Solution of Homework 2

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I. RENORMALIZATION GROUP FOR THE ASHKIN-TELLER MODEL

In this problem, we shall study the Ashkin-Teller model, i.e.,

\[ H = -K_{2} \sum_{\langle \vec{r}, \vec{r}^\prime \rangle} [\sigma (\vec{r}) \sigma (\vec{r}^\prime) + \tau (\vec{r}) \tau (\vec{r}^\prime)] - K_{4} \sum_{\langle \vec{r}, \vec{r}^\prime \rangle} \sigma (\vec{r}) \sigma (\vec{r}^\prime) \tau (\vec{r}) \tau (\vec{r}^\prime), \]

where $\sigma$ and $\tau$ are the Ising degrees of freedom with values $\pm 1$. Both $K_{2}$ and $K_{4}$ are positive.

1. Qualitative discussion

$K_{2} \to \infty$: In this limit, the first term dominates. Both $\sigma (\vec{r})$ and $\tau (\vec{r})$ must be homogeneous so as to minimize the ground state energy, i.e.,

- $\sigma = +1$, $\tau = +1$;
- $\sigma = +1$, $\tau = -1$;
- $\sigma = -1$, $\tau = +1$;
- $\sigma = -1$, $\tau = -1$.

$K_{4} \to \infty$: In this limit, there is $\sigma (\vec{r}) \sigma (\vec{r}^\prime) = \tau (\vec{r}) \tau (\vec{r}^\prime)$ so as to minimize the energy. That is, the $K_{4}$ term acts essentially as a Dirac-Delta function and the Hamiltonian becomes

\[ H = -2K_{2} \sum_{\langle \vec{r}, \vec{r}^\prime \rangle} \sigma (\vec{r}) \sigma (\vec{r}^\prime), \]

which is equivalent to a single Ising model with $K = 2K_{2}$. 
Figure 1: The RG flow with $x$-axis for $K_2$ and $y$-axis for $K_4$.

$K_2 \to 0$: If $K_2 \to 0$, then the Hamiltonian becomes

$$H \to -K_4 \sum_{\langle \vec{r}, \vec{r}' \rangle} \equiv \Sigma (\vec{r}) [\sigma (\vec{r}) \tau (\vec{r}') \Sigma (\vec{r}') \sigma (\vec{r}') \tau (\vec{r})],$$

and the partition function can be written as

$$Z = \sum_{\{\sigma=\pm 1, \tau=\pm 1\}} \exp \left[ +K_4 \sum_{\langle \vec{r}, \vec{r}' \rangle} \Sigma (\vec{r}) \Sigma (\vec{r}') \right]$$

$$= 2^N \sum_{\{\gamma=\pm 1\}} \exp \left[ +K_4 \sum_{\langle \vec{r}, \vec{r}' \rangle} \Sigma (\vec{r}) \Sigma (\vec{r}') \right],$$

where $N$ is the site number and the effective coupling constant is $K_{\text{eff}} = K_4$.

2. One-dimensional Ashkin-Teller model

Decimation: Now we define the even lattice of $\sigma$ and $\tau$ as

$$\mu (r) \equiv \sigma (2r),$$

and

$$\gamma (r) \equiv \tau (2r).$$

Similar to the lecture note, we sum over the both $\sigma$ and $\tau$ at site $(2r+1)$, i.e.,
\begin{align*}
A & \equiv \sum_{\sigma(2r+1)} \sum_{\tau(2r+1)} e^{\{K_2 \sigma(2r+1) \mu(r)+\mu(r+1)+K_2 \tau(2r+1) \gamma(r)+\gamma(r+1)\}} \\
& \times e^{\{K_4\sigma(2r+1)\mu(r)\tau(2r+1)\gamma(r)+\sigma(2r+1)\mu(r+1)\tau(2r+1)\gamma(r+1)\}} \\
& = e^{K_2[\mu(r)+\mu(r+1)]} + K_2[\gamma(r)+\gamma(r+1)] \cdot e^{K_4[\mu(r)\gamma(r)+\mu(r+1)\gamma(r+1)]} \\
& \quad + e^{-K_2[\mu(r)+\mu(r+1)]} - K_2[\gamma(r)+\gamma(r+1)] \cdot e^{-K_4[\mu(r)\gamma(r)+\mu(r+1)\gamma(r+1)]} \\
& \quad + e^{-K_2[\mu(r)+\mu(r+1)]} + K_2[\gamma(r)+\gamma(r+1)] \cdot e^{-K_4[\mu(r)\gamma(r)+\mu(r+1)\gamma(r+1)]} \\
& = 2e^{K_4[\mu(r)\gamma(r)+\mu(r+1)\gamma(r+1)]} \cosh \{K_2 [\mu (r) + \mu (r + 1)] + K_2 [\gamma (r) + \gamma (r + 1)]\} \\
& \quad + 2e^{-K_4[\mu(r)\gamma(r)+\mu(r+1)\gamma(r+1)]} \cosh \{K_2 [\mu (r) + \mu (r + 1)] - K_2 [\gamma (r) + \gamma (r + 1)]\}. \tag{1}
\end{align*}

Note that $A$ is invariant under the following three sets of transformations

\begin{align*}
\mu (r) & \rightarrow -\mu (r), \quad \mu (r + 1) \rightarrow -\mu (r + 1) \\
\tau (r) & \rightarrow -\tau (r), \quad \tau (r + 1) \rightarrow -\tau (r + 1)
\end{align*}

and

\begin{align*}
\sigma & \leftrightarrow \tau,
\end{align*}

respectively. That is, the number of independent values in $A$ is 3, i.e.,

1. $\mu (r) = \mu (r + 1) = 1, \gamma (r) = \gamma (r + 1) = 1$;
2. $\mu (r) = \mu (r + 1) = 1, \gamma (r) = -\gamma (r + 1) = 1$;
3. $\mu (r) = -\mu (r + 1) = 1, \gamma (r) = -\gamma (r + 1) = 1$.

