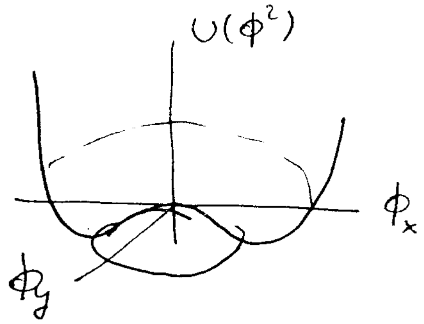
Perturbation Theory and Feynman Graphs

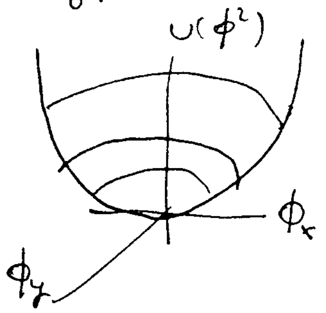
We have seen how to evaluate functional integrals for free fields, i.e. Gaussian models. In general all interesting systems may (or will) ~~be~~ contain interactions. These interactions may appear explicitely or as a result of constraints.

Example: $\lambda\phi^4$ thy: $\mathcal{H} = \frac{1}{2}(\nabla\vec{\phi})^2 + \frac{m_0^2}{2}\vec{\phi}^2 + \frac{\lambda}{4!}(\vec{\phi}^2)^2$

For $m_0^2<0$ the potential looks like (O(U))



while for $m_0^2>0$



We can reparametrize $\mathcal{H}$ as follows.

$$\mathcal{H}(\phi) = \frac{1}{2}(\nabla\vec{\phi})^2 + \frac{\lambda}{4!}(\vec{\phi}^2 - f^2)^2$$

with $2\cdot\frac{\lambda}{4!} f^2 \equiv -\frac{m_0^2}{2}$

If the potential is ∞ deep $\lambda\to\infty$ then the system is effectively constrained $\vec{\phi}^2 = f^2$ ("hard spin")

a rescaling $\vec{\phi} = f\vec{n}$ yields

$$\mathcal{L} = \frac{f^2}{2}(\vec{\nabla}\vec{n})^2 \qquad \text{with the constraint } \vec{n}^2 = 1$$

and $$Z = \int \mathcal{D}n_a \; e^{-\int d^dx \frac{1}{2T}(\vec{\nabla}n_a)^2} \; \delta(\vec{n}^2 - 1)$$

$$f^2 \equiv \frac{1}{T}$$

This model, known as the <u>non-linear sigma model</u>, is infact a continuum approximation to the Heisenberg model of a ferromagnet

$$\mathcal{H}_{\text{Lattice}} = \frac{1}{T}\sum_{\langle\vec{r},\vec{r}'\rangle} \vec{S}(\vec{r})\cdot\vec{S}(\vec{r}') \qquad \vec{S}^2 = 1$$

It also arises in High Energy Physics in the description of phenomena such that chiral symmetry braking, pion physics, etc.

Interactions may arise from the coupling of two fields otherwise free: e.g. QED

$$\mathcal{L}_{QED} = \bar{\Psi}(i\not{\partial} - m)\Psi - e\bar{\Psi}\gamma_\mu\Psi A^\mu - \frac{1}{4}F_{\mu\nu}^2$$

Other examples: Yang-Mills, etc.

I will describe here a technique to generate in a systematic fashion an expansion of physical quantities in powers of a coupling constant.

Example: ϕ^4 theory $m_0^2 > 0$ $(N=1)$

$$\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

The ground state configuration satisfies $\delta\mathcal{H}=0$

$$-\nabla^2\phi + m_0^2\phi + \frac{\lambda}{3!}\phi^3 = 0$$

$$\Rightarrow \quad \phi = \text{const} = 0 \qquad \underline{\text{for } m_0^2 > 0}$$

I'll expand around this ground state. This happens to coincide with an expansion in powers in λ.

$$Z[J] = N\int\mathcal{D}\phi \; e^{-\int \mathcal{L}\, d^dx + \int d^dx\, J\phi}$$

$$N^{-1} = Z[0]$$

$$e^{-\int\mathcal{L}\,d^dx} = e^{-\int[\mathcal{L}_0 + \frac{\lambda}{4!}\phi^4]\,d^dx} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\int d^dx\,\frac{\lambda}{4!}\phi^4(x)\right]^n e^{-\int\mathcal{L}_0}$$

$$\Rightarrow Z[J] = N\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int\mathcal{D}\phi\; e^{-\int(\mathcal{L}_0 - J\phi)}\left[\int d^dx\,\frac{\lambda}{4!}\phi^4(x)\right]^n$$

$$= N\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\lambda}{4!}\right)^n\int\mathcal{D}\phi\; e^{-S_0[J,\phi]}\int d^dx_1\cdots\int d^dx_n\;\phi^4(x_1)\cdots\phi^4(x_n)$$

$$= N\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda}{4!}\right)^n\left(\int\mathcal{D}\phi\; e^{-S_0[J,\phi]}\right)\int d^dx_1\cdots\int d^dx_n\,\langle\phi^4(x_1)\cdots\phi^4(x_n)\rangle_0$$

$$\text{where } \langle A(\phi)\rangle_0 = \frac{\int\mathcal{D}\phi\; A(\phi)\; e^{-S_0[J,\phi]}}{\int\mathcal{D}\phi\; e^{-S_0[J,\phi]}}$$

