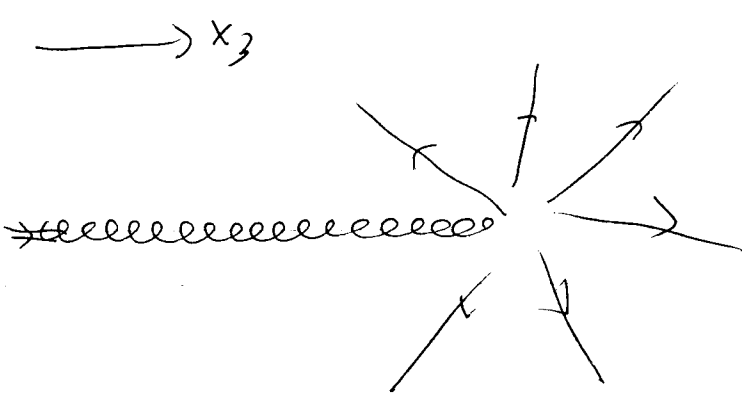
Instantons in non-Abelian Gauge Theories

We now discuss what happens in the case of non-Abelian gauge theory.

Let us first contruct the analog of the $U(1)$ vortex, It is the Dirac monopole

The vortex was regarded as a single solution of the phase field $\theta(x)$. Those solutions were multi-valued, and had branch cuts. Dirac proposed to look at

configurations of electric and magnetic fields created by an infinitely long (semi-infinite) solenoid of infinitesimal thickness. The ~~fie~~ magnetic field "radiating" out of the end of the solenoid looks as if



there was a magnetic charge at that point. From magnetostatics we know that the equations are

$$\vec{\nabla} \times \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad B_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda} F_{\nu\lambda}$$

These equations <u>do not</u> have monopoles.

But consider now the configuration

$$B_\mu(x) = \frac{q}{2} \frac{x_\mu}{|\vec{x}|^3} - 2\pi q \, \delta_{\mu 3} \, \delta(x_1) \delta(x_2) \, \Theta(x_3)$$

It represents a magnetic charge q at

the origin and "string" of magnetic flux with equal an opposite flux, a Dirac string.

$\delta(x$

(in the $D=2$ vortex $A_\mu = \partial_\mu \Theta = n \varepsilon_{\mu\nu} \frac{x_\nu}{x^2} - 2\pi n\, \theta(x_2)$

We also saw that vortices arise as solutions of some gauge theories with matter fields ("spontaneously broken"). The analog here is the 't Hooft - Polyakov monopole which arises in the Georgi-Glashow model of weak interactions:

$$S = \int d^D x \quad \frac{1}{2}(D_\mu \vec{\varphi})^2 - \frac{1}{2} m_0^2 \vec{\varphi}^2 + \frac{\lambda}{4!}(\vec{\varphi}^2)^2$$

$$+ \frac{1}{4} \mathrm{tr}\, F_{\mu\nu}^2$$

(Euclidean!)

for $D = 3$ (Euclidean) dimensions.

here $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ is a triplet (in the adjoint rep. of $SU(2)$) and

the gauge group is SU(2)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, \varepsilon^{abc} A^b_\mu A^c_\nu$$

$$D_\mu \phi^a = \partial_\mu \phi^a + g\, \varepsilon^{abc} A^b_\mu \phi^c$$

(in short $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\, A \times A$

$D_\mu \phi = \partial_\mu \phi + g\, A \times \phi$)

Here we are interested in finite action classical solutions of the Euclidean 2+1-dimensional theory, which are the same as the static finite energy classical solutions of the Minkowski theory in 3+1.

$i, j = 1, 2, 3$

$a, b, c = 1, 2, 3$

Equations of motion:

(1) $$D_i F^{aij} = g\, \varepsilon^{abc} (D^j \phi^b)\, \phi^c$$

and

(2) $$D_i D^i \phi^a = -\lambda (\phi^b \phi^b)\, \phi^a + \lambda f^2 \phi^a$$

(where $\lambda f^2 = |m_0^2|$)

The energy (or action, depending of the case) is

$$E = \int d^3x \left[\frac{1}{4} F^a_{ij} F^{aij} + \frac{1}{2} D_i \phi^a D_i \phi^a + \frac{\lambda}{4} (\phi^a \phi^a - f^2)^2 \right.$$

zero-energy solutions: the classical vacuum.

$$A_i^a(x) = 0 \qquad \phi^a(x)\phi^a(x) = f^2 \text{ and } D_i\phi^a = 0$$

which in this case $\Rightarrow$ $\partial_i \phi^a = 0 \Rightarrow \phi^a = \text{const.}$

