What we showed is that

$$\pi_2(G/H) = \pi_1(H)$$

For \( H \cong U(1) \), \( \pi_1(U(1)) = \mathbb{Z} \)

One example is the \( \mathbb{CP}^{N-1} \) model we discussed before whose target space is the coset \( \frac{SU(N)}{SU(N-1) \otimes U(1)} \)

which have instantons for all \( N \).

Instantons in non-Abelian Gauge Theories

We now discuss what happens in the case of non-Abelian gauge theory.

Let us first construct the analogy of the vortex. It is the Dirac monopole. The vortex was regarded as a single solution of the phase field \( \Theta(x) \). These solutions were multi-valued, and had branch cuts. Dirac proposed to look at
configuration of electric and magnetic fields created by an infinitely long (semi-infinite) solenoid of infinitesimal thickness. The magnetic field "radiating" out of the end of the solenoid looks as if there was a magnetic charge at that point. From magnetostatics we know that the equations are

\[ \nabla \times \vec{B} = 0 \quad \text{and} \quad \nabla \cdot \vec{B} = 0 \]

\[ \Rightarrow \vec{B} = \nabla \times \vec{A}, \quad \vec{B} \propto \frac{1}{2} \vec{E} \times \vec{H} \times F_{\mu \nu} \]

These equations do not have monopoles.

But consider now the configuration

\[ B_\mu(x) = \frac{q}{2} \frac{\delta_\mu_3}{|x|^3} \]

It represents a magnetic charge \( q \) at
the origin and "string" of magnetic flux with equal and opposite flux, a Dirac string.

\[ \delta(x) \]

the D = 2 vortex \( A_\mu = \Phi_\mu \theta = \sum_{\mu} E_{\mu \nu} \frac{xe^{-2\pi i n \theta(x)}}{x^2} \)

We also saw that vortices arise as solution of some gauge theories with matter fields ("spontaneously broken"). The analogy here is the 't Hooft--Polyakov monopole which arises in the Georgi--Glashow model of weak interactions:

\[ S = \int d^3x \left( \frac{1}{2} (D_\mu \Phi)^2 - \frac{1}{2} m_0^2 \Phi^2 + \frac{\lambda (\Phi^2)^2}{4!} \right) + \frac{1}{4} \text{Tr} F_{\mu \nu}^2 \]

(Euclidean !)

for D = 3 (Euclidean) dimensions.

Here \( \Phi = (\Phi_1, \Phi_2, \Phi_3) \) is a triplet (in the adjoint rep. of SU(2)) and
The gauge group is $SU(2)$

$$F^{a}_{\mu\nu} = \partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a} + g \, \varepsilon^{abc} A_{\mu}^{b} A_{\nu}^{c}$$

$$D_{\mu} \phi^{a} = \partial_{\mu} \phi^{a} + g \, \varepsilon^{abc} A_{\mu}^{b} \phi^{c}$$

(1) Using

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + g \, A \times A$$

$$D_{\mu} \phi = \partial_{\mu} \phi + g \, A \times \phi$$

Here we are interested in classical solutions.

of the Euclidean 2+1-dimensional theory, which are the same as the static finite energy classical solution of the Minkowski theory in 3+1.

$c, j = 1, 2, 3$

$a, b, c = 1, 2, 3$

\[ \Phi: \text{Equations of Motion:} \]

1. $D_{i} F_{i}^{a} = g \, \varepsilon^{abc} (D_{i} \phi^{b}) \phi^{c}$

and

2. $D_{i} D_{i} \phi^{a} = - \lambda (\phi^{b} \phi^{b}) \phi^{a} + \lambda f^{2} \phi^{a}$

(Where $\lambda f^{2} = |m_{0}|$)

The energy (or action, depending on the case) is

$$E = \int d^{3}x \left[ \frac{1}{4} F_{i}^{a} F_{i}^{a} + \frac{1}{2} D_{i} \phi^{a} D_{i} \phi^{a} + \frac{\lambda}{4} (\phi^{a} \phi^{a}) \right]$$
zero-energy solutions: the classical vacuum.

\[ A_i^a (x) = 0 \quad \phi^a (x) \phi^a (x) = f^2 \quad \text{and} \quad D_i \phi^a = 0 \]

which in this case \( \Rightarrow D_i \phi^a = 0 \Rightarrow \phi^a = \text{const.} \)

Finite action solutions: the action will be finite provided the fields approach sufficient fast the vacuum solution as \( r \to \infty \). \((|r^2| = r) \)

\[ \Rightarrow \quad r^{3/2} D_i \phi^a \to 0 \quad (r \to \infty) \]

and \( \phi^a \phi^a \to f^2 \quad (r \to \infty) \)

In spherical coordinates \((r, \theta, \varphi)\)

\[ (D \phi^a)_\theta = \frac{1}{r} \frac{\partial \phi^a}{\partial \theta} + g^{abc} A^b_{\theta} \phi^c \]

\[ \Rightarrow \quad (D \phi^a)_\theta \to 0 \quad \text{provided} \quad A^b_{\theta} \sim \frac{1}{r} \quad (r \to \infty) \]

(The same apply to the other components)

Also if \( A \sim \frac{1}{r} \Rightarrow F \sim \frac{1}{r^2} \) and \( F^2 \sim \frac{1}{k^4} \)

which is integrable in three dimensions.

\( \Rightarrow \) unlike the case of the non-linear \( \phi \)-model.

Finite action solutions may have fields \( \phi^a \) which are not equivalent at \( \infty \) via
they may point in different directions. Hence the boundary conditions we just derived imply that at the boundary, the surface of a large 2-sphere $S_2$ of large radius $R$, the fields $\phi^a$ may point in any direction of their target space $S_2 \rightarrow \text{target}$. The asymptotic fields are mappings of $S_2 \rightarrow S_2^{\text{target}}$ and we know that these mappings can be classified in topological classes of $\text{H}_2(S_2) = \mathbb{Z}$. The topological charge $Q$ used to classify the instantons of the D=2 $O(3)$ non-linear sigma-model, can now be used as the topological charge for the field configuration of this gauge theory. For instance, the instanton with $Q=1$ has a $\phi^a$ configuration which asymptotically is a "hairy ball" or a
How is this related to a monopole? The picture we just discussed is gauge-dependent. There is a gauge-invariant way to look at this due to 't Hooft. Define

\[ F_{\mu \nu} = \hat{\phi}^a F_{\mu \nu}^a - \frac{1}{g} \epsilon^{abc} \phi^b \phi^c D_\mu \phi^d D_\nu \phi^c \]

\[ (\hat{\phi}^a = \phi^a / ||\phi||) \]

which is gauge-invariant and for \( \hat{\phi}^a = (0,1) \) reduces to

\[ F_{\mu \nu} = \partial_\mu A^3 - \partial_\nu A^3 \]