The values of $A$ for other $\mu$ and $\gamma$ can be determined by performing the symmetry transformations given above. Because $A$ is positively defined, we can recast $A$ as

\begin{align*}
A & = e^{J_0 + J_1 \mu(r)\mu(r+1) + J_2 \gamma(r)\gamma(r+1) + J_3 \mu(r)\gamma(r+1)\gamma(r+1)}, \tag{2}
\end{align*}

where $J_1 = J_2 \equiv J$ due to the symmetry between $\mu (r)$ and $\gamma (r)$ in $A$. Now let us fix coefficients $J_0, J_1, J_2$ and $J_3$ by inserting the values of $\mu$ and $\gamma$ back to Eq. (2), i.e.,

\begin{align*}
e^{J_0+2J_3} & = 2e^{2K_4} \cosh (4K_2) + 2e^{-2K_4} \tag{3} \\
e^{J_0-J_3} & = 4 \cosh (2K_2) \tag{4} \\
e^{J_0-2J_3} & = 4 \cosh (2K_4), \tag{5}
\end{align*}

whose solutions are

\begin{align*}
J_0 & = \frac{1}{4} \left\{ 2 \ln [4 \cosh (2K_2)] + \ln [2e^{-2K_4} + 2e^{2K_4} \cosh (4K_2)] + \ln [4 \cosh (2K_4)] \right\} \tag{6} \\
J & = \frac{1}{4} \left\{ \ln [2e^{-2K_4} + 2e^{2K_4} \cosh (4K_2)] - \ln [4 \cosh (2K_4)] \right\} \tag{7} \\
J_3 & = \frac{1}{4} \left\{ -2 \ln [4 \cosh (2K_2)] + \ln [2e^{-2K_4} + 2e^{2K_4} \cosh (4K_2)] + \ln [4 \cosh (2K_4)] \right\}. \tag{8}
\end{align*}

That is, we have obtained

\begin{align*}
K_2' & = \frac{1}{4} \left\{ \ln [2e^{-2K_4} + 2e^{2K_4} \cosh (4K_2)] - \ln [4 \cosh (2K_4)] \right\} \\
& = \frac{1}{4} \ln \left[ \frac{e^{-2K_4} + e^{2K_4} \cosh (4K_2)}{2 \cosh (2K_4)} \right], \tag{9}
\end{align*}

whence
whose RG flow is shown in Fig. 1.

and the effective Hamiltonian is

\[ H = -\frac{N}{2} J_0 - K_2' \sum_{(r, r')} |\sigma(r) \sigma(r') + \tau(r) \tau(r')| - K_4' \sum_{(r, r')} \sigma(r) \sigma(r') \tau(r) \tau(r'). \]  

**β functions for \( K_2 \) and \( K_4 \):** The corresponding β-functions are given as

\[
\beta_{K_2} = \frac{K_2' - K_2}{\ln 2} = \frac{1}{4} \ln \left[ e^{-2K_4 + e^{2K_4} \cosh(4K_2)} \right] - K_2 = \frac{1}{4} \ln \left[ \frac{e^{-2K_4 + e^{2K_4} \cosh(4K_2)}}{2 \cosh^2(2K_2)} \right] - K_2.
\]  

\[
\beta_{K_4} = \frac{K_4' - K_4}{\ln 2} = \frac{1}{4} \ln \left[ e^{-2K_4 + e^{2K_4} \cosh(4K_2)} \right] - K_4 = \frac{1}{4} \ln \left[ \frac{e^{-2K_4 + e^{2K_4} \cosh(4K_2)}}{2 \cosh^2(2K_2)} \right] - K_4,
\]  

whose RG flow is shown in Fig. 1.

**Fixed points:** Fig. 1 suggests that there are four fixed points located at \((K_{2*}, K_{4*}) = (0, 0), (0, \infty), (\infty, 0), \) and \((\infty, \infty)\), respectively.

The linearized β-functions around the fixed point \( K_{2*} = K_{4*} = 0 \) are given as

\[
\beta_{K_2} = -\frac{1}{\ln 2} K_2,
\]  

and

\[
\beta_{K_4} = -\frac{1}{\ln 2} K_4,
\]  

which means that the eigenvalues are \(- (\ln 2)^{-1}\) and \(- (\ln 2)^{-1}\).