Finite action solutions: the action will be finite provided the fields approach sufficiently fast the vacuum solution as $r \to \infty$. $(|\vec{r}| = r)$

$$\Rightarrow \quad r^{3/2} D_i \phi^a \to 0 \qquad (r \to \infty)$$

$$\text{and} \quad \phi^a \phi^a \to f^2 \qquad (r \to \infty)$$

In spherical coordinates (r, θ, φ)

$$(D\phi^a)_\theta = \frac{1}{r}\frac{\partial \phi^a}{\partial \theta} + g\,\varepsilon^{abc} A_\theta^b \phi^c$$

$$\Rightarrow (D\phi^a)_\theta \underset{r\to\infty}{\to} 0 \quad \text{provided} \quad A_\theta^b \sim \frac{1}{r} \quad (r \to \infty)$$

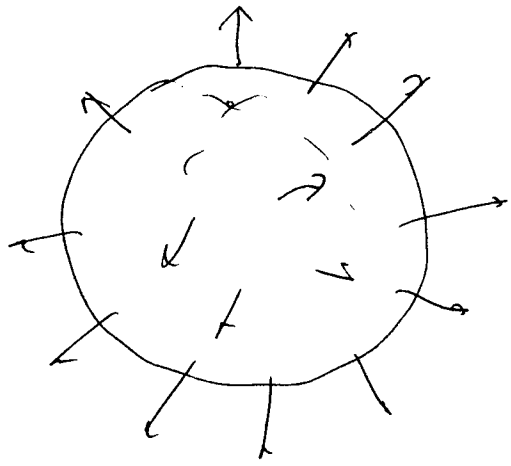
(the same applies to the other components)

also if $A \sim \frac{1}{r} \to F \sim \frac{1}{r^2}$ and $F^2 \sim \frac{1}{r^4}$ (for large r),

which is integrable in three dimensions.

$\Rightarrow$ unlike the case of the non-linear σ-model finite action solutions may have fields ϕ^a which are not equivalent at ∞ such a

they may point in ≠ directions. Hence the boundary conditions are just derived in play that at the boundary, the surface of a large 2-sphere S_2 of large radius R, the fields ϕ^a may point on any direction of their target space S_2 ⇒ the asymptotic fields are mappings of $S_2^R \longrightarrow S_2^{target}$ and we know that these mappings can be classified in topological classes of $\pi_2(S_2) = \mathbb{Z}$. The topological charge Q we used to classify the instantons of the D=2 O(3) non-linear σ-model, can now be used as the topological charge for the field configs. of this gauge theory. For instance, the instanton with Q=1 has a ϕ^a config. which asymptotically is a "hairy ball" or a



a "hedgehog".

How is this related to a magnetic monopole?

How is this related to a monopole? The picture we just discussed is gauge-dependent. There is a gauge-invariant way to look at this due to 't Hooft. Define

$$F_{\mu\nu} \equiv \hat{\phi}^a F^a_{\mu\nu} - \frac{1}{g} \varepsilon^{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D_\nu \hat{\phi}^c$$

$$\left(\hat{\phi}^a = \frac{\phi^a}{\|\phi\|} \right)$$ which is gauge-invariant

and for $\hat{\phi}^a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ reduces to

$$F_{\mu\nu} \equiv \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$$

Let us compute ~~(D=4)~~

dual tensor $\rightarrow$ $F^*_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}$ $(D=4)$

(dual vector) $F^*_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda} F^{\nu\lambda}$ $(D=3)$

If $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (as in Maxwell's)

$\Rightarrow \partial^\mu F^*_\mu = 0$ and $\partial^\mu F^*_{\mu\nu} = 0$

$(D=3)$ $(D=4)$

This is the Bianchi Identity which means that there are <u>no monopoles</u>.
$(\vec{\nabla}\cdot\vec{B}=0)$

But $(D=4)$

$$\partial^\nu F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\,\partial^\nu F^{\lambda\rho} = \frac{1}{2g}\varepsilon_{\mu\nu\lambda\rho}\,\varepsilon_{abc}\,\partial^\nu\hat{\phi}^a\partial^\lambda\hat{\phi}^b\partial^\rho\hat{\phi}^c$$

$$\equiv \frac{4\pi}{g}\underbrace{j_\mu}_{\text{topological current}}$$

$D=3 \Rightarrow$

$$\partial^\mu F^*_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\,\partial^\mu F^{\nu\lambda}$$

$$= \frac{4\pi}{g}\,j_0(x)$$

where $\int d^3x\, j_0(x) = \frac{1}{8\pi}\int d^3x\,\varepsilon_{ijk}\,\varepsilon^{abc}\,\partial_i\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}$

$$= \frac{1}{8\pi}\int d^3x\,\varepsilon_{ijk}\,\varepsilon^{abc}\,\partial_i(\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c)$$

$$= \frac{1}{8\pi}\int_{S_2} d^2S_i\,\varepsilon_{ijk}\,\varepsilon^{abc}\,\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c \equiv Q$$