Let us compute \((D=4)\)

\[ F_{\mu \nu}^* = \frac{1}{2} \epsilon_{\mu \nu \lambda \sigma} F_{\lambda \sigma} \quad (D=4) \]

\[ F_{\mu}^* = \frac{1}{2} \epsilon_{\mu \nu} F_{\nu \lambda} \quad (D=3) \]
If \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) (as in Maxwell's)

\[ \partial^\mu F^*_{\mu} = 0 \quad \text{and} \quad \partial^\mu F^*_{\mu \nu} = 0 \]

\((D=3)\) \hspace{1cm} \((D=4)\)

This is the Bianchi Identity which means that there are no monopoles. \((\mathcal{B} \neq 0)\)

But \((D=4)\)

\[ \partial^\mu F^*_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \lambda \rho} \partial^\lambda F^{\rho} \delta = \frac{1}{2g} \varepsilon_{\mu \nu \lambda \rho} \varepsilon^{a b c} \delta \phi^a \delta \phi^b \delta \phi^c \]

\[ = \frac{4\pi}{4\pi} \frac{4\pi}{g} \int \rightarrow \delta^\mu M \]

topological current

\(D=3 \Rightarrow \)

\[ \partial^\mu F^*_{\mu} = \frac{1}{2} \varepsilon_{\mu \nu \lambda} \partial^\mu F_{\nu \lambda} \]

\[ = \frac{4\pi}{g} \int \mathcal{J}_0 (x) \]

\(\text{where}\)

\[ \int d^3x \mathcal{J}_0 (x) = \frac{1}{8\pi} \int d^3x \varepsilon_{ij k} E^{a b c} \varepsilon \hat{\phi}^i \hat{\phi}^j \hat{\phi}^k \]

\[ = \frac{1}{8\pi} \int d^3x \varepsilon_{ij k} E^{a b c} \varepsilon \hat{\phi}^i \hat{\phi}^j \hat{\phi}^k \]

\[ = \frac{1}{8\pi} \int S_t \varepsilon_{ij k} E^{a b c} \hat{\phi}^i \hat{\phi}^j \hat{\phi}^k \]

\[ = 0 \]
\( \Rightarrow \) \( D=3 \) we get that

\[ \partial^\mu \mathbf{F}_\mu = \text{topological charge density} \]

\[ \equiv \frac{4\pi}{g} j_0(x) \]

\[ Q = \int d^3x \ j_0(x) \]

In particular,

and \( D=4 \Rightarrow \partial^\mu \mathbf{F}_\mu = \frac{4\pi}{g} j_\mu \]

\( j_\mu \) is the topological current

\[ j_\mu = \frac{1}{8\pi} \varepsilon_{\mu \nu \lambda \sigma} E_{\lambda \sigma} \partial^\nu \phi \partial^\mu \phi \]

\( \Rightarrow D=3 \) monopoles are "point-like" objects (instantons) whereas in \( D=4 \) they are finite energy solutions and hence define 4-dimensional currents.

If we define the magnetic field

\[ B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk} \]

\[ \Rightarrow \partial_i B_i = \frac{4\pi}{g} j_0(x) \]

\[ \Rightarrow \int d^3x \ \frac{1}{g} j_0(x) = m = \text{magnetic charge} = \frac{Q}{g} \]
The solution for $Q=1$ is of the form:

$$
\phi^a(x) = \delta^{ia} \frac{x^i}{r} F(r)
$$

$$
A^a_i(x) = \epsilon_{abc} \frac{x^b}{r} W(r)
$$

such that:

$$
F(r) \rightarrow f \quad \text{as} \quad r \rightarrow \infty
$$

$$
W(r) \rightarrow \frac{1}{gr} \quad \text{as} \quad r \rightarrow \infty
$$

$$
\Rightarrow \quad \vec{B}(x) \rightarrow \frac{x}{gr^3} \quad \text{i.e. a magnetic pole of strength} \ \frac{e}{\mu}
$$

The field equations become:

$$
K(r) \equiv 1 - gr W(r) \equiv K
$$

$$
H(r) \equiv gr F(r) \equiv H
$$

$$
\frac{r^2 d^2 K}{dr^2} = K (K^2 - 1) + H^2 K
$$

$$
\frac{r^2 d^2 H}{dr^2} = 2 H K^2 + \lambda H \left( \frac{H^2}{g^2} - r^2 f^2 \right)
$$

which are quite difficult to solve. However, in the limit $\lambda \rightarrow 0$ (BPS) (Bogomolnyi, Prasad, Sommerfeld):

$$
\Rightarrow \quad K(r) = \frac{rg f}{\sinh(gfr)} \quad , \quad H(r) = \frac{gr f}{\tanh(gfr)} - 1
$$
It can be shown (see Rajaraman) that these solutions satisfy (and saturate) the bound
\[ E \geq \frac{4\pi G f}{g} \]
and satisfy the Bogomolnyi Equation.

\[ F_{ij} = \varepsilon_{ijk} D_k \phi^a \]

**Yang-Mills in D=4**

We summarize the discussion of monopoles with some general conditions for instanton solutions in D=4 (Euclidean) Yang-Mills gauge theory (without matter).

Let \( G \) be a gauge group and consider the YM action (Euclidean)
\[ S = \frac{1}{4} \int d^4x \, tr F^2 \]
\( \text{faster than} \)

\[ \Rightarrow S < \infty \text{ if } F_{\mu \nu} \sim O\left(\frac{1}{r^2}\right) \quad r \to 0 \]

\[ \Rightarrow A_\mu \sim g^{-1} \phi \gamma^\mu + O\left(\frac{1}{r}\right) \quad r \to \infty \]

\( \text{gauge transformation} \)
where the gauge transfer manifold $g(x) \in G$

$\Rightarrow$ at the boundary of $\mathbb{R}^4$, which is $S_3$
a large 3-sphere of radius $R \to \infty$

the field configurations are parametrized by $g(x) \in G$, the gauge transformations

which is an element of $G$. Thus the
configurations are in a one-to-one correspondence with the mappings

$g(x) : S_3 \rightarrow G$

For instance, if $G = SU(2) \Rightarrow g \in SU(2)$

$g = n_4 + i \overrightarrow{n} \cdot \sigma / n_4^2 + n^2 = 1$

$g^* g = I \Rightarrow \text{this is the sphere } S_3$

$\Rightarrow$ we have the maps $g : S_3 \rightarrow S_3$

The topological charge or winding number
is now called the Pontrjagin index which
measures the number of times $S_3$ covers $S_3$

$= \text{integral of the Jacobian of the map } g(x)$
In general, \( g \in G \), we define

\[
Q = \frac{1}{24 \pi^2} \int_{S_3} \epsilon_{\mu \nu \lambda} \text{Tr} \left( L_\mu L_\nu L_\lambda \right)
\]

\[
L_\mu (x) = g^{-1} \partial_\mu g \quad (a, b, c, d = 1, \ldots, n)
\]

\[
= \frac{1}{12 \pi^2} \int_{S_3} \epsilon^{abcd} \epsilon_{\mu \nu \lambda} (n^a \partial_\mu h^b \partial_\nu h^c \partial_\lambda h^d)
\]

