1 Vertex Functions, Effective Potential and Ward Identities

In this problem you will study the effects of interactions on the physical properties of a system of interacting relativistic fermions in 1 + 1 dimensions known as the chiral Gross-Neveu model. This model is a reasonable description of the physics in quasi-one-dimensional systems, and is also of interest to investigate the behavior of quantum field theories of relativistic Fermi fields.

In 1 + 1 dimensions fermion fields $\psi_a(x)$ are two-component spinors of the form

$$\psi_a = \begin{pmatrix} \psi_{R,a} \\ \psi_{L,a} \end{pmatrix}$$  \hspace{1cm} (1)

The upper component, $\psi_{R,a}$, is the right-moving component of the fermion, and the lower component $\psi_{L,a}$ is the left-moving component of the fermion. In what follows we will consider the case in which there are $N$ species of fermions labeled by an index $a = 1, \ldots, N$.

Next, we define the following set of two-dimensional Dirac $\gamma$ matrices

$$\gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1 = i \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_5 = \gamma_0 \gamma_1 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$  \hspace{1cm} (2)

with the notation

$$\partial = \partial_\mu \gamma^\mu = \gamma_0 \partial_0 - \gamma_1 \partial_1$$  \hspace{1cm} (3)

where $\mu = 0, 1$ denote the indices of a 1 + 1-dimensional Minkowski space-time.

Let us introduce the operators

$$\bar{\psi} \psi = \psi^\dagger \gamma_0 \psi \equiv \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R$$  \hspace{1cm} (4)

and

$$\bar{\psi} \gamma_5 \psi = \psi^\dagger \gamma_1 \psi \equiv \psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R$$  \hspace{1cm} (5)

Using this notation, the Lagrangian density of the chiral Gross-Neveu model is (we set the speed of light to $c = 1$)

$$\mathcal{L} = \bar{\psi}_a(x) i \partial \psi_a(x) + \frac{g}{2} \left[ (\bar{\psi}_a(x) \psi_a(x))^2 - (\bar{\psi}_a(x) \gamma^5 \psi_a(x))^2 \right]$$  \hspace{1cm} (6)

where we have not written down the spinor indices explicitly. In all the sections which follow below you have to use path-integral methods.
1. Derive an expression for the free fermion propagator in momentum space.

2. Derive the Feynman rules for a perturbation expansion of the fermion two-point function

\[ S_{\alpha,\beta}^{a,b}(p) \] (7)

(with \( p = (p^0, p^1) \)) in powers of the coupling constant \( g \).

3. Derive an expression for the quantities listed below up to, and including, their second order corrections, i.e. \( O(g^2) \). Draw a Feynman diagram for each contribution. Check the cancellation of the vacuum diagrams. Give a consistent sign to each contribution.

(a) the fermion two-point function

(b) the effective coupling constant \( g \). What correlation function should you consider?. Is it a connected Green function or a one-particle irreducible vertex function. Justify your answer. Do not do the integrals!

4. (a) Show that the Lagrangian of the chiral Gross-Neveu model is invariant under the continuous global chiral symmetry

\[ \psi_{\alpha a}(x) \rightarrow (e^{i\theta \gamma_5})_{\alpha\beta} \psi_{\beta a} \] (8)

(b) Find the transformation law obeyed by the operators \( \Delta_0 \equiv \bar{\psi}_a \psi_a \) and \( \Delta_5 \equiv i\bar{\psi}_a \gamma_5 \psi_a \) under this symmetry.

(c) Give a physical interpretation of this symmetry. What is the meaning of the operators \( \Delta_0 \) and \( \Delta_5 \) in terms of the right and left moving components of the fermions?

5. Now we add chiral symmetry-breaking terms to the Lagrangian of the form

\[ \mathcal{L}_{\text{chiral}} = H_0(x) \Delta_0(x) + H_5(x) \Delta_5(x) \] (9)

where \( H_0(x) \) and \( H_5(x) \) are the symmetry breaking fields. Consider now the path integral for this problem in the presence of the symmetry breaking terms. Assume that the operators \( \Delta_0 \) and \( \Delta_5 \) have uniform expectation values given by \( \bar{\Delta}_0 \) and \( \bar{\Delta}_5 \) respectively. Find the transformation law obeyed by the symmetry breaking fields \( H_0 \) and \( H_5 \) under a global chiral transformation of the Fermi fields by an angle \( \theta \).