$\Rightarrow$ $D=3$ we get that

$$\partial^\mu F_\mu^* = \underline{\text{topological charge density}}$$

$$\equiv \frac{4\pi}{g} j_0(x)$$

$$Q = \int d^3x \; j_0(x)$$

~~In particular~~

and $D=4 \Rightarrow \partial^\mu F^*_{\mu\nu} = \frac{4\pi}{g} j_\mu$

j_μ is the topological current

$$j_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\lambda\rho} \varepsilon_{abc} \partial_\nu \hat{\phi}^a \partial_\lambda \hat{\phi}^b \partial_\rho \hat{\phi}^c$$

$\Rightarrow$ $D=3$ monopoles are "point-like" objects (instantons) whereas in $D=4$ they are finite energy solutions and hence define 4-dimensional currents.

If we define the magnetic ~~field~~ field

$$B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$$

$$\Rightarrow \partial_i B_i = \frac{4\pi}{g} j_0(x)$$

$$\Rightarrow \int d^3x \, \frac{1}{g} j_0(x) = m = \text{magnetic charge} = \frac{Q}{g}$$

The solution for $Q=1$ is of the form:

$$\phi^a(x) = \delta_{ia} \frac{x^i}{r} F(r)$$

$$A_i^a(x) = \varepsilon_{aij} \frac{x^j}{r} W(r)$$

s.t. $F(r) \xrightarrow[r\to\infty]{} f$

$W(r) \xrightarrow[r\to\infty]{} \frac{1}{gr}$

$\Rightarrow \vec{B}(\vec{x}) \xrightarrow[r\to\infty]{} \frac{\vec{x}}{g r^3}$ i.e. a magnetic pole of strength $\frac{1}{g}$

The field equations become

$$K(r) \equiv 1 - g r W(r) \equiv K$$

$$H(r) \equiv g r F(r) \equiv H$$

$$r^2 \frac{d^2 K}{dr^2} = K(K^2-1) + H^2 K$$

$$r^2 \frac{d^2 H}{dr^2} = 2 H K^2 + \lambda H \left(\frac{H^2}{g^2} - r^2 f^2\right)$$

which are quite difficult to solve. However in the limit $\lambda \to 0$ (BPS) (Bogomolnyi, Prasad, Sommerfield)

$$\Rightarrow K(r) = \frac{r g f}{\sinh(g f r)}, \quad H(r) = \frac{g r f}{\tanh(g f r)} - 1$$

It can be shown (see Rajaraman) that these solutions satisfy (and saturate) the bound

$$E \geq \frac{4\pi Q f}{g}$$

and satisfy the Bogomolnyi Equation.

$$F^a_{ij} = \varepsilon_{ijk} D_k \phi^a$$

<u>Yang-Mills in D=4</u>

We summarize the discussion of monopoles with some general conditions for instanton solution in D=4 (Euclidean) Yang-Mills gauge theory (without matter).

Let G be a gauge group and consider the YM action (Euclidian)

$$S = \frac{1}{4} \int d^4x \; \mathrm{tr}\, F^2$$

(faster than

$$\Rightarrow \quad S < \infty \quad \text{if} \quad F_{\mu\nu} \sim O\left(\frac{1}{r^2}\right) \text{ as } x \to \infty$$

$$\Rightarrow \quad A_\mu \underset{r\to\infty}{\sim} g^{-1} \partial_\mu g + O\left(\frac{1}{r}\right)$$

gauge transformation

where the gauge transformation $g(x) \in G$

$\Rightarrow$ at the boundary of $\mathbb{R}^4$, which is S_3

a large 3-sphere of radius $R \to \infty$,

the field ~~for~~ configurations are parametrized

by $g(\infty) \in G$, the gauge transformation

which is an element of G. Thus the

configurations are in one-to-one

correspondence with the mappings

$$g(x) : S_3 \longrightarrow G$$

For instance, if $G = SU(2) \Rightarrow g \in SU(2)$

$$g = n_4 + i\,\vec{n}\cdot\vec{\sigma} \quad / \quad n_4^2 + \vec{n}^2 = 1$$

$$g^\dagger g = I \qquad \Rightarrow \text{this is the sphere } S_3$$

$\Rightarrow$ We have the maps $g : S_3 \longrightarrow S_3$

The topological charge or winding number

is now called the Pontriagin index which

measures the number of times S_3 covers S_3

= integral of the Jacobian of the map $g(x)$

In general, $g \in G$, we define

$$Q = \frac{1}{24\pi^2} \int_{S_3} d^3x \; \varepsilon_{\mu\nu\lambda} \, \mathrm{tr}\left(L_\mu L_\nu L_\lambda \right)$$