It can be shown (see Polyakov) that

\[
Q = \frac{1}{32 \pi^2} \int d^4 x \; \epsilon^{\mu \nu \lambda \sigma} \text{Tr} \left( F_{\mu \nu} F_{\lambda \sigma} \right)
\]

\[
= \frac{1}{8 \pi^2} \int d^4 x \; \text{Tr} \left( F \wedge F \right)
\]

\[
\Rightarrow Q = \frac{1}{8 \pi^2} \int d^4 x \; \text{Tr} \left( F_{\mu \nu} F_{\mu \nu}^* \right) \quad \text{Poincaré Index}
\]

Thus works for all compact simply connected gauge groups.
\[ S = \frac{1}{4g^2} \int d^4x \, \text{Tr} \, F_{\mu \nu}^2 \]
\[ = \frac{1}{8g^2} \int d^4x \, \text{Tr} \left( F_{\mu \nu} - F_{\mu \nu}^* \right)^2 + \frac{1}{4g^2} \int d^4x \, \text{Tr} \left( F_{\mu \nu} F_{\mu \nu}^* \right) \]
\[ \Rightarrow S = \frac{8 \pi^2 g^2}{g^2} + \frac{1}{8g^2} \int d^4x \, \text{Tr} \left( F_{\mu \nu} - F_{\mu \nu}^* \right)^2 \]
\[ \Rightarrow S \geq \frac{8 \pi^2 g^2}{g^2} \]

We can saturate the bound above if we require \( F_{\mu \nu} = F_{\mu \nu}^* \) (self-dual solution) and \( F_{\mu \nu} = F_{\mu \nu}^* \) ("Candy-Reimann")

Also if \( F = F^* \) \( \Rightarrow \) \( D_{\mu} F_{\mu \nu} = 0 \) an solution.

\( Q = 1 \) solution (for \( SU(2) \))

\[ A_\mu = -\frac{\eta_{\alpha \mu \nu} \left( x_\nu - a_\nu \right) \left( x_\alpha - a_\alpha \right)^2 + s^2}{\left( x_\alpha - a_\alpha \right)^2 + s^2} \]

\( s \): scale
\( a_\mu \): location
\( \eta_{\alpha \beta \gamma} = \varepsilon_{\alpha \beta \gamma} \) and \( \eta_{\alpha 0 0} = \delta_{\alpha 0} \) etc.

\[ F_{\mu \nu} = -4 \frac{\eta_{\alpha \mu \nu} s}{\left( x_\alpha - a_\alpha \right)^2 + s^2} \]
Instantons and the double-well potential

Let's consider a quantum mechanical double well problem

\[ L = \frac{1}{2} x^2 + \frac{\lambda}{2} x^2 - \frac{\lambda}{4} x^4 \]

(\( \lambda \) small)

Classically there are two degenerate ground states

\[ x_\pm = \pm \left( \frac{\lambda}{\lambda} \right)^{1/2} \]

and the \( \pm \) symmetry is (classically) broken.

In p.thy, \( x = + \left( \frac{\lambda^2}{\lambda} \right)^{1/2} + y \) (\( y \) small)

and nothing dramatic happens (the degeneracy is not lifted);

\[ \langle x(0) x(t) \rangle \rightarrow \text{constant} \]

Wick rotation \( t = \tau \rightarrow i \tau \)

\[ \langle x(0) x(\tau) \rangle \rightarrow \sum_n l_n \exp^{-\left( E_n - E_0 \right) \tau} \rightarrow e^{-\Delta E \tau} \]

\( \tau \rightarrow \infty \)

splitting.
\[
\langle x(\tau) x(\tau) \rangle = \frac{\int dx(\tau) e^{-E[x]} x(\sigma) x(\tau)}{\int dx(\tau) e^{-E[x]}}
\]

\[
E[x] = \int_{-\infty}^{+\infty} d\tau \left[ \frac{1}{2} \left( \frac{dx}{d\tau} \right)^2 - \frac{\mu^2 x^2}{2} + \frac{\lambda}{4} x^4 \right]
\]

Q.M. \iff 1D L.S.M.

Extrema of \( E[x] \): \( \delta E = 0 \Rightarrow \bar{x}(\tau) \)

\[
E(x) = E(\bar{x}(\tau)) + \int \frac{\delta E}{\delta x} \delta x + \frac{1}{2} \int \frac{\delta^2 E}{\delta x \delta x}, \delta x \delta x
\]

\[
\frac{\delta E}{\delta x} \bigg|_{x = \bar{x}} = 0
\]

\[
\delta x = x(\tau) - \bar{x}(\tau)
\]

\[
0 = \frac{\delta E}{\delta \bar{x}} - \frac{d}{d\tau} \frac{\delta E}{\delta \dot{x}}
\]

(Euler-Lagrange)

\[
\frac{\delta E}{\delta \dot{x}} = \ddot{x}, \quad \frac{\delta E}{\delta \bar{x}} = -\mu^2 \bar{x} + \lambda \bar{x}^3
\]

\[
\frac{d^2 \bar{x}}{d\tau^2} = -\mu^2 \bar{x} + \lambda \bar{x}^3
\]

Eq. of Mot. of Classical particle in pot. \( V(x) \)

\[
V(x) = \frac{\mu^2 \bar{x}^2}{2} - \frac{\lambda}{4} \bar{x}^4 = -U(x)
\]

\[
E = \frac{1}{2} \left( \frac{d\bar{x}}{d\tau} \right)^2 - V(\bar{x}) \text{ is "conserved"}
\]
(1) \( \tilde{X} = \pm \left( \frac{\mu^2}{\lambda} \right)^{1/2} \) is a pair of deg. solutions.

\[
E(\tilde{X}) = T \left( \frac{\lambda}{4} \tilde{X}^4 - \frac{\mu^2}{2} \tilde{X}^2 \right) = T \left[ \frac{\lambda}{4} \left( \frac{\mu^2}{\lambda} \right)^{1/2} - \frac{\mu^2}{\lambda} \right]
\]

\[
E(\tilde{X}) = - \frac{\mu^4}{4\lambda} T
\]

(2) \( E = 0 \)

\[
\frac{d\tilde{X}}{d\tau} = \sqrt{2V(\tilde{X})}
\]

\[
\tilde{X}(\tau) = \pm \left( \frac{\mu^2}{\lambda} \right)^{1/2} \tanh \frac{\mu (\tau - \tau_0)}{\sqrt{2}}
\]

\[
E(\tilde{X}(\tau)) - E(\left( \frac{\mu^2}{\lambda} \right)^{1/2}) = 2\sqrt{2} \frac{\mu^3}{\lambda} \quad \text{finite}
\]

\( \tau \) the solution \( \tilde{X}(\tau) \) has a contrib. \( e^{-2\sqrt{2} \frac{\mu^3}{\lambda} \tau} \)

\( \exists \) it is important if \( \tau > 2\sqrt{2} \frac{\mu^3}{\lambda} \)