On the other hand an operator ~~take~~ such as $\phi^4(x_1)\, e^{-S_0[\phi,J]}$ can be ~~gotten~~ obtained as follows

$$\phi^4(x_1)\, e^{-S_0[\phi,J]} = \cancel{\frac{\delta^4}{\delta J^4(x_1)}}\frac{\delta^4}{\delta J^4(x_1)}\, e^{-S_0[\phi,J]}$$

Thus we may write

$$Z[J] = \sum_{n=0}^{\infty}\left(-\frac{\lambda}{4!}\right)^n \frac{1}{n!}\int \mathcal{D}\phi \int dx_1^d \frac{\delta^4}{\delta J^4(x_1)} \cdots \int dx_n^d \frac{\delta^4}{\delta J^4(x_n)}\, e^{-S_0[\phi,J]}$$

by reexponentiating we get (formally)

$$Z[J] = N\, \cancel{Z_0}\, \exp\left\{-\frac{\lambda}{4!}\int dx^d \frac{\delta^4}{\delta J(x)^4}\right\}\int \mathcal{D}\phi\, e^{-S_0[\phi,J]}$$

$$\mathcal{L}_{int}(\phi) = \frac{\lambda}{4!}\int dx^d\, \phi^4(x)$$

$$\Rightarrow \quad Z[J] = N\, \exp\left\{-\int dx^d\, \mathcal{L}_{int}\left[\frac{\delta}{\delta J(x)}\right]\right\}\int \mathcal{D}\phi\, e^{-S_0[\phi,J]}$$

However $\displaystyle \int \mathcal{D}\phi\, e^{-S_0[\phi,J]} = Z_0[J] = Z_0(0) \exp\left\{\frac{1}{2}\int dx_1^d \int dx_2^d\, J(x_1)\, G_0(x_1,x_2)\, J(x_2)\right.$

$$\Rightarrow Z[J] = N\, \exp\left\{-\int dx^d\, \mathcal{L}_{int}\left[\frac{\delta}{\delta J(x)}\right]\right\}\exp\left\{\frac{1}{2}\int dx_1^d\int dx_2^d\, J(x_1)\, G_0(x_1,x_2)\, J(x_2)\right.$$

$Z[0] = 1 \Rightarrow$

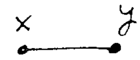
$$N^{-1} = \exp\left\{-\int dx^d\, \mathcal{L}_{int}\left[\frac{\delta}{\delta J(x)}\right]\right\}\exp\left\{\frac{1}{2}\int dx_1^d \int dx_2^d\, J\, G_0\, J\right\}\Bigg|_{J=0}$$

I will use this formula to derive an expansion for $G_2(x_1, x_2)$

$$G_2(x,y) = \langle \phi(x)\, \phi(y) \rangle = \frac{\delta^2 Z[J]}{\delta J(x)\, \delta J(y)}\bigg|_{J=0}$$

$$\Rightarrow \quad G_2(x,y) = N^{-1} \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4!}\right)^n \int dx_1 \cdots \int dx_n \frac{\delta^2}{\delta J(x)\, \delta J(y)} \frac{\delta^4}{\delta J^4(x_1)} \cdots \frac{\delta^4}{\delta J^4(x_n)} Z_0[J]$$

(a) $n = 0$

$$G_2(x,y) = N^{-1} \frac{\delta^2}{\delta J(x)\delta J(y)} Z_0[J]\bigg|_{J=0} = G_0(x,y) =$$ 

$$N^{-1} = 1$$

(b) $n=1$

$$G_2^{(1)} = G_0(x,y) + \cancel{\left(-\frac{\lambda}{4!}\right)\frac{1}{1!}}\, \delta G_2^{(1)}(x,y)$$

$$\delta G_2^{(1)}(x,y) = \left(-\frac{\lambda}{4!}\right)^1 \frac{N_1}{1!} \frac{\delta^2}{\delta J(x)\delta J(y)} \int d^d x_1 \frac{\delta^4}{\delta J^4(x_1)} e^{\frac{1}{2}\int d^d y_1 \int d^d y_2\, J(y_1)\, G_0(y_1,y_2)\, J(y_2)}\bigg|_J$$