$$L_\mu(x) = g^{-1} \partial_\mu g \qquad (a, b, c, d = 1, \ldots, 4)$$

$$\equiv \frac{1}{12\pi^2} \int_{S_3} d^3x \; \varepsilon^{abcd} \, \varepsilon_{\mu\nu\lambda} \left(n^a \partial_\mu n^b \partial_\nu n^c \partial_\lambda n^d \right)$$

It can be shown (see Polyakov) that

$$Q = \frac{1}{32\pi^2} \int d^4x \; \varepsilon^{\mu\nu\lambda\rho} \, \mathrm{Tr}\left(F_{\mu\nu} F_{\lambda\rho} \right)$$

$$\equiv \frac{1}{8\pi^2} \int_x \mathrm{Tr} \; F \wedge F$$

$$\Rightarrow Q = \frac{1}{8\pi^2} \int d^4x \; \mathrm{Tr}\left(F^{\mu\nu} F^*_{\mu\nu} \right)$$ Pontriagin Index

This works for all compact simply connected gauge groups.

$$S = \frac{1}{4g^2}\int d^4x \; \mathrm{Tr}\, F_{\mu\nu}^{\;2}$$

$$= \frac{1}{8g^2}\int d^4x \; \mathrm{Tr}\left(F_{\mu\nu} - F_{\mu\nu}^{*}\right)^2 + \frac{1}{4g^2}\int d^4x \; \mathrm{Tr}\left(F^{\mu\nu}F^{*}_{\mu\nu}\right)$$

$$\Rightarrow S = \frac{8\pi^2 Q}{g^2} + \frac{1}{8g^2}\int d^4x \; \mathrm{Tr}\left(F_{\mu\nu} - F^{*}_{\mu\nu}\right)^2$$

$$\Rightarrow S \geq \frac{8\pi^2 Q}{g^2}$$

We can saturate the bound ~~with~~ if we require $F_{\mu\nu} = F_{\mu\nu}^{*}$ (self-dual solution) ("Cauchy-Riemann")

Also if $F = F^{*} \Rightarrow D_\mu F^{\mu\nu} = 0$ are solutions.

$Q = 1$ solution (for $SU(2)$)

$$A^a_\mu = -\frac{\eta_{a\mu\nu}\,(x_\nu - a_\nu)}{(\vec{x}-\vec{a})^2 + \rho^2}$$

ρ : scale

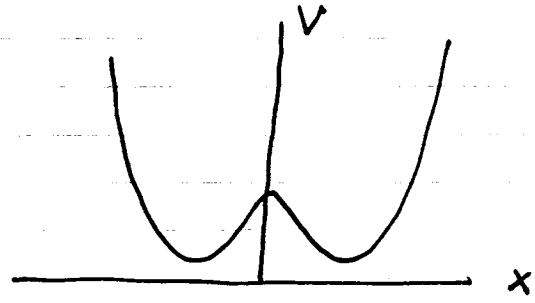
a_μ : location.

$\eta_{abc} = \varepsilon_{abc}$ and $\eta_{ab0} = \delta_{ab}$ etc.

$$\Rightarrow F^a_{\mu\nu} = -\frac{4\,\eta_{a\mu\nu}\,\rho^2}{(\vec{x}-\vec{a})^2 + \rho^2}$$

Instantons and the double-well potential

Let's consider a quantum mechanical double well problem



$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{\mu^2}{2}x^2 - \frac{\lambda}{4}x^4$$

(λ small)

Classically there are two degenerate ground states

$$x_\pm = \pm\left(\frac{\mu^2}{\lambda}\right)^{1/2}$$

and the $\pm$ symmetry is (classically) broken.