Stability:

\[
\frac{1}{2} \int d\tau \int d\tau' \frac{\delta^2 E}{\delta \tilde{X}(\tau) \delta \tilde{X}(\tau')} \bigg|_{\tilde{X}}
\]

\[
\frac{\delta^2 E}{\delta \tilde{X}(\tau) \delta \tilde{X}(\tau')} \bigg|_{\tilde{X}} = - \frac{d^2}{d\tau^2} \delta (\tau - \tau') \delta (\tilde{X}(\tau) - \tilde{X}(\tau'))
\]

\[
= \left[ - \frac{d^2}{d\tau^2} + \left( -\mu^2 + 3\lambda \tilde{X}^2(\tau) \right) \right] \delta (\tau - \tau')
\]
Let \( y_n(t) \) be a complete set of eigenstates of the operator

\[
- \frac{d^2}{dt^2} + (-\mu^2 + 3\lambda \bar{x}_c(t)) \tag{1}
\]

satisfying

\[
y(-\infty) = y(+\infty) = 0
\]

\[
\left[ - \frac{d^2}{dt^2} + \omega^2 + 3\lambda \bar{x}_c(t) \right] y_n(t) = \omega_n^2 y_n(t)
\]

\( \Rightarrow \) eigenvalues.

\( \Rightarrow \bar{x}_c(t) \) is stable iff \( \omega_n \geq 0 \)

Try

\[
x(t) = \bar{x}_c(t) + \sum_n \xi_n y_n(t) \tag{2}
\]

\( \xi_n \) arbitrary.

\( \Rightarrow \delta x(t) = N \prod_n d\xi_n \)

\[
\langle x(0) x(t) \rangle = e^{-\mathcal{E}[\bar{x}_c(t)]} \frac{\bar{x}_c(0) \bar{x}_c(t)}{1 + e^{-\mathcal{E}[\bar{x}_c(t)]}} \prod_n \omega_n^{-1}(\bar{x}_c) \tag{3}
\]

\( \mathcal{E}[\bar{x}_c] \) is independent of \( \bar{x}_c \).

Problem:

\[
\det \left( - \frac{d^2}{dt^2} + (-\mu^2 + 3\lambda \bar{x}_c(t)) \right) = \prod_n \omega_n^{-1}(\bar{x}_c)
\]

Vanishes because \( \omega_0 = 0 \) !

Symmetry: \( \bar{x}_c(t) \) has a zero-mode \( a \) (origin)

\( \mathcal{E}[\bar{x}_c] \) is independent of \( a \).

\( \Rightarrow \frac{\delta \mathcal{E}(\bar{x}_c)}{\delta a} = 0 \)
\[
\Rightarrow \frac{\delta}{\delta \bar{x}(\tau')} \frac{\delta \mathcal{E}}{\delta \bar{x}(\tau)} \frac{d\bar{x}(\tau)}{da} = 0 \Rightarrow \frac{d\bar{x}(\tau)}{da} \text{ is a zero-energy mode.}
\]

\[y_0 = \frac{d\bar{x}(\tau)}{da} \iff \omega^2_0 = 0 \quad \text{no restoring force.}\]

\(a: \text{Collective Coordinate. Needed to deal with it exactly.}\)

But \[
\begin{cases}
\mathcal{E}[x] = \mathcal{E}[x_a] \\
\delta x = \delta x_a \quad \text{(invariant measure)}
\end{cases}
\]

Introduce "\(\mathcal{I}\)"; let \(F(x)\) be a function

\[
1 = \int dF \delta(F(x_a)) = \int_{-\infty}^{+\infty} da \delta(F(x_a)) \frac{\partial F(x_a)}{\partial a} e^{-\mathcal{E}[x]}
\]

\[
\Rightarrow Z = \int \delta x e^{-\mathcal{E}[x]} = \int \delta x \int_{-\infty}^{+\infty} da \delta(F(x_a)) \frac{\partial F(x_a)}{\partial a} e^{-\mathcal{E}[x_a]}
\]

Change of variables: \(x \to x_a\) or \(x(\tau) \to x(\tau - a)\)

\[
\frac{\partial F(x_a)}{\partial a} = D[x, a]
\]

\[
Z = \int \delta x_a \int_{-\infty}^{+\infty} da \delta(F(x)) \ D[x_a, a] \ e^{-\mathcal{E}[x_a]}
\]

\(\langle x_a \rangle \rangle_a = x_0 \equiv x\)

Using translation invariance and the invariance of the measure.
\[ Z = \int dx \int da \quad D(x - a, a) \quad \delta(F(x)) \quad e^{-\frac{3}{2}[x]} \]

Let's calculate \( D \)

\[ D \quad \equiv \quad D[x - a, a] \]

\[ D[x - a, a] = \int_{-\infty}^{+\infty} dx \quad \frac{\partial F[x]}{\partial a} \quad \frac{\partial F[x]}{\partial a} \quad a = 0 \]

\[ F[k_a] = \int_{-\infty}^{+\infty} dx \quad \frac{\partial F[x]}{\partial a} \quad \frac{\partial F[x]}{\partial a} \quad a = 0 \]

\[ \frac{\partial F[k_a]}{\partial a} = \int_{-\infty}^{+\infty} dx \quad \frac{\partial F[x]}{\partial a} \quad \frac{\partial F[x]}{\partial a} \quad a = 0 \quad \frac{\partial x(x + a)}{\partial a} \quad a = 0 \]

\[ \frac{\partial D[k_a]}{\partial a} \quad a = 0 \quad \frac{\partial x(x + a)}{\partial a} \quad a = 0 \]

\[ D \quad \equiv \quad D[x - a, a] = \int_{-\infty}^{+\infty} dx \quad \frac{\partial F[x]}{\partial a} \quad \frac{\partial F[x]}{\partial a} \quad a = 0 \quad \frac{\partial x(x + a)}{\partial a} \quad a = 0 \]

\[ x(x) = x_{c1}(x, a) + \sum_{n \neq 0} \delta_n \quad y_n(x - a) \]

\[ x_{c1}(x, a) = x_{c1}(x - a) \]

\[ \frac{\partial x_{c1}}{\partial a} \quad x_{c1} \quad \frac{\partial x_{c1}}{\partial a} \quad x_{c1} \]

\[ \frac{\partial y_n(x - a)}{\partial a} \quad \frac{\partial y_n(x)}{\partial a} \]

\[ \Rightarrow \quad \frac{\partial x(x)}{\partial a} \quad a = 0 \quad \left[ \frac{\partial x_{c1}(x)}{\partial a} + \sum_{n \neq 0} \delta_n \quad \frac{\partial y_n(x)}{\partial a} \right] \quad a = 0 \]
\[ D = \int_{-\infty}^{+\infty} \left( \frac{\partial \xi_n}{\partial \tau} \right)^2 + \sum_{n \neq 0} \xi_n \int_{-\infty}^{+\infty} \frac{\partial \xi_n}{\partial \tau} \left| a=0 \right| \frac{\partial \eta_n}{\partial \tau} \left| a=0 \right| \tau \ dt \]

\[ A = \int_{-\infty}^{+\infty} \xi_c^2 \left( \frac{\partial \xi_c}{\partial \tau} \right) \left. \right|_{a=0} \]

\[ r_n = \int_{-\infty}^{+\infty} \xi_c \eta_n \left( \frac{\partial \eta_n}{\partial \tau} \right) \left. \right|_{a=0} \]

\[ D = A + \sum_{n \neq 0} \xi_n r_n \]