$$N_1^{-1} = 1 + \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!} \int d^d x_1 \frac{\delta^4}{\delta J^4(x_1)} e^{\frac{1}{2}\int J G_0 J}\bigg|_{J=0}$$

$$\frac{\delta}{\delta J(x_1)} e^{\frac{1}{2}\int J(y_1)\, G_0(y_1,y_2)\, J(y_2)\, dy_1 dy_2} =$$

$$= \frac{1}{2}\left[\int_{y_1} J(y_1)\, G_0(y_1, x_1) + \int_{y_2} G_0(x_1, y_2)\, J(y_2)\right] e^{\frac{1}{2} J G_0 J}$$

$$\cancel{\frac{\delta^2}{\delta J(x_1)} e^{\frac{1}{2}JG_0J} = \frac{1}{2}\left(G_0(x_1,x_1) + G_0(x_1,x_1)\right) e^{\frac{1}{2}JG_0J} + G_0(x_1,x_1)}$$

$$\frac{\delta^2}{\delta J(x_1)^2} e^{\frac{1}{2} J G_0 J} = \frac{1}{2} \overbrace{\left(G_0(x_1,x_1) + G_0(x_1,x_1)\right)}^{G_0(x_1,x_1)} e^{\frac{1}{2} J G_0 J}$$

$$+ \left[\frac{1}{2} \int_y \left(J(y)\, G_0(y,x_1) + G_0(x_1,y)\, J(y)\right)\right]^2 e^{\frac{1}{2} J G_0 J}$$

$$\frac{\delta^3}{\delta J(x_1)^3} e^{\frac{1}{2} J G_0 J} =$$

$$= G_0(x_1,x_1) \left[\frac{1}{2} \int_y \left(G_0(x_1,y)\, J(y) + J(y)\, G_0(y,x_1)\right)\right] e^{\frac{1}{2} J G_0 J}$$

$$+ \left[\frac{1}{2} \int_y \left(G_0(x_1,y)\, J(y) + J(y)\, G_0(y,x_1)\right)\right]^3 e^{\frac{1}{2} J G_0 J}$$

$$+ 2 \left[\frac{1}{2} \int_y \left(G_0(x_1,y)\, J(y) + J(y)\, G_0(y,x_1)\right)\right] \frac{1}{2} \overbrace{\left(G_0(x_1,x_1) + G_0(x_1,x_1)\right)}^{G_0(x_1,x_1)}$$

$$e^{\frac{1}{2} J G_0 J}$$

$$= 3 \left[\frac{1}{2} \int_y \left(G_0(x_1,y)\, J(y) + J(y)\, G_0(y,x_1)\right)\right] G_0(x_1,x_1)$$

$$e^{\frac{1}{2} J G_0 J}$$

$$+ \left[\frac{1}{2} \int_y \left(G_0(x_1,y)\, J(y) + J(y)\, G_0(y,x_1)\right)\right]^3 e^{\frac{1}{2} J G_0 J}$$

$$\frac{\delta^4}{\delta J(x_1)^4} e^{\frac{1}{2} J G_0 J} =$$

$$= 3 \left(G_0(x_1, x_1) \right)^2 e^{\frac{1}{2} J G_0 J}$$

$$+ 3 \; G_0(x_1, x_1) \left[\frac{1}{2} \int_y \left(G_0(x_1, y) J(y) + J(y) G_0(y, x_1) \right) \right]^2 e^{\frac{1}{2} J G_0 J}$$

$$+ 3 \left[\frac{1}{2} \int_y \left(G_0(x_1, y) J(y) + J(y) G_0(y, x_1) \right) \right]^2 e^{\frac{1}{2} J G_0 J} \frac{1}{2} \left(G_0(x_1, x_1) + G_0(x_1, x_1) \right)$$

$$+ \left[\frac{1}{2} \int_y \left(G_0(x_1, y) J(y) + J(y) G_0(y, x_1) \right) \right]^4 e^{\frac{1}{2} J G_0 J}$$

$$\Rightarrow \left. \frac{\delta^4}{\delta J(x_1)^4} e^{\frac{1}{2} J G_0 J} \right|_{J=0} =$$

$$= 3 \left(G_0(x_1, x_1) \right)^2$$

$$\left. \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(x')} \frac{\delta^4}{\delta J(x_1)^4} e^{\frac{1}{2} J G_0 J} \right|_{J=0} =$$

$$= 3 \left(G_0(x_1, x_1) \right)^2 \frac{1}{2} \left(G_0(x, x') + G_0(x', x) \right)$$

$$+ 3 \; G_0(x_1, x_1) \; 2 \; \frac{1}{2} \left(G_0(x, x_1) + G_0(x_1, x) \right) \frac{1}{2} \left(G_0(x', x_1) + G_0(x_1$$

