \implies \mathcal{M} \sim |h|^{1/\delta}, \quad \text{with} \quad \delta = \frac{y_h}{D-y_h}
\]  

(91)

Using the expression for \( y_h \) from section 5 we have:

\[
\delta = \frac{4(N-2)+(N-3)\epsilon}{(N-1)\epsilon}
\]  

(92)

8. (a) Let us recall eq. (71) for \( D = 2 \) (\( \epsilon = 0 \)).

\[
\beta_u(u,h) = \frac{du}{dl} = \frac{du}{d\ln \Lambda} = \frac{N-2}{2\pi} u^2
\]  

(93)

Integrating both sides of the previous equation we get:

\[
u(u) = \frac{u(\mu)}{1 + u(\mu) \frac{(N-2)}{2\pi} \ln \left( \frac{\mu}{\Lambda} \right)}
\]  

(94)

(b) For \( \Lambda \gg \mu \) we have that \( u(\Lambda) \sim \frac{1}{\ln \left( \frac{\mu}{\Lambda} \right)} \implies |u(\Lambda)| \ll 1 \). The coupling constant get smaller and smaller \( u(\Lambda) \xrightarrow{\Lambda \gg \mu} 0 \). This behavior is called asymptotic freedom.

(c) Going back to the Callan-Symanzik equation we have:

\[
f + \frac{\partial f}{\partial u} \beta_u = f + \frac{(N-2)}{2\pi} \frac{\partial f}{\partial u} u^2 = 0
\]  

(95)

\[
\implies \frac{df}{f} = -\frac{2\pi}{(N-2)} \frac{du}{u^2}
\]  

(96)

Integrating the previous equation we get:

\[
\xi \sim \exp \left[ \frac{2\pi}{u(N-2)} \right]
\]  

(97)

There is a singularity as \( u \to 0 \), so \( \xi \xrightarrow{u \to 0} \infty \).