In p.thy. $\quad x = +\left(\frac{\mu^2}{\lambda}\right)^{1/2} + y \quad$ (y small)

and nothing dramatic happens (the degeneracy is not lifted)

$$\Rightarrow \quad \langle x(0)\, x(t)\rangle \xrightarrow[t\to\infty]{} \text{constant.}$$

Wick rotation $\tau = it$

$$\langle x(0)\, x(\tau)\rangle = \sum_n |x_{n0}|^2 e^{-(E_n - E_0)\tau} \xrightarrow[\tau\to\infty]{} e^{-\Delta E\, \tau}$$

$\uparrow$ splitting.

$$\langle x(0)\, x(\tau) \rangle = \frac{\int \mathcal{D}x(\tau)\; e^{-\mathcal{E}[x]}\; x(0)\, x(\tau)}{\int \mathcal{D}x(\tau)\; e^{-\mathcal{E}[x]}}$$

$$\mathcal{E}[x] = \int_{-\infty}^{+\infty} d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 - \frac{\mu^2 x^2}{2} + \frac{\lambda}{4} x^4 \right]$$

Q.M. $\leftrightarrow$ 1 D C.S.M.

Extrema of $\mathcal{E}[x]$: $\quad \delta\mathcal{E} = 0 \quad \Rightarrow \bar{x}(\tau)$

$$\mathcal{E}(x) = \mathcal{E}(\bar{x}(\tau)) + \int \frac{\delta\mathcal{E}}{\delta x}\bigg|_{\bar{x}} \delta x + \frac{1}{2}\int \frac{\delta^2\mathcal{E}}{\delta x\, \delta x'}\, \delta x\, \delta x'$$

$$\frac{\delta\mathcal{E}}{\delta x}\bigg|_{x=\bar{x}} = 0 \qquad \delta x = x(\tau) - \bar{x}(\tau)$$

$$0 = \frac{\delta\mathcal{E}}{\delta \bar{x}} - \frac{d}{d\tau}\frac{\delta\mathcal{E}}{\delta \dot{\bar{x}}} \qquad \text{(Euler-Lagrange)}$$

$$\frac{\delta\mathcal{E}}{\delta \dot{\bar{x}}} = \dot{\bar{x}} \qquad \frac{\delta\mathcal{E}}{\delta \bar{x}} = -\mu^2 \bar{x} + \lambda \bar{x}^3$$

$$\frac{d^2\bar{x}}{d\tau^2} = -\mu^2 \bar{x} + \lambda \bar{x}^3$$

Eq. of. Mot. of Classical particle in pot. $V(x)$

$$V(x) = \frac{\mu^2}{2}\bar{x} - \frac{\lambda}{4}\bar{x}^4 = -U(x$$

$$E = \frac{1}{2}\left(\frac{d\bar{x}}{d\tau}\right)^2 - V(\bar{x}) \text{ is "conserved"}$$

(1) $\bar{x} = \pm \left(\frac{\mu^2}{\lambda}\right)^{1/2}$ is a pair of deg. solutions.

$$\mathcal{E}(\bar{x}) = T\left(\frac{\lambda}{4}\bar{x}^4 - \frac{\mu^2}{2}\bar{x}^2\right) = T\left[\frac{\lambda}{4}\left(\frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^2}{2}\frac{\mu^2}{\lambda}\right]$$

$$\mathcal{E}(\bar{x}) = -\frac{\mu^4}{4\lambda}T$$

$\boxed{130}$ (2) $E=0$ $\qquad \frac{d\bar{x}}{d\tau} = \sqrt{2V(\bar{x})}$

$$\bar{x}_c(\tau) = \pm\left(\frac{\mu^2}{\lambda}\right)^{1/2} \tanh\frac{\mu(\tau-a)}{\sqrt{2}}$$

$$\mathcal{E}(\bar{x}_c(\tau)) - \mathcal{E}\left(\left(\frac{\mu^2}{\lambda}\right)^{1/2}\right) = 2\sqrt{2}\,\frac{\mu^3}{\lambda} \qquad \text{finite !}$$

$\Rightarrow$ the solution $\bar{x}_c(\tau)$ has a contrib. $\sim e^{-2\sqrt{2}\frac{\mu^3}{\lambda}\tau}$

$\Rightarrow$ it is important if $\tau > e^{+2\sqrt{2}\mu^3/\lambda}$

$$\text{Stability} = \frac{1}{2}\int d\tau \int d\tau' \left.\frac{\delta^2\mathcal{E}}{\delta x(\tau)\,\delta x(\tau')}\right|_{\bar{x}} (x(\tau)-\bar{x}(\tau))\,(x(\tau')-\bar{x}(\tau'))$$

$$\left.\frac{\delta^2\mathcal{E}}{\delta x(\tau)\,\delta x(\tau')}\right|_{\bar{x}_c} = -\frac{d^2}{d\tau^2}\delta(\tau-\tau') + \left(-\mu^2 + 3\lambda\,\bar{x}_c^2(\tau)\right)\delta(\tau-\tau')$$

$$\equiv \left[-\frac{d^2}{d\tau^2} + \left(-\mu^2 + 3\lambda\,\bar{x}_c^2(\tau)\right)\right]\delta(\tau-\tau')$$

Let $y_n(\tau)$ be a complete set of eigenstates of the operator $-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda\, \bar{x}_c^2(\tau))$ satisfying

$$y(-\infty) = y(+\infty) = 0$$

$$\left[-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda\, \bar{x}_c^2(\tau))\right] y_n(\tau) = \omega_n^2\, y_n(\tau)$$

↳ eigenvalues.