As for \( K_{2*} = K_{4*} = \infty \), we can define \( T_2 = \frac{1}{K_{2*}} \) and \( T_4 = \frac{1}{K_{4*}} \), and then linearized the β functions around \( T_{2*} = T_{4*} = 0 \). By assuming \( \frac{\delta T_2}{T_2} \ll 1, \frac{\delta T_4}{T_4} \ll 1 \), the renormalization transformation of \( K_2 \) and \( K_4 \) in Eq. (10) and (9) implies

\[
\delta T_2 = \frac{T_2^2}{4} \ln 2,
\]  

and

\[
\delta T_4 = \frac{T_4^2}{4} \ln 2.
\]  

The corresponding β-functions are

\[
\beta_{T_2} \equiv \frac{\delta T_2}{\ln 2} = \frac{T_2^2}{4},
\]  

and

\[
\beta_{T_4} \equiv \frac{\delta T_4}{\ln 2} = \frac{T_4^2}{4}.
\]  

That is, the eigenvalues of the renormalization group transformation around this fixed point is zero.

Similarly, for \( K_{2*} = 0, K_{4*} = \infty \) \((K_{2*} = \infty, K_{4*} = 0)\), the eigenvalues are \(- \frac{1}{\ln 2}\) and \(0 (0 \text{ and } -\frac{1}{\ln 2})\), respectively.

**Similarity and difference compared to the Ising model:** Similar to the Ising model, the stable fixed point is at \( K_2 = 0 \) and the perturbative fixed point \((K_2 \rightarrow \infty)\) is unstable. Compared to the Ising case, there is one more coupling constant, \( K_4 \), whose stable fixed-point value is \( K_4 = 0 \), but not the perturbative fixed point \( K_4 = \infty \) (zero-temperature limit).
Table I: Fixed points, eigenvalues and critical exponents. Note that $T_2 \equiv K_2^{-1}, T_4 \equiv K_4^{-1}$.

<table>
<thead>
<tr>
<th>Fixed Points</th>
<th>Stable?</th>
<th>Eigenvalues</th>
<th>Critical Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_2 = K_4 = 0$</td>
<td>Yes</td>
<td>$(-\frac{1}{16}, -\frac{1}{16})$ for $(K_2, K_4)$</td>
<td>$(-\ln 2, -\ln 2)$</td>
</tr>
<tr>
<td>$K_2 = K_4 = \frac{1}{2}$</td>
<td>No</td>
<td>$(\epsilon, \epsilon)$ for $(T_2, T_4)$</td>
<td>$(\frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td>$K_2 = 0, K_4 = \frac{1}{2}$</td>
<td>No</td>
<td>$(-\frac{1}{16}, \epsilon)$ for $(K_2, T_4)$</td>
<td>$(-\ln 2, \frac{1}{2})$</td>
</tr>
<tr>
<td>$K_2 = \frac{1}{2}, K_4 = 0$</td>
<td>No</td>
<td>$(\epsilon, -\frac{1}{16})$ for $(T_2, K_4)$</td>
<td>$(\frac{1}{2}, -\ln 2)$</td>
</tr>
<tr>
<td>$K_2 = 0, K_4 = \infty$</td>
<td>Yes</td>
<td>$(-\frac{1}{16}, -\epsilon)$ for $(K_2, T_4)$</td>
<td>$(-\ln 2, -\frac{1}{2})$</td>
</tr>
<tr>
<td>$K_2 = \infty, K_4 = 0$</td>
<td>Yes</td>
<td>$(-\epsilon, -\frac{1}{16})$ for $(T_2, K_4)$</td>
<td>$(-\frac{1}{2}, -\ln 2)$</td>
</tr>
<tr>
<td>$K_2 = \frac{1}{2}, K_4 = \infty$</td>
<td>No</td>
<td>$(\epsilon, -\epsilon)$ for $(T_2, T_4)$</td>
<td>$(\frac{1}{2}, -\frac{1}{2})$</td>
</tr>
<tr>
<td>$K_2 = \infty, K_4 = \infty$</td>
<td>Yes</td>
<td>$(-\epsilon, -\epsilon)$ for $(T_2, T_4)$</td>
<td>$(\frac{1}{2}, -\frac{1}{2})$</td>
</tr>
</tbody>
</table>

3. Migdal-Kadanoff Approximation

Renormalized coupling constants: By using the Migdal-Kadanoff approximation, we can derive the renormalization group equations in higher dimension, for example, $D$. Assuming we are around the ordered phase, the decimation process can be approximated by moving the bonds to the left-over bonds. Consequently, in $D$-dimensions, there are

\[
K_2' = \frac{1}{4} \ln \left[ \frac{e^{-2^{D-1}K_4} + e^{2^{D-1}K_4} \cosh (2^{D-1}4K_2)}{2 \cosh (2^{D-1}2K_4)} \right],
\]

and

\[
K_4' = \frac{1}{4} \ln \left[ \frac{\cosh (2^{D-1}2K_4) \left[ e^{-2^{D-1}2K_4} + e^{2^{D-1}2K_4} \cosh (2^{D-1}4K_2) \right]}{2 \cosh^2 (2^{D-1}2K_4)} \right]
\]

whose RG-flow in the case $\epsilon = 0.1$ is shown in Fig. 2.