$$+ 3 \cdot 2 \; \frac{1}{2} \left(G_0(x_1, x) + G_0(x, x_1) \right) \frac{1}{2} \left(G_0(x_1, x_1) + G_0(x_1, x_1) \right)$$

$$\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(x')} \frac{\delta^4}{\delta J(x_1)^4} e^{\frac{1}{2} J G_0 J} \Big|_{J=0} =$$

since $G_0(x, x') = G_0(x', x) \Rightarrow$

$$= 3 \left(G_0(x_1, x_1) \right)^2 G_0(x, x')$$

$$+ 12 \; G_0(x_1, x_1) \; G_0(x, x_1) \; G_0(x_1, x')$$

Note: equivalently we can derive this result by noticing that

$$\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(x')} \frac{\delta^4}{\delta J(x_1)^4} e^{\frac{1}{2} J G_0 J} \Big|_{J=0} =$$

$$\equiv \cancel{\tfrac{4}{8}} \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(x')} \frac{\delta^4}{\delta J(x_1)^4} \frac{1}{6!} \left[\int_{y y'} J(y) \, G_0(y, y') J(y') \right]^6 \Big|_{J=0}$$

since all other terms are necessarily zero. This amounts to an expression in which pairs of points, say x and x' or x with x_1, or x' with x_1, or x_1 with itself, are connected in all possible ways and each connection (or <u>contraction</u>) is represented by a propagator factor.

$$\Rightarrow \delta G_2^{(1)}(x,y) = N\left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!}\left\{ \int d^d x_1 \; 12 \; G_0(x,x_1) G_0(x_1,y) G_0(x_1,x_1) + \right.$$

$$\left. + \; 3 \int d^d x_1 \; [G_0(x_1,x_1)]^2 G_0(x,y) \right\}$$

$$\Rightarrow G_2^{(1)}(x,y) = N\left\{ G_0(x,y) + \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!}\left[\int d^d x_1 \; 12 \; G_0(x,x_1) G_0(x_1,y) G_0(x_1,x_1) + \right.\right.$$

$$\left.\left. + \int d^d x_1 \; 3 \; [G_0(x_1,x_1)]^2 G_0(x,y) \right]\right\}$$

$$N^{-1} = 1 + \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!} \int d^d x_1 \; 3 \; [G_0(x_1,x_1)]^2$$

$\Rightarrow$ up to terms $O(\lambda^2)$

$$N G_0 = G_0(x,y) - G_0(x,y) \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!} \int d^d x \; 3 [G_0(x,x)]^2 + O(\lambda$$

$$\Rightarrow$$

$$G_2^{(1)}(x,y) = G_0(x,y) - \cancel{G_0(x,y) \left(-\frac{\lambda}{4!}\right)^1} \frac{1}{1!} \int d^d x \; 3 [G_0(x_1,x_1)]^2 +$$

$$+ \left(-\frac{\lambda}{4!}\right) \int d^d x_1 \, 12 \;\; G_0(x,x_1) G_0(x_1,y) G_0(x_1,x_1)$$

$$+ \left(-\frac{\lambda}{4!}\right) \int d^d x_1 \;\; 3 \; \cancel{[G_0(x_1,x_1)]^2} \; G_0(x,y) \;\; + O(\lambda^2)$$

$G_0(x_1,x_1)$

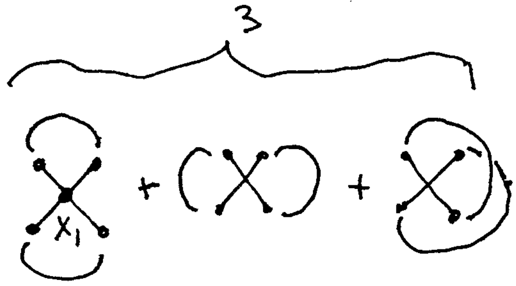
$$G_2^{(1)}(x,y) = G_0(x,y) + \left(-\frac{\lambda}{4!}\right) 12 \int d^d x_1 \; G_0(x,x_1) G_0(x,y) + O(\lambda^2)$$

Example:

(a) vacuum graphs

$n=0$ $\qquad 1$

$n=1$  $\rightarrow \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!}$



contraction  $S \int d^dz \, [G_0(z,z)]^2 \quad \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!}$

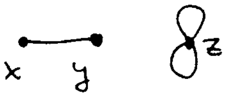
$$S = 3$$

$$N^{-1} = 1 + \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!} \; 3 \int d^dz \, [G_0(z,z)]^2$$

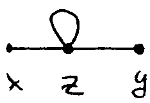
(b) 2-point function.