$\Rightarrow$ $\bar{x}_c(\tau)$ is stable iff $\omega_n^2 \geq 0$

Try

$$x(\tau) = \bar{x}_c(\tau) + \sum_n \xi_n\, y_n(\tau) \qquad \xi_n \text{ arbitrary}$$

$$\Rightarrow \mathcal{D}x(\tau) = N \prod_n d\xi_n$$

$$\langle x(0)\, x(\tau)\rangle \cong \frac{\overset{\text{1 kink}}{e^{-\mathcal{E}[\bar{x}_c(\tau)]}}\; \bar{x}_c(0)\, \bar{x}_c(\tau) \prod_n \omega_n^{-1}(\bar{x}_c) + \overset{\text{no kink}}{\frac{\mu^2}{\lambda}}}{1 + e^{-\mathcal{E}[\bar{x}_c(\tau)]} \prod_n \omega_n^{-1}(\bar{x}_c)}$$

problem: $\det\left(-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda\, \bar{x}_c^2(\tau))\right)^{1/2} = \prod_n \omega_n^{-1}(\bar{x}_c)$

Vanishes because $\omega_0 = 0$!

Symmetry: $\bar{x}_c(\tau)$ has a zero-mode : a (origin)

$\mathcal{E}[\bar{x}_c]$ is indep. of a

$$\Rightarrow \frac{\delta \mathcal{E}(\bar{x}_c)}{\delta a} = 0$$

$$\Rightarrow \frac{\delta}{\delta \bar{x}_c(\tau')} \frac{\delta \mathcal{E}}{\delta a} = 0$$

$$\frac{\delta^2 \mathcal{E}}{\delta \bar{x}_c(\tau') \delta \bar{x}_c(\tau)} \frac{d\bar{x}_c(\tau)}{da} = 0 \Rightarrow \frac{d\bar{x}(\tau)}{da} \text{ is a zero-energy mode.}$$

$$y_0 = \frac{d\bar{x}_c(\tau)}{da} \Leftrightarrow \omega_0^2 = 0 \quad \text{no restoring force.}$$

a: Collective Coordinate. Need to deal with it exactly.

But $\begin{cases} \mathcal{E}[x] = \mathcal{E}[x_a] \\ \mathcal{D}x = \mathcal{D}x_a \quad \text{(invariant measure)} \end{cases}$ $\quad x(\tau) \mapsto x_a \equiv x(\tau + a)$ translation

Introduce "1"; Let $F(x)$ be a function

$$1 = \int dF \; \delta(F(x_a)) = \int_{-\infty}^{+\infty} da \; \delta(F(x_a)) \frac{\partial F}{\partial a}(x_a)$$

$$\Rightarrow Z = \int \mathcal{D}x \; e^{-\mathcal{E}[x]} \equiv \int \mathcal{D}x \int_{-\infty}^{+\infty} da \; \delta(F(x_a)) \frac{\partial F}{\partial a}(x_a) \, e^{-\mathcal{E}[x]}$$

Change of variables: $x \to x_{-a}$ or $x(\tau) \to x(\tau - a)$

$$\frac{\partial F}{\partial a}(x_a) = D[x, a]$$

$$Z = \int \mathcal{D}x_{-a} \int_{-\infty}^{+\infty} da \; \delta(F(x)) \; D[x_{-a}, a] \; e^{-\mathcal{E}[x_a]}$$

$$(x_a)_{-a} = x_0 \equiv x$$

Using translation invariance and the invariance of the measure =

$$Z = \int \mathcal{D}x \int_{-\infty}^{+\infty} da \; D(x_{-a}, a) \; \delta(F(x)) \; e^{-\mathcal{E}[x]}$$