Fixed points: Fig. 2 suggests that there are eight-fixed points. The calculations is similar to that presented in last part, so we summarized our result in Table. I.
Phase diagrams and order parameter: The phase diagram and corresponding order parameters are shown in Fig. 2. As for the order parameters, let us fix them by considering some limits. In the limit that $K_2 = 0$, the Ashkin-Teller model is equivalent to one single model with effective coupling $K_{\text{eff}} = K_4$ and effective spin $\sigma(r)\tau(r)$, so the order parameter for phase 1 is $\langle \sigma \tau \rangle \neq 0$, but for phase 3 $\langle \sigma \tau \rangle = 0$. In the limit $K_4 = \infty$, the model is equivalent to one single Ising model with effective coupling $K_{\text{eff}} = 2K_2$ and spin $\sigma$ or $\tau$. So the order parameter for phase 2 is $\langle \sigma \rangle \neq 0$, $\langle \tau \rangle \neq 0$ and thus $\langle \sigma \tau \rangle \neq 0$, but for phase 1, $\langle \sigma \rangle = \langle \tau \rangle = 0$. In the limit $K_4 = 0$, this model is equivalent to two independent Ising model, so the order parameters for phase 3 are $\langle \sigma \rangle = \langle \tau \rangle = 0$.

II. MOMENTUM-SHELL RG FOR THE NON-LINEAR $\sigma$-MODEL

In this problem, we shall study the $O(N)$ non-linear sigma model in the Euclidean spacetime with dimension $D = 2 + \epsilon$,

$$\mathcal{L} = \frac{1}{2g} (\partial_\mu \vec{n}) (\partial^\mu \vec{n}) + \frac{1}{2g} \vec{H} \cdot \vec{n},$$

(18)

where $\vec{n} = (n^1, \ldots, n^a, \ldots n^N)$ with $a = 1, \ldots, N$ and $|\vec{n}|^2 = 1$. That is, the target manifold is a $(N-1)$-sphere, $S^{N-1}$.

Assuming the classical solution of the field equation is $n^a_0$, then it satisfies

$$n^a_0 n^a_0 = 1.$$

In the $R^N$, we can further choose a set of frame fields $e^a_i$ perpendicular to $n^a_0$, i.e.,

$$e^a_i n^a_0 = 0, \quad e^a_i e^a_j = \delta_{ij}.$$

Now we can recast $n^a$ as

$$n^a = \sqrt{1 - |\phi|^2 n^a_0} + \sum_{i=1}^{N-1} \phi_i e^a_i.$$

Clearly, $\partial_\mu n^a_0$ must be perpendicular to $n^a_0$, so we can define

$$\partial_\mu n^a_0 = B^i_\mu e^a_i,$$

and similarly,

$$\partial_\mu e^a_i = A^a_{ij} e^a_j + \tilde{B}^i_\mu n^a_0,$$

where $e^a_i n^a_0 = 0$ implies $A^a_{ij} = -e^a_i \partial_\mu e^a_j$ and $B^i_\mu = -\tilde{B}^i_\mu = e^a_i \partial_\mu n^a_0$.

1. Dimensionless coupling constants and external sources

By dimensional analysis, there are

$$[g] = E^{-\epsilon},$$

and

$$[H] = E^2,$$

where $E$ stands for the energy. We can define the dimensionless coupling constant $u$ and external sources $\tilde{h}$ as

$$g = \Lambda^{-\epsilon} u,$$

(19)

and

$$\tilde{H} = \Lambda^2 \tilde{h}.$$
2. Expansion around background fields

Because the classical solution is \( n_0^a \), the magnitude of \( \phi_i \) is assumed small. We can perform expansion of \( \phi \) up to the order \( \mathcal{O}(\phi^2) \) and the action becomes

\[
S = \int \left\{ \frac{1}{2g} \left[ \left( (D_\mu \phi_i + B_\mu^i) \right)^2 + \left( \phi_i \phi_j - |\phi|^2 \delta_{ij} \right) B_\mu^i B_\mu^j \right] + \frac{1}{2g} \left[ \left( 1 - \frac{1}{2} |\phi|^2 \right) H^a n_0^a + e_i^a \phi_i H^a \right] \right\},
\]

(21)

where

\[
D_\mu \phi_i \equiv \left( \partial_\mu \phi_i - A_{ij}^\mu \phi_j \right).
\]

The detailed derivations is given in Append. A. \( A_{ij}^\mu \) looks like the spin connection, so let us export the local symmetry of the action above.

Due to the following definitions

\[
A_{ij}^\mu = -e_i^a \partial_\mu e_j^a,
\]

and

\[
n^a = \sqrt{1 - \sum_i (\phi_i)^2 n_0^a + \phi_i e_i^a},
\]

under the local rotation,

\[
\phi_i \to R_{ij}(x) \phi_j,
\]

and

\[
e_i^a \to e_j^a R_{ji},
\]

the action in Eq. (18) is invariant. In addition, the transformation law of \( A_{ij}^\mu \) is

\[
A_{ij}^\mu \to (R^{-1} A_{ij}^\mu R)^{ij} - (R^{-1} \partial_\mu R)^{ij},
\]

which is similar to that for the spin connection under the local Lorentz transformation (or rotation in the Euclidean spacetime). Because \( \phi^i \) can be regarded as the tangent vector in the target manifold \( S^{N-1} \) defined in the local chart around \( n_0^a \), \( A_{ij}^\mu \) is the connection for the “vector field” \( \phi^i \).