$n=0$ 

$= G_0(x,y)$

$n=1$ 

$N \; 3 . \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!} \int d^dz \left[G_0(z,z)\right]^2 G_0(x,y)$



$N \; S \; \left(-\frac{\lambda}{4!}\right)^1 \frac{1}{1!} \int d^dz \, G_0(x,z) \, G_0(z,y) \, G_0(z,z)$

$$S = 4 \times 3$$

$$G_2(x,y) = N \left\{ G_0(x,y) + \left(-\frac{\lambda}{4!}\right) 3 \int d^dz \, [G_0(z,z)]^2 G_0(x,y) + \left(-\frac{\lambda}{4!}\right) 4\times 3 \int d^dz \, G_0(x,z) \, G_0(z,y) \, G_0(z,z)\right.$$

$$G_2(x,y) = \left[G_0(x,y) + G_2^{(1)}(x,y) + O(\lambda^2) \right] \left[1 + N_1 + O(\lambda^2) \right]$$

$$G_2(x,y) \cong G_0(x,y) + G_2^{(1)}(x,y) + N_1 \, G_0(x,y) + O(\lambda^2)$$

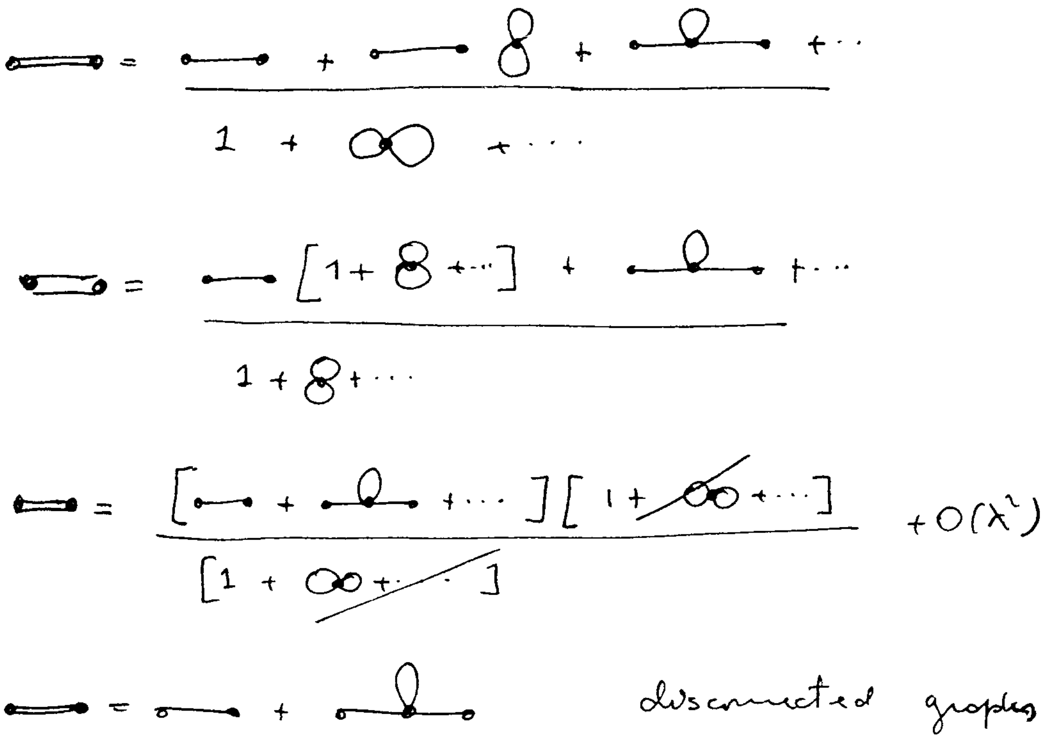
$$N_1 = -\left(-\frac{\lambda}{4!}\right) 3 \int d^dz \, [G_0(z,z)]^2$$

$\cancel{G_2(x,y) = G_0(x,y) + G_2^{(1)}(x,y) +}$

$$G_2(x,y) = G_0(x,y) + \left(\frac{-\lambda}{4!}\right) 4\times 3 \int d^dz\; G_0(x,z)\, G_0(z,y)\, G_0(z,z) +$$

$$+\left(-\frac{\lambda}{4!}\right) 3 \cancel{\int d^dz\; \left(G_0(z,z)\right)^2}\, G_0(x,y) +$$

$$-\left(-\frac{\lambda}{4!}\right) 3 \cancel{\int d^dz\; \left(G_0(z,z)\right)^2}\, G_0(x,y) + O(\lambda^2)$$



disconnected graphs cancel against the denominator.

$$G_2(x,y) = G_0(x,y) + \left(\frac{-\lambda}{4!}\right) 4\times 3 \int d^dz\; G_0(x,z) \overset{G_0(z,z)}{\downarrow} G_0(z,y) + O(\lambda^2)$$

<u>Cancellation of vacuum graphs</u>: The cancellation found in 1st order is not accidental. It happens to all orders.