Hence

\[ Z = \int \Theta \chi \int_{-\infty}^{+\infty} \delta \left[ F(\omega) \right] \left[ A + \sum_{n \neq 0} \xi_n r_n \right] e^{-E[\chi]} \]

\[ = \int \prod_{n \neq 0} ^{+\infty} \delta \chi_n \int_{-\infty}^{+\infty} \left[ A + \sum_{n \neq 0} \xi_n r_n \right] e^{-E[\chi_c]} e^{-\left[ \int \frac{\partial \eta_n}{\partial \chi_n} \frac{\partial E}{\partial \chi_n} \right]_{\chi_n}} \]

\[ \delta \chi_c \delta \chi \]

\[ \tilde{\chi} = \chi(\tau) - \xi_c(\tau, \alpha) = \sum_{n \neq 0} \xi_n \eta_n(\tau, \alpha) \]

Note that \( \eta_n \)'s are the e-functions with non-vanishing e-values. They are usually \( \tau > 0 \) for stability. Also note that since \( \xi_c(\tau, \alpha) \) is monotonic \( \Rightarrow \frac{\partial \xi_c(\tau, \alpha)}{\partial \alpha} \) has no zeroes \( \Rightarrow \) the ground state w.f. \( \Rightarrow \) it is the lowest energy state \( = 0 \).
Now $F(x) = 0$ is automatically satisfied.

$$x_{a=0}(t) = x_{c1}(t, a)|_{a=0} + \sum_{n \neq 0} \xi_n y_n(t) \quad \text{and} \quad y_0(t) = \frac{N \partial \xi_1}{\partial a}$$

$$F[x] = \int_{-\infty}^{+\infty} dt \sum_{n \neq 0} \xi_n y_n(t) = -\sum_{n \neq 0} \int_{-\infty}^{+\infty} \frac{d\tau}{\pi} J_0(\xi_1 \tau) y_n(\tau) \approx 0 \quad \text{(orthogonality)}.$$

$$\Rightarrow \mathcal{Z} = \int_{-\infty}^{+\infty} \prod \int_{-\infty}^{+\infty} \left( A + \sum_{n \neq 0} \xi_n r_n \right) e^{i \mathcal{E}[\xi_1]} e^{-\mathcal{S}_{eff}[\xi_n]}$$

To leading order we can neglect $\sum_{n \neq 0} \xi_n r_n$.

$$\mathcal{Z} \approx \left[ \int_{-\infty}^{+\infty} da \ A \right] e^{-\mathcal{E}[\xi_1]} \left[ \prod_{n \neq 0} \omega_n^{-1} \right] \text{const.}$$

and

$$\langle x(0), x(t) \rangle \approx A \int_{-\infty}^{+\infty} da \ x_{c1}(0, a) x_{c1}(t, a) \ e^{-\mathcal{E}[\xi_1]} \prod_{n \neq 0} \omega_n^{-1}$$

$$\Rightarrow \sum \frac{\pi}{\lambda} \prod_{n \neq 0} \omega_n^{-1} + A \left[ \int_{-\infty}^{+\infty} da \ x_{c1}(0, a) x_{c1}(t) \right] e^{-\mathcal{E}[\xi_1]} \prod_{n \neq 0} \omega_n^{-1}$$

$$\frac{\pi}{\lambda} \prod_{n \neq 0} \omega_n^{-1} + A \left( \int_{-\infty}^{+\infty} da \right) e^{-\mathcal{E}[\xi_1]} \prod_{n \neq 0} \omega_n^{-1}$$
\[ \langle x(0) x(t) \rangle \approx \frac{\mu^2}{\lambda} + A e^{-\mathbb{E}[x_1]} \int_{-\infty}^{+\infty} da \left[ x_{11}(0, a) x_{11}(\tau, a) - \frac{\mu^2}{\lambda} \right]. \]

\[ \prod_{n \neq 0} \omega_n^{-1} = \text{det} \left[ -\frac{d^2}{dt^2} + (-\mu^2 + 3\lambda x_1^2(t)) \right]^{-1/2} \]

\[ \prod_{n \neq 0} \omega_{n, 0}^{-1} = \text{det} \left[ -\frac{d^2}{dt^2} + (3\mu^2) \right]^{-1/2} \]

This ratio (call it $K$) can be calculated in terms of phase shifts:

\[ \int_{-\infty}^{+\infty} da \left[ x_{11}(0, a) x_{11}(\tau, a) - \frac{\mu^2}{\lambda} \right] = -\frac{\mu^2 \tau}{\lambda} = \frac{\theta}{\tanh \frac{\mu \tau}{\sqrt{2}}} \]

as $\tau \to \infty$

\[ f(t) \sim \frac{\mu^2}{\lambda} - KA e^{-\mathbb{E}[x_1]} \left( \frac{2\mu^2}{\lambda} \right) t \]

This approximation fails for $t > \frac{\sqrt{2} \mathbb{E}[x_1]}{2KA}$

\[ A = \int x^2 \, dt \equiv \mathbb{E}[x_1] = \frac{2\sqrt{2} \mu^3}{\lambda} \]
we have problems for $\tau \rightarrow \infty$

The width of the kink (or instanton) is $n^{-\frac{2}{\sqrt{2}} \mu^3/\lambda}$

If $\lambda$ is small $\Rightarrow e^{2 \frac{\mu^3}{\lambda}} > \frac{1}{\mu}$

Hence we can sum over an ensemble of weakly interacting instantons (exponentially weak)

Let $x_{\infty}(\tau) = \sqrt{\frac{\mu^2}{\lambda}} \prod_{j=1}^{N} \text{sgn}(\tau - a_j)$

$a_j$: location of the $j$-th instanton.

$E_{\infty}$ for $N$ instantons $\sim N \frac{2 \sqrt{2} \mu^3}{\lambda} + \text{exp. small terms}$

$\Rightarrow \langle x(0) x(\tau) \rangle = \sum_{N=0}^{\infty} C^N \frac{\mu^2}{\lambda} \left[ \sum_{N=0}^{\infty} C^N \frac{2 \sqrt{2} \mu^3}{\lambda} \int d q_1 \ldots d q_N \prod_{j=1}^{N} \text{sgn}(\tau - a_j) \right]$}

$C = \frac{A K \frac{2 \tau}{\lambda}}{\tanh \left( \frac{\mu \tau}{\lambda} \right)} \sim 2 A K \frac{2 \tau}{\lambda} \quad (\tau \rightarrow \infty)$

$\int d q_1 \ldots d q_N = \frac{T^N}{N!}$
\[ \langle x(0) x(\tau) \rangle = \frac{\mu^2}{\lambda} e^{-\langle \Delta E \rangle}. \]

\[ \Delta E \leq 2AK e^{-\frac{2\sqrt{2} \mu^3}{A}} \]

\( \Rightarrow \) The ground state and the 1st excited state are split by an amount 

\[ \sim 2AK e^{-A} \]

\[ A = \frac{2\sqrt{2} \mu^3}{\lambda} \]

which has an essential singularity in \( \lambda \).