3. Equation of motion for \( A_{ij}^\mu \) and \( B_\mu^i \)

From Eq. (21), we can derive the equation of motion for \( \phi_i \), i.e.,

\[
-\frac{1}{2g} 2D^\mu \left( D_\mu \phi_i + B_\mu^i \right) + 2 \frac{1}{2g} \left( \phi_j B_{\mu}^i B_{\mu}^j - \phi_i \delta_{mn} B_{\mu}^m B_{\mu}^n \right) + \frac{1}{2g} \left( -\phi^i n_0^a + e_i^a \right) H^a = 0.
\]

Because \( \phi_i = 0 \) is the solution of the equation of motion, there is

\[
D^\mu B_\mu^i = -\frac{1}{2} e_i^a H^a = 0,
\]

(22)

which reduces to

\[
D^\mu B_\mu^i = 0
\]

in the absence of external sources.

Because \( B_{ij}^\mu \) is a classical field, the equation of motion in Eq. (22) can be applied to the action. In other words, the action must depend on \( \phi^i \) quadratically, i.e.,

\[
S = \int \left\{ \frac{1}{2g} \left[ \left( (D_\mu \phi_i) \right)^2 + B_{\mu}^i B_{\mu}^i + \left( \phi_i \phi_j - |\phi|^2 \delta_{ij} \right) B_{\mu}^i B_{\mu}^j \right] + \frac{1}{2g} \left( 1 - \frac{1}{2} |\phi|^2 \right) H^a n_0^a \right\}.
\]

(23)
Notice that Eq. (23) depends on $\phi^i$ quadratically, so we can integrate out $\phi^i$ to obtain the effective action, i.e.,

$$Z = \int D\phi^i \exp (-S)$$

$$= e^{-S_0} \det \left\{ \frac{1}{2g} \left[ -\delta_{ij} D^\mu D_\mu + B_\mu^m B_\mu^m (\delta_{im} \delta_{jn} - \delta_{ij} \delta_{mn}) - \frac{1}{2} \delta_{ij} H^a n_0^a \right] \right\}^{-1/2}$$

$$= \exp \left\{ -S_0 - \frac{1}{2} \text{Tr} \ln \frac{1}{2g} \left[ -\delta_{ij} D^\mu D_\mu + B_\mu^m B_\mu^m (\delta_{im} \delta_{jn} - \delta_{ij} \delta_{mn}) - \frac{1}{2} \delta_{ij} H^a n_0^a \right] \right\},$$

where $S$ in the first line is given as

$$S = S_0 + S',$$

$$S_0 = \int \frac{1}{2g} B_\mu^i B_\mu^i + \int \frac{1}{2g} H^a n_0^a,$$

and

$$S' = \int \frac{1}{2g} \left[ (D_\mu \phi^i)^2 + (\phi_i \phi_j - |\phi|^2 \delta_{ij}) B_\mu^i B_\mu^j \right] - \frac{1}{2} |\phi|^2 H^a n_0^a$$

$$= \int \frac{1}{2g} \phi^i \left[ -\delta_{ij} D^\mu D_\mu + B_\mu^m B_\mu^m (\delta_{im} \delta_{jn} - \delta_{ij} \delta_{mn}) - \frac{1}{2} \delta_{ij} H^a n_0^a \right] \phi^j.$$ 

The effective action is

$$S_{\text{eff}} = S_0 + \frac{1}{2} \text{Tr} \ln \frac{1}{2g} \left[ -\delta_{ij} D^\mu D_\mu + B_\mu^m B_\mu^m (\delta_{im} \delta_{jn} - \delta_{ij} \delta_{mn}) - \frac{1}{2} \delta_{ij} H^a n_0^a \right].$$

By performing expansion of $S_{\text{eff}}$ for fields $B_\mu^i$ and $H^a$, we can obtain the quantum corrections to $S_0$. For example, the corrections to the $B^i$ term is

$$-\frac{1}{2} \int d^D x \int \frac{d^D p}{(2\pi)^D} \frac{\delta_{ij}}{p^2} B_\mu^m B_\mu^m (\delta_{im} \delta_{jn} - \delta_{ij} \delta_{mn})$$

$$= -\frac{1}{2} (-1) \int d^D x (N - 2) B_\mu^i B_\mu^i \left[ \int_{b\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} \right]$$

$$= \frac{1}{2} \int d^D x (N - 2) B_\mu^i B_\mu^i \frac{S_D \Lambda^{D-2}}{(2\pi)^D - \delta l}, \quad (24)$$

and the corrections to the $H^a n_0^a$ term is

$$-\frac{1}{2} \int d^D x \int \frac{d^D p}{(2\pi)^D} \frac{\delta_{ij}}{p^2} \left( -\frac{1}{2} \delta_{ij} H^a n_0^a \right)$$

$$= \frac{1}{4} \int d^D x H^a n_0^a (N - 1) \int_{b\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2}$$

$$= \frac{1}{4} \int d^D x H^a n_0^a (N - 1) \frac{S_D \Lambda^{D-2}}{(2\pi)^D - \delta l}, \quad (25)$$

where

$$\int_{b\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} = \frac{S_D}{(2\pi)^D} \int_{\Lambda} dpp^{D-3}$$

$$= \frac{S_D}{(2\pi)^D} \frac{1}{D - 2} (1 - b^{D-2}) \Lambda^{D-2}$$

$$\simeq \frac{S_D}{(2\pi)^D} \frac{1}{b^{D-2}} \Lambda^{D-2}.$$
\( N = 4, \ \epsilon = -1 \)