6. Derive a formal expression for the Effective Potential \( U(\bar{\Delta}_0, \bar{\Delta}_5) \) in terms of a series expansion in powers of the expectation values. Relate the coefficients of this expansion in terms of vertex functions. What is the meaning of these vertex functions in terms of fermion operators?

7. Find a perturbation theory expression for all the coefficients of the effective potential \( U \) to leading order (i.e. the first non-vanishing term) in an
expansion in terms of the coupling constant $g$. Do not do the integrals. 
Draw the Feynman diagram associated with each contribution. 

**Beware of the signs!!!!**

8. Derive the chiral Ward Identity for the generating functional of the vertex functions for the operators $\Delta_0$ and $\Delta_5$.

9. Derive a Ward Identity that relates the vertex functions $\Gamma_{00}$ and $\Gamma_{55}$ in the limit $p \to 0$ ($p \equiv p_{\mu}$). Assume that, as $H_0 \to 0$ and $H_5 \to 0$, only $\Delta_0$ has a non zero expectation value, i.e. that the chiral symmetry is **spontaneously broken**.

10. Is there a Goldstone boson in this system?. Justify your answer. Explain the significance of your answer to the original fermion problem. What does it say about its two-particle spectrum? And of the one-particle spectrum?.

**Comment:** In this problem you were not required to do any integrals. Therefore you cannot determine if the chiral symmetry is spontaneously broken or not. In the next problem set we’ll come back to this issue and do the integrals!

2 **Renormalization of the $O(N)$ scalar $\phi^4$ theory**

Consider the $\phi^4$ theory in a regime in which the renormalized mass is non-zero for an $N$-component real scalar field $\phi_a$ where $a = 1, \ldots, N$, with an $O(N)$-invariant Lagrangian density in $D$ Euclidean dimensions of the form

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi_a)^2 + \frac{m_0^2}{2} \phi_a^2 + \frac{\lambda}{4!} (\phi_a^2)^2 \quad (10)$$

As usual, repeated indices are summed over.

In the symmetric theory, i.e. in the absence of spontaneous symmetry breaking, the two-point and four-point 1PI vertex functions $\Gamma^{(2)}_{ab}(p)$ and $\Gamma^{(4)}_{abcd}(p_1, \ldots, p_4)$ take the symmetric form

$$\Gamma^{(2)}_{ab}(p) = \delta_{ab} \tilde{\Gamma}^{(2)}(p) \quad (11)$$

and

$$\Gamma^{(4)}_{abcd}(p_1, \ldots, p_4) = S_{abcd} \tilde{\Gamma}^{(4)}(p_1, \ldots, p_4) \quad (12)$$

where

$$S_{abcd} = \frac{1}{3} (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) \quad (13)$$

1. Find all the contributions to $\Gamma^{(2)}$ and $\Gamma^{(4)}$ to **one loop order** for this $N$-component theory. Write down and to draw all Feynman diagrams and their associated analytic expressions. Write your results in terms of the integrals discussed in class (for the one-component theory). Do not do the integrals. Derive the explicit dependence of each diagram on the number of components $N$. 

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2. Define a set of renormalization conditions, at zero external momentum, for the vertex functions $\Gamma^{(2)}(p)$ and $\Gamma^{(4)}(p_1,\ldots,p_4)$ in the symmetric massive theory.

3. Determine $m^2_0$, and $\lambda$ in terms of the renormalized mass $\mu$ and coupling constant $g$ at fixed momentum cutoff $\Lambda$, to one loop order. Express your answers in terms of the integrals defined in class. Do not do the integrals!

4. Show that the renormalization conditions of part 2) and the renormalization constants obtained in part 3) yields finite vertex functions at arbitrary values of the external momentum, to one loop order in perturbation theory.

Useful identities:

\[
\sum_c S_{abcc} = \frac{N + 2}{3} \delta_{ab}
\]

\[
\sum_{ij} S_{abij} S_{ijcd} = \frac{2}{3} S_{abcd} + \frac{N + 2}{3} \delta_{ab} \delta_{cd}
\]