$$\langle \phi(x_1)\, \phi(x_2) \rangle = \frac{\int \mathcal{D}\phi\; \phi(x_1)\, \phi(x_2)\, e^{-\int \mathcal{L}_0(\phi) - \int \mathcal{L}_{int}(\phi)}}{\int \mathcal{D}\phi\; e^{-\int \mathcal{L}_0(\phi) - \int \mathcal{L}_{int}(\phi)}}$$

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int \mathcal{D}\phi \; e^{-\int \mathcal{L}_0} \phi(x_1)\phi(x_2) \left[\int \mathcal{L}_{int}(\phi)\right]^p}{\text{denom.}}$$

$$= \frac{\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \langle \phi(x_1)\phi(x_2) \left[\int \mathcal{L}_{int}(\phi)\right]^p \rangle_0 \int \mathcal{D}\phi \, e^{-\int \mathcal{L}_0(\phi)}}{\text{denom.}}$$

$$= \frac{\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int dy_1 \cdots dy_p \langle \phi(x_1)\phi(x_2) \mathcal{L}_{int}(\phi(y_1)) \cdots \mathcal{L}_{int}(\phi(y_p)) \rangle_0 \cancel{Z_0}}{\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int dy_1 \cdots dy_p \langle \mathcal{L}_{int}(\phi(y_1)) \cdots \mathcal{L}_{int}(\phi(y_p)) \rangle_0 \cancel{Z_0(0)}}$$

In general

$$\langle \phi(x_1)\phi(x_2) \mathcal{L}_{int}(\phi(y_1)) \cdots \mathcal{L}_{int}(\phi(y_p)) \rangle_0 =$$

$$= \sum_{k=0}^{p} \binom{p}{k} \underbrace{\langle \phi(x_1)\phi(x_2) \mathcal{L}_{int}(\phi(y_1)) \cdots \mathcal{L}_{int}(\phi(y_k)) \rangle_0^{\text{linked}}}_{\text{linked graphs}} \underbrace{\langle \mathcal{L}_{int}(\phi(y_{k+1})) \mathcal{L}_{int}^{} \cdots}_{\text{vac. graphs}}$$

numerator of $\langle \phi(x_1)\phi(x_2)\rangle = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \sum_{k=0}^{p} \binom{p}{k} \int_y \langle \mathcal{L}_{int}(\phi(y_{k+1})) \cdots \mathcal{L}_{int}(\phi(y_p)) \rangle_0 \langle \phi(x_1)\phi(x_2) \cdots$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int dy_1 \cdots \int dy_k \; \langle \phi(x_1)\phi(x_2) \mathcal{L}_1^{int} \cdots \mathcal{L}_k^{int} \rangle_0^{l} \underbrace{\int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle \mathcal{L}_{k+1}^{int} \cdots \mathcal{L}_p^{int} \rangle_0}_{\text{denominator}}$$

$$\Rightarrow \langle \phi(x_1)\phi(x_2) \cdots \phi(x_N)\rangle = \sum_{k=0}^{\infty} \int dy_1 \cdots \int dy_k \frac{(-1)^k}{k!} \langle \phi(x_1) \cdots \phi(x_N) \mathcal{L}_{int}(1) \cdots \mathcal{L}_{int}(k) \rangle_0^{\text{linked}}$$

Rule (Wick's Theorem) ~~Every~~ After all derivatives are done ~~we~~ must set $J=0$

Obviously the only terms to ~~which~~ survive are those with no J dependence.

Every derivative brings down one factor ~~power~~ of J and a propagator factor.

Thus another derivative will be needed to cancel the factor J. Thus all derivatives come in pairs (contractions) in all possible ways and for each pair one assigns a prop. factor.

Feynman Graphs:

For computing $G_2(x,y)$ at order n-th in P.T. we must proceed ~~do a~~ as follows: a graph consists of $N\,(=2)$ external points and n internal vertices

(i) assign a factor $\left(-\frac{\lambda}{4!}\right)^n \frac{1}{n!}$

~~draw all possible pairwise contractions and~~

(ii) assign to every contraction a line (propagator) $G_0(x_1,x_2)$

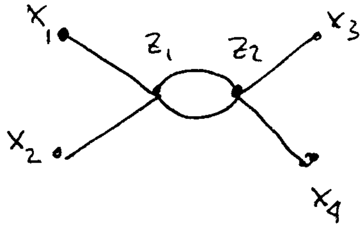
(iii) using those lines draw all possible contractions between the internal vertices themselves and between them and the external points x, y (or legs)

(iv) Integrate over all the coordinates of the internal vertices

(v) Rule (iii) can be simplified by ~~provided one~~ drawing ~~only~~ graphs of different topology provided one multiplies each ~~such~~ graph by its multiplicity S (symmetry factor)

(vi) multiply the above contribution by N (the vacuum graphs) calculated up to the same order

Example: 2nd order graph for 4 point function.