References:

- Aspects of Symmetry (Coleman)
- Solitons and Instantons (Rajaraman)
- Gauge Fields and Strings (Polyakov)
Topological Excitations and Phase Transitions

i.e. Vortices and Disorder,
Monopoles and Confinement

We will now discuss two specific examples in which topological excitations (i.e. instantons) drive the phase transition. We will discussed the $U(1)$ non-linear O-model (i.e. a compactified or periodic scalar) and $U(1)$ compact electrodynamics.

Vortices and phase transitions (Kosterlitz and Thouless)

Consider the following problem in classical $\mathbb{Z}_2$ statistical mechanics.

On every site of a lattice we define a two-component unit vector $\vec{\hat{n}} = (\cos \Theta, \sin \Theta)$ where $\langle,\rangle$ denotes the nearest neighbor.

The interaction energy is

$$E = - J \sum \vec{\hat{n}}(\vec{r}) \cdot \vec{\hat{n}}(\vec{r}') = - J \sum \cos(\Theta(\vec{r}) - \Theta(\vec{r}')) \langle \vec{r}, \vec{r}' \rangle$$

$$\text{nearest}$$
\[ E = - J \sum_{\vec{r}, \mu} \cos(\Delta \mu \Theta(\vec{r})) \]

\[ \mu = 1, 2 \]

Note: the naive continuum limit is

\[ E \propto \int d^2 x \left( \nabla \Theta \right)^4 \]

which would be a free (Gaussian) field if it were not for the periodicity (symmetry)

\[ \Theta(\vec{r}) \rightarrow \Theta(\vec{r}) + 2\pi N(\vec{r}) \]

where \( N(\vec{r}) \in \mathbb{Z} \). This local symmetry requires that all observables must obey the periodicity condition.

We saw before that in this case the theory must have vortices. We will now see that it does and we'll find out what role they play. We will investigate this question using a method called (lattice) duality first introduced by Kramers and Wannier for the 2D
Using models and closely related to topology

The P.F. is

$$Z = \prod_{r} \int_{0}^{2\pi} \frac{d\theta(r)}{2\pi} e^{-\frac{E(\theta)}{\tau}} + \beta \sum f(r, \mu)$$

$$\beta = \frac{\sqrt{Z}}{\tau}$$

Consider an expression of the form $e^{iV(\theta)}$, which is periodic under $\theta \rightarrow \theta + 2\pi$.

$e^{iV(\theta)}$ can be expanded in Fourier series:

$$e^{iV(\theta)} = \sum_{\ell \in \mathbb{Z}} e^{i\ell\theta} V_{\ell}$$

Note: $e^{i\theta} \in U(1)$ and $\frac{e^{-i\ell\theta}}{\sqrt{2\pi}}$ are vectors in the $\ell$-th irreducible representation of $U(1)$.

$e^{i\theta} \in U(1)$ and $e^{i\theta_{2}} \in U(1)$ is $e^{i(\theta + \theta_{2})} \in U(1)$
but \( e^{i\beta} e^{i\theta} \in U(1) \) and \( e^{i\beta} e^{i\theta} \in U(1) \)

\[ e^{i\beta} e^{il\theta} = e^{i(l+\theta)} \in U(1) \]

\( \Rightarrow \) The representations form a group under the addition. The representations of \( U(1) \) are labelled by integers, i.e., the representation forms a group \( \mathbb{Z} \). ("dual" group)

In particular,

\[ e^{\beta \cos \theta} = \sum_{l=-\infty}^{\infty} I_l(\beta) e^{il\theta} \]

\[ I_l(\beta) = \frac{2\pi}{\beta} e^{\beta \cos \theta} e^{-il\theta} \]

is the modified Bessel function.

For \( \beta \) large,

\[ I_l(\beta) \approx \frac{\beta^l}{\sqrt{2\pi}} e^{-\frac{l^2}{2\beta}} \]

\[ Z = \# \left( \prod_{r} \frac{1}{2\pi} \int_{0}^{2\pi} e^{-\frac{r^2}{2\beta} + i \sum_{\mu} \frac{r^{2\mu}(\beta)}{2\beta} \Delta \xi} \right) \]

i.e., we assign an \( l_{\mu}(r) \in \mathbb{Z} \) to each link \((r,\mu)\).
Since
\[ \int_0^{2\pi} d\theta(r) \delta(\Theta(r)) \Delta_\mu \epsilon_\mu(r) = \delta(\Delta_\mu \epsilon_\mu(r)) \]

\[ \Delta_\mu \epsilon_\mu = \sum_{\mu=1}^{2} \left[ \epsilon_\mu(r) - \epsilon_\mu(r - e_\mu) \right] \]

\[ Z = \# \sum_{\{\epsilon_\mu(r)\}} \prod_r \delta(\Delta_\mu \epsilon_\mu(r)) \; e^{-\frac{1}{2\beta} \sum_{(R,\mu)} \left( \Delta_\mu S(R) \right)^2} \]

Solve the constraint by going to the dual lattice

\[ \Delta_\mu \epsilon_\mu = 0 \Rightarrow \epsilon_\mu = \epsilon_\mu A_\nu S \]

where \( S \in \mathbb{Z} \)

defined on the sites of the dual lattice \( \mathbb{R}^* \)

\[ Z = \# \sum_{\{S(r)\}} \; e^{-\frac{1}{2\beta} \sum_{(R,\mu)} \left( \Delta_\mu S(r) \right)^2} \]

Note: \( \beta \leftrightarrow \frac{1}{\beta} \) (strong \( \leftrightarrow \) weak coupling)

\( U(1) \leftrightarrow \mathbb{Z} \)

direct \( \leftrightarrow \) dual lattice.
Alternative procedure (Villain) (equivalent)

We can also make the replacement

\[ e^{\beta \cos \Theta} \rightarrow \sum_l e^{-\frac{\beta}{2} (\Theta - 2\pi l)^2} \]

and we recover periodicity if

\[ \Theta \rightarrow \Theta + 2\pi k \]
\[ l \rightarrow l + k \]

\[ \Rightarrow \] The integers \( l \) are associated with vortices.

\[ \Rightarrow \int \rho \left( e^{\beta \sum_{\mu \nu} \cos (\alpha_{\mu \nu} \Theta)} \rightarrow \sum \int \sin (\Theta) e^{-\frac{\beta}{2} (\Theta - 2\pi l)^2} \right) \]

\[ \Theta \text{ has no monopoles. Consider a config. If } \Theta \text{ is a} \]

\[ \Rightarrow \Theta \text{ must jump by } 2\pi m \text{ across the cut} \]
The dual theory has degrees of freedom which are integers. It can be regarded as a "height" of some column of atoms $\{S(R)\}$ is a discretized surface (surface roughening or discrete gaussian)