\( N = 4, \ \epsilon = 0 \)

\( N = 4, \ \epsilon = 1 \)

Figure 3: Renormalization group flow

Alternatively, one can calculate these quantum corrections by using the Feynman diagrams, which are given in App. B. Finally, the effective action becomes

\[
S_{\text{eff}} = \int \frac{1}{2g} \left[ 1 - g(N - 2) \frac{S_D \Lambda^{D-2}}{(2\pi)^D} \delta l \right] B^i \partial B^i + \int \frac{1}{2g} h^a n_0 \left[ 1 - g \frac{2}{N - 1} \frac{S_D \Lambda^{D-2}}{(2\pi)^D} \delta l \right] + \ldots
\]

\[
\simeq \int \frac{\Lambda^\epsilon}{2u} \left[ 1 + u(N - 2) \frac{S_D}{(2\pi)^D} \delta l \right] B^i \partial B^i + \int \frac{\Lambda^\epsilon}{2u} h^a n_0 \left[ 1 + u(N - 2) \frac{S_D}{(2\pi)^D} \delta l - \frac{u}{2} (N - 1) \frac{S_D}{(2\pi)^D} \delta l \right] + \ldots
\]

Combined with rescaling, one can obtain the following \( \beta \) functions,

\[
\beta_{\tilde{u}} \equiv \frac{\delta \tilde{u}}{\delta l} = -\epsilon \tilde{u} + (N - 2) \tilde{u}^2, \quad (26)
\]

and

\[
\beta_{\hat{h}^a} \equiv \frac{\delta \hat{h}^a}{\delta l} = 2 \hat{h}^a + \frac{N - 3}{2} \hat{h}^a \hat{\tilde{u}}, \quad (27)
\]

where

\[
\tilde{u} = u \frac{S_D}{(2\pi)^D}, \text{ and, } \hat{h}^a = h^a \frac{S_D}{(2\pi)^D}.
\]

The corresponding renormalization flow of the parameters \( \tilde{u} \) and \( \hat{h}^a \) is shown in Fig. 3.

5. Fixed points

For later convenience, we shall assume \( N \geq 3 \). The fixed points of the \( \beta \) functions can be obtained by requiring \( \beta_{\tilde{u}} = \beta_{\hat{h}^a} = 0 \), which locate at

\[
\tilde{u}_* = 0, \ \hat{h}^a = 0,
\]
or
\[ \tilde{u}_* = \frac{\epsilon}{N-2}, \quad \tilde{h}^a_* = 0. \]

In the Euclidean spacetime, the action is required to be positive-definite, so for \( \epsilon < 0 \), the fixed point located at \( \tilde{u}_* < 0 \) is not physical.

Now let us calculate the eigenvalues around these fixed points. For \( \tilde{u}_* = \tilde{h}^a_* = 0 \), the \( \beta \) functions can be approximated by

\[ \beta^{(1)}_u \simeq -\epsilon \tilde{u}, \quad \beta^{(1)}_{\tilde{h}^a} \simeq 2\tilde{h}^a, \]

which means for \( \epsilon > 0 \), \( \tilde{u} \) is irrelevant, while \( \tilde{h}^a \) is always relevant. For \( \tilde{u}_* = \frac{\epsilon}{N-2} \) and \( \tilde{h}^a_* = 0 \), the \( \beta \) functions are

\[ \beta^{(2)}_u \simeq \epsilon \tilde{u}, \quad \text{and} \quad \beta^{(2)}_{\tilde{h}^a} = \left(2 + \frac{\epsilon}{2} \frac{N-3}{N-2}\right)\tilde{h}^a, \]

which means that \( \tilde{h}^a \) is relevant and for \( \epsilon > 0 \), \( \tilde{u} \) is relevant as well.

For \( D = 2 \), or \( \epsilon = 0 \), at the tree-level, \( g \) has zero dimension, so it is marginal. However, in the presence of quantum fluctuations, \( g \) becomes marginally relevant.

6. Correlation length and critical exponents

By dimensional analysis, the correlation length \( \xi \) can be written as

\[ \xi = \Lambda^{-1} f (\tilde{u}). \]

By requiring \( \xi \) independent of the energy scale, there is

\[ 0 = \Lambda \frac{d}{d\Lambda} \xi, \]
\[ 1 = -\beta_u \frac{d}{d\tilde{u}} \ln f (\tilde{u}) \]
\[ 1 = -\beta'_u (\tilde{u} - \tilde{u}_*) \frac{d}{d\tilde{u}} \ln f (\tilde{u}), \]

where around the fixed points \( \tilde{u} = \tilde{u}_* \), there is

\[ \beta_u = \beta'_u (\tilde{u} - \tilde{u}_*). \]

Hence, \( f (\tilde{u}) \) is given as

\[ f (\tilde{u}) = f (\tilde{u}_0) \left( \frac{\tilde{u} - \tilde{u}_*}{\tilde{u}_0 - \tilde{u}_*} \right)^{-\beta'_u - 1}, \]

or

\[ \xi \sim (\tilde{u} - \tilde{u}_*)^{-\frac{1}{\beta'_u}}. \]