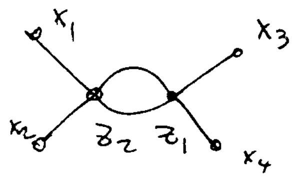
$$S\cdot\left(-\frac{\lambda}{4!}\right)^2 \frac{1}{2!}\int d^d z_1 \int d^d z_2\; G_0(x_1,z_1)\,G_0(x_2,z_1)$$

$$G_0(x_3,z_2)\,G_0(x_4,z_2)\quad \left(G_0(z_1,z_2)\right)^2$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{G_0(z_1,z_2)\,G_0(z_2,z_1)}$$

$$S = (4\times 3)^2\times 2$$

note that



is not topologically equivalent but gives the same result if $G_0(z_1,z_2)=G_0(z_2,z_1)$

Feynman Rules in Momentum space

If $\mathcal{L}$ is translationally invariant we can F.T.

$$\phi(x) = \int_{|\vec{k}|<\Lambda} \frac{d^d k}{(2\pi)^d}\, e^{-ik\cdot x}\,\phi(k)$$

$$J(x) = \int \frac{d^d k}{(2\pi)^d}\, e^{-ik\cdot x}\, J(k)$$

$$Z_0(J) = \int \mathcal{D}\phi(k)\,\exp\left\{-\frac{1}{2}\int\frac{d^d k}{(2\pi)^d}\left[\phi(k)\phi(-k)(k^2+m_0^2) - J(k)\,\phi(-k)\right]\right.$$

$$\int \mathcal{L}_{int} = \frac{\lambda}{4!}\int\frac{dk_1^d}{(2\pi)^d}\cdots\int\frac{dk_4^d}{(2\pi)^d}\,(2\pi)^d\,\delta^d\!\left(\sum_{i=1}^4 \vec{k}_i\right)\phi(k_1)\cdots\phi(k_4)$$

and

$$G_N(x_1,\dots x_N) = \int\frac{dp_1^d}{(2\pi)^d}\cdots\int\frac{dp_N^d}{(2\pi)^d}\, e^{-i\sum_{j=1}^N \vec{p}_j\cdot\vec{x}_j}\; G_N(p_1\cdots p_N)$$

$$G_N(p_1 \cdots p_N) = \langle \phi(p_1) \cdots \phi(p_N) \rangle =$$

$$= \frac{1}{Z[J]} \frac{\delta^N Z[J]}{\delta J(-p_1) \cdots \delta J(-p_N)} \Bigg|_{J=0}$$

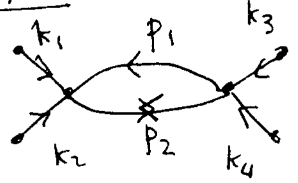
Translation invariance $\Rightarrow$ $G_N(p_1 \cdots p_N) = (2\pi)^d \delta^d\left(\sum_{i=1}^{N} p_i\right) \bar{G}_N[p]$

$$Z[J] = N \exp\left\{ -\frac{\lambda}{4!} \int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_4}{(2\pi)^d} (2\pi)^d \delta\left(\sum_{i=1}^{4} k_i\right) \frac{\delta}{\delta J(-k_1)} \cdots \frac{\delta}{\delta J(-k_4)} \right\}$$

$$\times \exp\left\{ \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} J(p) \, G_0(p) \, J(-p) \right\}$$

$$G_0(p) = \frac{1}{p^2 + m_0^2}$$

Example:



$$\left(-\frac{\lambda}{4!}\right)^2 \frac{1}{2!} (4 \times 3)^2 \times 2 \times 2 \; G_0(k_1) G_0(k_2) G_0(k_3) G_0(k_4) \times$$

$$\times \int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} (2\pi)^d \delta^d(k_1 + k_2 + p_1 + p_2) (2\pi)^d \delta(k_3 + k_4 - p_1 - p_2) G_0(p_1) G_0$$

$$\equiv (2\pi)^d \delta^d(k_1 + k_2 + k_3 + k_4) \left\{ \left(-\frac{\lambda}{4!}\right)^2 \frac{1}{2!} \right\} S \times 2 \; G_0(k_1) \cdots G_0(k_4) \times$$

$$\times \int \frac{d^d p_4}{(2\pi)^d} G_0(p) \, G_0(k_1 + k_2 - p)$$

Thus the rules are very similar to those in position space.