Let's calculate D

$$D = D[x_{-a}, a]$$

$$\cancel{D[x_a, a] = \frac{\partial F[x_a]}{\partial a} = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0} (x}$$

$$F[x_a] = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0} \left(x(\tau+a) - x_{cl}(\tau)\big|_{a=0} \right)$$

$$\frac{\partial F}{\partial a}[x_a] = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0} \; \frac{\partial x(\tau+a)}{\partial a} = D[x_a, a]$$

$$D = D[x_{-a}, a] = \int_{-\infty}^{+\infty} d\tau \; \frac{\partial x_{cl}}{\partial a}\Big|_{a=0}(\tau) \; \frac{\partial x}{\partial a}\Big|_{a=0}(\tau)$$

$$x(\tau) = x_{cl}(\tau, a) + \sum_{n \neq 0} \xi_n \, y_n(\tau - a)$$

$$x_{cl}(\tau, a) \equiv x_{cl}(\tau - a)$$

$$\frac{\partial x_{cl}}{\partial a} \equiv - \frac{\partial x_{cl}}{\partial \tau}$$

$$\frac{\partial y_n(\tau - a)}{\partial a} = - \frac{\partial y_n}{\partial \tau}$$

$$\Rightarrow \frac{\partial x(\tau)}{\partial a}\Big|_{a=0} = - \left[\frac{\partial x_{cl}}{\partial \tau}(\tau) + \sum_{n \neq 0} \xi_n \frac{\partial y_n(\tau)}{\partial \tau} \right]_{a=0}$$

$$\Rightarrow D = \int_{-\infty}^{+\infty} d\tau \left(\frac{\partial x_{cl}}{\partial \tau}\right)^2 + \sum_{n\neq 0} \xi_n \int_{-\infty}^{+\infty} d\tau \left.\frac{\partial x_{cl}}{\partial \tau}\right|_{a=0}(\tau) \left.\frac{\partial y_n}{\partial \tau}\right|_{a=0}(\tau)$$

$$A = \int_{-\infty}^{+\infty} d\tau \; \dot{x}_{cl}^{\,2} \qquad \dot{x}_{cl} = \left.\frac{\partial x_{cl}}{\partial \tau}\right|_{a=0}$$

$$r_n = \int_{-\infty}^{+\infty} d\tau \; \dot{x}_{cl}\, \dot{y}_n \qquad \dot{y}_n = \left.\frac{\partial y_n}{\partial \tau}\right|_{a=0}$$

$$D = A + \sum_{n\neq 0} \xi_n r_n$$

Hence

$$Z = \int \mathcal{D}x \int_{-\infty}^{+\infty} da \; \delta[F(x)] \; \left[A + \sum_{n\neq 0} \xi_n r_n\right] e^{-\mathcal{E}[x]}$$

$$\equiv \int \prod_n d\xi_n \int_{-\infty}^{+\infty} da \; \left[A + \sum_{n\neq 0} \xi_n r_n\right] e^{-\mathcal{E}[x_{cl}]} \, e^{-\left[\int d\tau\, d\tau' \frac{1}{2} \frac{\delta^2 \mathcal{E}}{\delta x(\tau)\, \delta x(\tau')}\right]_1 \cdot \tilde{x}(\tau)\, \tilde{x}(\tau')}$$

$$\tilde{x} = x(\tau) - x_{cl}(\tau, a) \equiv \sum_{n\neq 0}^{\infty} \xi_n \, y_n(\tau - a)$$

Note that y_n's are the e. functions with non-vanishing e. values. These are necessarily $\xi > 0$ for stability. Also note that since $x_{cl}(\tau, a)$ is monotonic $\Rightarrow \frac{\partial x_{cl}(\tau, a)}{\partial a}$ has no zeroes (nodeless) $\Rightarrow$ is the gnd. state w.f. $\Rightarrow$ it is the lowest energy state $= 0$

Now $F(x)=0$ is automatically satisfied.

$$x_{a=0}(\tau) = x_{cl}(\tau,a)\big|_{a=0} + \sum_{n\neq 0} \xi_n y_n(\tau) \quad ; \quad y_0(\tau) = N\frac{\partial x_{cl}}{\partial a}$$

$$F[x] = \int_{-\infty}^{+\infty} d\tau \; y_0(\tau) \sum_{n\neq 0} \xi_n y_n(\tau) =$$

$$= \sum_{n\neq 0} \xi_n \int_{-\infty}^{+\infty} d\tau \; y_0(\tau)\, y_n(\tau) = 0 \qquad \text{(orthogonality)}.$$

$$\Rightarrow Z = \int_{-\infty}^{+\infty} \prod_n d\xi_n \int_{-\infty}^{+\infty} da \; \left(A + \sum_{n\neq 0} \xi_n r_n\right) e^{-\mathcal{E}[x_{cl}]} \; e^{-S_{eff}[\xi_n]}$$