The critical exponent is

\[ \nu = \frac{1}{\beta'_u} = \begin{cases} -\frac{1}{\epsilon} & \tilde{u}_* = 0 \\ \frac{1}{\epsilon} & \tilde{u}_* = \frac{\epsilon}{N-2}. \end{cases} \]
7. $M \sim (\tilde{u} - \tilde{u}_*)^\beta$ and $M \sim |h|^{1/\delta}$

$M \sim (\tilde{u} - \tilde{u}_*)^\beta$: First of all, we shall determine the scaling dimension of $M$ in presence of quantum fluctuations. This can be done by calculating the field strength renormalization. By definition, there is

$$n^a = \sqrt{1 - \sum_i (\phi_i)^2 n_0^a + \phi_i \epsilon_i^a}$$

$$\approx \left[1 - \frac{1}{2} \sum_i (\phi_i)^2 \right] n_0^a + \phi_i \epsilon_i^a.$$  

So after integrating over the momentum shell, the new field $(n_0^a)'$ is

$$(n_0^a)' = \left[1 - \frac{1}{2} (N-1) \int_{\Lambda} d^D p \frac{g}{(2\pi)^D p^2} \right] n_0^a$$

$$= \left[1 - \frac{1}{2} (N-1) \tilde{u} \delta l \right] n_0^a$$

$$= Z^{1/2} n_0^a,$$

where

$$Z^{1/2} = 1 - \frac{1}{2} (N-1) \tilde{u} \delta l$$

is the field strength renormalization. That is, the scaling dimension of $n_0^a$ is $[n_0^a] = \Lambda^{\frac{1}{2} (N-1)^a}$, while the engineering scaling dimension is 0. Alternatively, we can define

$$\gamma_n = \frac{\delta Z}{\delta \tilde{l}} = -(N-1) \tilde{u}.$$  

Now we are ready to derive the critical exponent $\beta$. By dimensional analysis, there is

$$M = \Lambda^{-\frac{1}{2} \gamma_n} g (\tilde{u}),$$

which should be independent of the energy scale, i.e.,

$$\frac{d}{d\Lambda} M = 0$$

$$-\frac{1}{2} \gamma_n g (\tilde{u}) - \beta'_n (\tilde{u} - \tilde{u}_*) \frac{d}{d\tilde{u}} g (\tilde{u}) = 0$$

$$g (\tilde{u}) \sim (\tilde{u} - \tilde{u}_*)^{-\frac{\gamma_n}{2}}.$$

or

$$M \sim (\tilde{u} - \tilde{u}_*)^{-\frac{\gamma_n}{2}}.$$  

The value of $\beta$ is thus given as

$$\beta = \begin{cases} 0 & \tilde{u}_* = 0 \\ \frac{(N-1)}{2(N-2)} & \tilde{u}_* = \frac{\epsilon}{N-2} \end{cases}.$$

$M \sim |h|^{1/\delta}$: Similarly, by dimensional analysis, there is

$$M = \Lambda^{-\frac{1}{2} \gamma_n} g (|h|),$$

which implies

$$M \sim |h|^{-\frac{\gamma_n}{2}}.$$  

That is, we have obtained

$$\frac{1}{\delta} = \begin{cases} 0 & \tilde{u}_* = 0 \\ \frac{\epsilon(N-1)}{4(N-2)} & \tilde{u}_* = \frac{\epsilon}{N-2} \end{cases}.$$
8. $D = 2$

$\tilde{u}(\Lambda)$: The $\beta$ function for $\tilde{u}$ implies

$$\int_{\tilde{u}(\mu)}^{\tilde{u}(\Lambda)} \tilde{u}^{-1} = (N - 2) \int_{\mu}^{\Lambda} d\ln \Lambda$$

$$\tilde{u}(\Lambda) = \frac{\tilde{u}(\mu)}{1 + \tilde{u}(\mu) (N - 2) \ln \frac{\Lambda}{\mu}}.$$ 

Note that at the scale $\Lambda_{LP} = \mu \exp \left[\frac{-1}{N - 2}\ln(\mu)\right]$, $\tilde{u}(\Lambda)$ becomes divergent. This scale is known as the Landau pole.

$\Lambda \gg \mu$: For $\Lambda \gg \mu$, there is

$$\tilde{u}(\Lambda) \rightarrow 1$$

so the interaction becomes weak.

**Correlation length again:** Similar to Sec. II 6, $\frac{d}{d\Lambda} \xi = 0$ implies

$$\frac{\xi[\tilde{u}(\Lambda)]}{\xi[\tilde{u}(\mu)]} = \exp \left[ - \int_{\tilde{u}(\mu)}^{\tilde{u}(\Lambda)} \frac{1}{\beta_a} d\tilde{u} \right].$$

$$= \exp \left\{ \frac{1}{N - 2} \left[ \frac{1}{\tilde{u}(\Lambda)} - \frac{1}{\tilde{u}(\mu)} \right] \right\}.$$ 

Hence, for $\Lambda \rightarrow \infty$, there are $\tilde{u}(\Lambda) \rightarrow 0$ and thus $\xi[\tilde{u}(\Lambda)] \rightarrow \infty$. By contrast for $\Lambda \rightarrow \Lambda_{LP}$, the correlation length $\xi[\tilde{u}(\Lambda)]$ becomes finite.

**Appendix A: Derivation of Eq. (21)**

In this section, we shall provided the detailed derivations of Eq. (21).