Suppose we wanted to compute

$$\langle e^{i\Theta(x)} e^{-i\Theta(x')} \rangle$$

The dual of this expectation value requires that we now solve the constraint

$$\Delta_\mu \epsilon_\mu = n(x)$$

$$n(x) = \delta(x-x') - \delta(x-x'')$$

The solution is

$$\epsilon_\mu = \epsilon_{\mu\nu} (\Delta_\nu S + B_\nu)$$

where $B_\mu (R)$ is a vector field on the links of the dual lattice

$$\epsilon_{\mu\nu} \Delta_\mu B_\nu = n(x) \quad (flux)$$
At us work on the Discrete Gaussian Model

\[ Z = \sum_{\{S(R)\}} e^{-\frac{1}{2\beta} \sum_{(R,\mu)} (\Delta \phi)_{S(R)}} \]

Poisson Summation Formula:

\[ \sum_{n \in \mathbb{Z}} f(n) = \lim_{\gamma \to \infty} \int_{-\gamma}^{\gamma} d\phi \sum_{m = -\infty}^{\infty} e^{2\pi i m \phi - y m^2} f(\phi) \]

\[ \Rightarrow Z = \sum_{\{m(R)\}} \prod_{R} \int d\phi(e) \ e^{-\frac{1}{2\beta} \sum_{(R,\mu)} (\Delta \phi)_{S(R)}^2 + i \sum_{R} 2\pi i m(R) \phi} \]

\[ e^{-y \sum_{R} m^2(R)} \]

Version 1

For \( Z \) small, \( Z \approx e^{-y} \), we can approximate

\[ \sum_{m(R) \in \mathbb{Z}} e^{-y m^2(R) + i 2\pi m(R) \phi(R)} = 1 + 2e^{-y} \cos(2\pi \phi(R)) + O(e^{-4y}) \]

\[ \approx e^{2\pi \phi(R)} + O(\pi^2) \]
\[ Z = \int \pi \, d\phi(x) \, e^{-\frac{1}{2\beta} \sum \frac{(\Delta \phi)^2}{\mu^2} + Z \sum \cos (2\pi \phi) / R} \]

\[ \approx \int d\phi \, e^{-S[\phi]} \]

\[ (a \rightarrow 0) \quad S[\phi] = \frac{k}{2} \int d^2x \left( \Theta \phi \right)^2 + \frac{g}{2a^2} \int d^2x \cos (2\pi \phi) \]

\[ \frac{K}{\beta} = \frac{1}{\beta} = \frac{T}{J} \quad 2Z = \frac{g}{2} \Rightarrow g = 4Z \]

\[ \sqrt{K} \phi = \varphi \]

\[ S[\phi] = \int d^2x \left( \frac{1}{2} \Theta^2 \phi \right)^2 + \frac{g}{2a^2} \cos (\beta \varphi) \]

\[ \text{Sin-Gordon Field Theory!} \]

\[ \sqrt{K} \beta = 2\pi \]

\[ \beta = \frac{2\pi}{\sqrt{K}} \]

\[ \Rightarrow \beta = \frac{2\pi}{\sqrt{J}} \]
Version 2: Integrate \( \phi \) out:

\[
\int \frac{d\phi(r)}{I^2} e^{-\frac{1}{2\beta} \sum \frac{1}{R_i^2} \left( \Delta_i \phi \right)^2} \times \sum \frac{2\pi \gamma^2 m(r) \phi(r)}{\gamma^2} - \sum y m_i^2
\]

\[
e^{-\sum \frac{y^2 m_i^2}{R_i^2}} e^{-\frac{1}{2} \sum \frac{(2\pi)^2 m(r) G(r-r') m(r')}{R_i^2}} \times \left( \int \frac{d\phi(r)}{I^2} e^{-\frac{1}{2\beta} \sum \frac{1}{R_i^2} \left( \Delta_i \phi \right)^2} \right)
\]

where \( \Delta^2 G = \beta \delta_{r,r'} \)

\[
\text{for } |r-r'| \gg \alpha
\]

\[
G(r-r') \approx \frac{\beta}{2\pi} \ln \left( \frac{r-r'}{\alpha} \right)
\]

Lattice Coulomb

\[
e^{-\sum \frac{y^2 m_i^2}{R_i^2}} e^{-\frac{1}{2} \sum \frac{(2\pi)^2 m(r) G(r-r') m(r')}{R_i^2}} \times e^{-\sum \frac{y^2 m(r)^2}{R_i^2}} \approx 1
\]

Core energy

\[
\sum m(r) = 0
\]

\[
Z = \sum \left\{ m(r) \right\} e^{-\sum \frac{\pi}{2\pi} \frac{(2\pi)^2 m(r) G(r-r') m(r')}{R_i^2}} \ln \left( \frac{r-r'}{\alpha} \right) m(r')
\]
This is a 2D (logarithmic) Coulomb gas. We saw before that this precisely the contribution of vortices.

\[ Z \propto \sum e^{-\frac{1}{2} \sum_{(r,r')} m(r) m(r') \frac{U(r-r')}{|r-r'|}} \]

\[ U(r-r') = \frac{1}{\pi} \log \left| \frac{r-r'}{a} \right| \]

\[ t = \frac{I}{\pi n^2} \quad \Rightarrow \quad \gamma = \frac{\mu}{t} \]

What is the self energy (or action) of one vortex?

\[ E = \mu + U(L) \]

where \( L \) is the linear size of the system.

In 3D \( U(R) \sim \frac{1}{R^2} \rightarrow 0 \)

In 2D \( U(L) \sim \frac{1}{2\pi} \ln\left(\frac{R}{a}\right) \rightarrow \text{diverges as } R \rightarrow \infty \)

(\( \Rightarrow \) this favors the neutrality condition)

What is the entropy for 1 vortex?
Entropy $= \ln L^2$

$= \ln$ (number of available states)

$\Rightarrow$ (Free energy) $= E - tS$

$= (\mu + \frac{1}{2\pi} \ln \left( \frac{L}{a} \right) )$

$- t \Theta \ln \left( \frac{L}{a} \right)^2$

$F < 0 \text{ if } t > t_c$

$2t_c = \frac{1}{2\pi} \Rightarrow \frac{1}{4\pi}$

For $t > t_c = \frac{1}{4\pi}$ free vortices proliferate a condensate of vortex plasma.

For $t < t_c$ free vortices are suppressed and exist only in vortex-antivortex pairs.

$\Rightarrow$ "chiral order"

We can ask a similar question in the sine-Gordon language $\frac{\phi}{2\pi}$. For what values of $\beta$ is the $\cos \beta$ relevant?

We will see in a few lectures that the
Scaling dimension of $\exp\beta \psi$:

We will show below that

$$\langle e^{i \beta \psi(x)} e^{-i \beta \psi(y)} \rangle \propto \frac{1}{|x-y|^{2+\Delta}}$$

$$\Delta = \frac{\beta^2}{4\pi}$$ is the scaling dimension.