ⓐ Construction of graphs: a general graph has N external ~~legs~~ points from each of which emanates a line (a leg) labelled $\vec{k}_i$ and n_r vertices of type r, represented by points from which r lines emanate labelled $\vec{q}_1 \cdots \vec{q}_r$ (one such set per vertex). All lines are connected pair-wise, indices of a paired couple of the are identical. No vacuum parts should be considered

(b) ~~for~~ every vertex of type r there's a factor $(2\pi)^d \left(-\frac{\lambda}{r!}\right) \delta(\Sigma q_i$ where all the q_i emanate from that vertex

For every line labelled $\vec{q}$ a factor $G_0(\vec{q})$

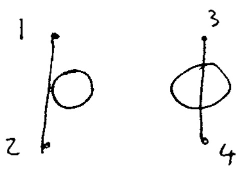
a multiplicity factor

sum (integrate) over all internal momenta and indices

L.4 Connected and disconnected GF's

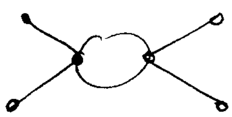
Suppose I want to compute $G_4(x_1, x_2, x_3, x_4)$. Obviously there's a set of graphs s.t. $G_4 \sim G_2(x_1 x_2)\, G_2(x_3, x_4) +$ other combinations

e.g



note: this graph is linked (i.e. it has no vacuum part) however it is disconnected.

unlike



We saw that

$$G_N(x_1,\dots,x_N) = \frac{1}{Z[J]} \frac{\delta^{(N)} Z[J]}{\delta J(x_1)\cdots\delta J(x_N)}\Bigg|_{J=0}$$

Let's compute

$$G_N^c(x_1,\dots,x_N) = \frac{\delta^{(N)}}{\delta J(x_1)\dots\delta J(x_N)} \ln Z[J]\Bigg|_{J=0}$$

example

$$G_2^c(x_1,x_2) = \frac{\delta^{(2)}}{\delta J(x_1)\,\delta J(x_2)} \ln Z[J]\Bigg|_{J=0} =$$

$$= \frac{\delta}{\delta J(x_1)} \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(x_2)}\Bigg|_{J=0} =$$

$$= \frac{1}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\Bigg|_{J=0} - \frac{1}{Z^2[J]} \frac{\delta Z}{\delta J(x_1)}\Bigg|_{J=0} \frac{\delta Z}{\delta J(x_2)}\Bigg|_{J=0}$$

$$\Rightarrow \quad G_2^c(x_1,x_2) = G_2(x_1,x_2) - G_1(x_1)\, G_1(x_2)$$

or $\langle\langle \phi(x_1)\,\phi(x_2)\rangle\rangle \equiv \langle \phi(x_1)\,\phi(x_2)\rangle_c = \langle\phi(x_1)\,\phi(x_2)\rangle - \langle\phi(x_1)\rangle\langle\phi$

or $\langle\phi(x_1)\,\phi(x_2)\rangle_c = \langle [\phi(x_1) - \langle\phi(x_1)\rangle]\,[\phi(x_2) - \langle\phi(x_2)\rangle]\rangle$

is called the connected green's function

The generating functional of the <u>connected</u> green's function

is the "Free energy" (or vacuum energy) $F[J]$

$F[J] \equiv \ln Z[J]$

$$G_N^c(x_1,\dots,x_N) = \frac{\delta^N F[J]}{\delta J(x_1)\cdots\delta J(x_N)}\Big|_{J=0}$$

are the "cumulants" or connected G.F.

Remember $J(x) \equiv J \equiv H$ is the magnetic field (e.g. in the Landau theory of magnetism)

$$\frac{\delta F}{\delta J} \equiv \frac{dF}{dH} = \left\langle \int d^dx\, \phi(x) \right\rangle = \int d^dx \langle \phi(x)\rangle = V\langle \phi\rangle .$$

$$\Rightarrow f = \frac{F}{V} \Rightarrow \frac{df}{dH} = \langle\phi\rangle = M \quad \text{magnetization.}$$

$$\frac{d^2f}{dH^2} = \frac{1}{V}\frac{d}{dH}\left\langle \int d^dx_1\, \phi(x_1)\right\rangle =$$

$$= \frac{1}{V}\left\langle \int d^dx_1 \int d^dx_2\, \phi(x_1)\phi(x_2)\right\rangle - \frac{1}{V}\left\langle \int d^dx_1\, \phi(x_1)\right\rangle \left\langle \int d^dx_2\, \phi(x_2)\right\rangle$$

$$\chi = \frac{d^2f}{dH^2} = \frac{1}{V}\int dx_1 \int dx_2\, G_2(x_1,x_2) - \frac{1}{\cancel{V}} V^{\cancel{2}}\langle\phi\rangle^2$$

$$\chi = \frac{1}{\cancel{V}}\cancel{V}\int dy\, G_2(y) - V\langle\phi\rangle^2$$

$$\chi = \int d^dy\, G_2(|\vec{y}|) - V\langle\phi\rangle^2$$

or $\chi = \int d^dy\, G_2^c(|\vec{y}|)$ suceptibility