Because of the following decomposition,

$$n^a = \sqrt{1 - \phi_i \cdot \phi_i n_0^a} + \sum_{i=1}^{N-1} \phi_i e_i^a,$$

there are

$$\partial_{\mu} n^a$$

$$= \frac{1}{2} \frac{1}{\sqrt{1 - |\phi|^2}} \left( -2\phi_i \partial_{\mu} \phi_i n_0^a + \sqrt{1 - |\phi|^2} \partial_{\mu} n_0^a + \partial_{\mu} \phi_i e_i^a + \phi_i \partial_{\mu} e_i^a \right)$$

$$\simeq -\phi_i \partial_{\mu} \phi_i n_0^a + \sqrt{1 - |\phi|^2} \partial_{\mu} n_0^a + \partial_{\mu} \phi_i e_i^a + \phi_i \partial_{\mu} e_i^a$$

and
\[\partial_\mu n^\alpha \partial^\mu n^\alpha\]

\[= (-\phi_1 \partial_\mu \phi_i n_0^a) (\partial_\mu n_0^a) + \left(1 - |\phi|^2\right) \partial_\mu n_0^a \partial_\mu n_0^a + \partial_\mu n_0^a \left(-\phi_1 \partial_\mu \phi_i n_0^a + \partial_\mu \phi_j e_j^a + \phi_j \partial_\mu e_j^a \right) + \partial_\mu \phi_i e_i^a \left(\sqrt{1 - |\phi|^2} \partial_\mu n_0^a + \partial_\mu \phi_j e_j^a + \phi_j \partial_\mu e_j^a \right) + \phi_i \partial_\mu e_i^a \left(\partial_\mu n_0^a + \partial_\mu \phi_j e_j^a + \phi_j \partial_\mu e_j^a \right)\]

\[= \left(1 - |\phi|^2\right) \partial_\mu n_0^a \partial_\mu n_0^a + \partial_\mu \phi_i e_i^a \left(\partial_\mu n_0^a + \partial_\mu \phi_j e_j^a + \phi_j \partial_\mu e_j^a \right) + \phi_i \partial_\mu e_i^a \left(\partial_\mu n_0^a + \partial_\mu \phi_j e_j^a + \phi_j \partial_\mu e_j^a \right)\]

\[= \left(1 - |\phi|^2\right) B_\mu^a B_\mu^a + \partial_\mu \phi_i e_i^a \left(\partial_\mu n_0^a + \partial_\mu \phi_j e_j^a + \phi_j \partial_\mu e_j^a \right) + \phi_i \partial_\mu e_i^a \left(\partial_\mu n_0^a + \partial_\mu \phi_j e_j^a + \phi_j \partial_\mu e_j^a \right)\]

where

\[\partial_\mu e_i^a \partial_\mu n_0^a = (A_\mu^a e_0^a - B_\mu^a n_0^a) B_\mu^a e_i^a = A_\mu^a B_\mu^a,\]

and

\[\partial_\mu e_j^a \partial_\mu e_i^a = (A_\mu^a e_0^a - B_\mu^a n_0^a) (A_\mu^a e_0^a - B_\mu^a n_0^a) = A_\mu^a A_\mu^a + B_\mu^a B_\mu^a.\]

Notice that

\[\vec{h} \cdot \vec{n} \simeq \left(1 - \frac{1}{2} |\phi|^2\right) h^a n_0^a + e_i^a \phi_i h^a,\]

the action up to order \(O(\phi^2)\) can be written as

\[S = \int \left\{ \frac{\Lambda^4}{2u} \left( |(D_\mu \phi_i + B_\mu^a)|^2 + (\phi_i \phi_j - |\phi|^2 \delta_{ij}) B_\mu^a B_\mu^a \right) + \Lambda^2 \left[ \left(1 - \frac{1}{2} |\phi|^2\right) h^a n_0^a + e_i^a \phi_i h^a \right] \right\},\]

where

\[D_\mu \phi_i \equiv (\partial_\mu \phi_i + A_\mu^{ij} \phi_j).\]

Appendix B: Feynman diagrams and momentum-shell integration

In this section, we shall derive the coefficient of \(B_\mu^a B_\mu^a\) and \(n_0^a h^a\) by using the Feynman diagrams.

Both the Feynman rules and Feynman diagrams are shown in Fig. 4. The loop diagrams shown in Fig. 4 can be calculated as follow.
\begin{align*}
- \frac{1}{2g} \delta_{\mu\nu} \left[ \frac{1}{2} \left( \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} \right) - \delta_{ij}\delta_{mn} \right] & \int_{\Lambda \to \Lambda} \frac{d^D p}{(2\pi)^D} \frac{g\delta_{mn}}{p^2} \\
& = - \frac{1}{2g} \delta_{\mu\nu} \left[ \delta_{ij} - \delta_{ij} (N - 1) \right] \int_{\Lambda \to \Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} \\
& = \frac{1}{2} \delta_{\mu\nu}\delta_{ij} (N - 2) \int_{\Lambda \to \Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2},
\end{align*}

and

\begin{align*}
& \frac{1}{2} \delta_{mn} \int \frac{d^D p}{(2\pi)^D} \frac{g\delta_{mn}}{p^2} \\
& = \frac{1}{2} g (N - 1) \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2},
\end{align*}

which match the result shown in Sec. II 4.