Since we are in $D=2$ dimension, $\exp\beta \psi$ is irrelevant if $\Delta > 2$

$$\Rightarrow \frac{\beta^2}{4\pi} > 2 \Rightarrow \beta^2 > \beta_c^2 = 8\pi$$

Since $\beta^2 = \frac{4\pi^2}{T}$ we have $\frac{4\pi^2}{T} > 8\pi$

$$T < T_c = T_{KT} = \frac{\pi}{2}$$

Hence for $T < T_c$, $\exp\beta \psi$ is irrelevant and we can work with the free field model $\Rightarrow$ no bounded vortices. For $T > T_c$ $\exp\beta \psi$ is relevant, vortices proliferate $\Rightarrow \phi$ gets pinned at minima $\frac{2\pi n}{\beta}, n \in \mathbb{Z}$.
Monopoles (Polyakov)

We will now discuss compact QED. We saw before that this theory has monopoles which we will see play a role analogous to the notion of the 2D system.

We define compact QED on a lattice (as we did before) in terms of an angle-valued mesh field \( A_\mu (r) \) defined on the links \((r, l, \mu)\) of a cubic lattice.

The partition function is

\[
Z = \int_{\mathbb{T}^3} \prod_{(r, l, \mu)} \frac{dA_\mu (r)}{2\pi} \ e^{-S[A]}
\]

\[
- S(A) = \frac{1}{2g^2} \sum_{\text{plaq.}} \cos(F_{\mu \nu}(r))
\]

\[
F_{\mu \nu}(r) = \sum_{\text{plaq.}} A_\mu - A_\mu - A_\nu A_\mu
\]
Note: due to the periodicity $A_\mu \rightarrow A_\mu + 2\pi l_\mu$

the Bianchi identity for $F_{\mu \nu}$ cannot
be satisfied but it is violated modulo $2\pi$.

⇒ we can have monopoles with a
quantized magnetic charge. We can
use an analogous procedure to what
we did for vortices

$\mathcal{S} = \frac{1}{2g^2} \sum \omega \rightarrow F_{\mu \nu}$

plaquette

$\int DA_\mu e^{-\mathcal{S}} = \int \prod \frac{dA_\mu(x)}{(2\pi)^{2}} \sum e^{-\frac{1}{g^2} (F_{\mu \nu} + 2\pi m_{\mu \nu})^2}$

on plaquettes!

\[ A_\mu \rightarrow A_\mu + 2\pi l_\mu \quad \Rightarrow \quad \text{"discrete" one-form} \]

\[ F_{\mu \nu} \rightarrow F_{\mu \nu} + 2\pi (\partial_\mu A_\nu - \partial_\nu A_\mu) \]

\[ m_{\mu \nu} \rightarrow (\partial_\mu l_\nu - \partial_\nu l_\mu) \quad \text{to enforce periodicity} \]

"discrete" 2-form

⇒ $F_{\mu \nu}$ may satisfy Bianchi but $m_{\mu \nu}$ does not
In fact $\frac{1}{2} \varepsilon_{\mu \nu \lambda} \Delta_{\mu} m_{\nu \lambda} = \Phi$ measures the violation of the Bianchi identity.

Notice that $\Phi \in \mathbb{Z}$. The integers $\{m_{\mu \nu}\}$ live on plaquettes and represent Dirac strings.

Dirac string $\leadsto$ monopole

\[ \sum_{\{m_{\mu \nu}\}} e^{-\frac{g^2}{4} \sum_{(r, \mu \nu)} \frac{m_{\mu \nu}^2}{g}} \]

\[ \int DA \times e^{-\frac{i}{2} \sum_{(r, \mu \nu)} m_{\mu \nu} F_{\mu \nu}} \]

\[ \int DA_{\mu} + \frac{i}{2} \sum_{(r, \mu \nu)} A_{\mu} \Delta_{\nu} m_{\mu \nu} = \delta(\Delta_{\nu} m_{\mu \nu}) \]

is the constraint on every brane.

Solution

$\Delta_{\nu} m_{\mu \nu} = 0 \Rightarrow m_{\mu \nu} = \varepsilon_{\mu \nu \lambda} A_{\lambda} S$, $S(\mathcal{R}) \varepsilon_{\mu \nu \lambda}$

dual cubic
\[
\Rightarrow Z = \sum \left\{ m_{\mu}(r) \right\} e^{-\frac{\sum}{(r_{\mu\nu})^2} \left( m_{\mu\nu}(r) \right)^2} \prod \delta (\Delta_{\nu}, m_{\mu\nu}) (r_{\mu}\nu) \\
= \sum \left\{ S(R) \right\} e^{-\frac{g^2}{2} \sum \left( \Delta_{\mu} S(R) \right)^2}
\]

It is again a discrete Gaussian Model!

but in $D=3$ dimensions!

The same line of argument shows that we can write this as a sum over monopole configurations or as a Sine-Gordon theory. \&
For the monopole gas we get (using the PSF)

\[ Z = \sum' e^{-\frac{1}{2} \sum_{R,R'} \left( \frac{2\pi}{\hbar} \right)^2 m(R) G(R-R') m(R') } \]

\[ \{ m(R) \} \]

(neutrality)

Where \( G(R-R') = \frac{1}{4\pi |R-R'|} \)

In this case we have a three-dimensional (neutral) Coulomb gas. It is well known that this gas is a plasma phase at all temperatures \( t = \frac{g^2}{4\pi\hbar^2} \)

\[ \Rightarrow \text{energy} < \infty \text{ but entropy still \( \sim \text{volume} \)} \Rightarrow \text{entropy dominates} \]

\[ \Rightarrow \text{monopoles proliferate (or "condense") for all values of } g \]

\[ \Rightarrow \text{Polyakov showed that this leads to an Area Law for Wilson loop operators} \Rightarrow \text{Confinement} \]

\[ \Rightarrow \text{Monopole "condensation" } \Rightarrow \text{Confinement.} \]
Once again, the Coulomb gas is equivalent to a pure Gaussian theory in the same Euclidean dimension.

\[ S_D = \int dx^D \left[ \frac{1}{2} \left( \partial \phi \right)^D - \frac{y}{a} \cos \beta \phi \right] \]

Monopole gas: \( \beta = \frac{2\pi}{\log y} \)

Vortex gas: \( \beta = \frac{2\pi}{\sqrt{n}} \)

\( y \) : fugacity.

Main difference:

\( D = 2 \) RG Flow \( \Rightarrow \) Coster-Kronig

\[ \frac{\beta^2}{8\pi} - 1 \]

\( D > 2 \) \[ |\phi| \neq 1 \] \( \Rightarrow \) \[ \phi_j \rightarrow \phi_j (1 - \frac{1}{\nu}) \]

\( \beta \) is not dimensionless \( \Rightarrow \) no fixed line

\( \Rightarrow \) \( y \) always grows \( \Rightarrow \) plasma phase \( \Rightarrow \) screening of non-local Screening and screening length \( \Rightarrow \) confinement.