to leading order we can neglect $\sum_{n\neq 0} \xi_n r_n$

$$Z \simeq \underbrace{\left[\int_{-\infty}^{+\infty} da \; A\right]}_{\text{inv. measure}} e^{-\mathcal{E}[x_{cl}(a)]} \left[\prod_{n\neq 0} \omega_n^{-1}\right] \text{const.}$$

and

$$\langle x(0)\, x(\tau)\rangle \simeq A \int_{-\infty}^{+\infty} da \; x_{cl}(0,a)\, x_{cl}(\tau,a) \; e^{-\mathcal{E}[x_{cl}]} \prod_n \omega_n^{-1}$$

$$Z \approx A \left[\int_{-\infty}^{+\infty} da\right] e^{-\mathcal{E}[x_{cl}]} \prod_n \omega_n^{-1}$$

$\Rightarrow$ Sum

$$\langle x(0), x(\tau)\rangle \simeq \frac{\frac{\mu^2}{\lambda} \prod_{n\neq 0} \omega_{n,0}^{-1} + A\left[\int_{-\infty}^{+\infty} da \; x_{cl}(0)\, x_{cl}(\tau)\right] e^{-\mathcal{E}[x_{cl}]} \prod_{n\neq 0} \omega_n^{-1}}{\prod_{n\neq 0} \omega_{n,0}^{-1} + A\left(\int_{-\infty}^{+\infty} da\right) e^{-\mathcal{E}[x_{cl}]} \prod_{n\neq 0} \omega_n^{-1}}$$

$$\langle x(0)\, x(\tau)\rangle \simeq \frac{\mu^2}{\lambda} + A\, e^{-\mathcal{E}[x_{cl}]} \int_{-\infty}^{+\infty} da \left[x_{cl}(0;a)\, x_{cl}(\tau,a) - \frac{\mu^2}{\lambda} \right] \cdot$$

$$\cdot \left[\frac{\prod_{n\neq 0} \omega_n^{-1}}{\prod_{n\neq 0} \omega_{n,0}^{-1}} \right]$$

$$\prod_{n\neq 0} \omega_n^{-1} = \det{}' \left[-\frac{d^2}{d\tau^2} + (-\mu^2 + 3\lambda\, x_{cl}^2(\tau)) \right]^{-1/2}$$ without zero modes

$$\prod_{n\neq 0} \omega_{n,0}^{-1} = \det{}' \left[-\frac{d^2}{d\tau^2} + (2\mu^2) \right]^{-1/2}$$

This ratio (call it K) can be calculated in terms of phase shifts

$$\int_{-\infty}^{+\infty} da \left[x_{cl}(0,a)\, x_{cl}(\tau,a) - \frac{\mu^2}{\lambda} \right] = -\frac{\mu^2}{\lambda}\tau \frac{2}{\tanh \frac{\mu\tau}{\sqrt{2}}}$$

$$\underset{\tau\to\infty}{\sim} -\frac{2\mu^2}{\lambda}\tau$$

$$f(\tau) \underset{\tau\to\infty}{\sim} \frac{\mu^2}{\lambda} - K A\, e^{-\mathcal{E}[x_{cl}]} \left(\frac{2\mu^2}{\lambda}\right)\tau$$

this approx. fails for $\tau \gtrsim \frac{e^{\mathcal{E}[x_{cl}]}}{2KA} = \frac{e^{2\sqrt{2}\mu^3/\lambda}}{2KA}$

$$A = \int \dot{x}^2\, d\tau \equiv \mathcal{E}[x_{cl}] = \frac{2\sqrt{2}\,\mu^3}{\lambda}$$

$\Rightarrow$ we have problems for $\tau \to \infty$

The width of the kink (or instanton) is $\sim \frac{1}{\mu}$

If λ is small $\Rightarrow$ $e^{2\sqrt{2}\mu^3/\lambda} \gg \frac{1}{\mu}$

Hence we can sum over an ensemble of weakly interacting instantons (exponentially weak!)

Let $$x_{cl}(\tau) \simeq \sqrt{\frac{\mu^2}{\lambda}} \prod_{j=1}^{N} \operatorname{sgn}(\tau - a_j)$$

a_j: location of the j-th instanton.

$$\mathcal{E}_{cl} \text{ for } N \text{ instantons} \simeq N \frac{2\sqrt{2}\mu^3}{\lambda} + \text{exp. small terms}$$

$$\Rightarrow \langle x(0)\, x(\tau) \rangle = \frac{\frac{\mu^2}{\lambda}\left[\sum_{N=0}^{\infty} C^N e^{-\frac{2\sqrt{2}\mu^3}{\lambda}} \int_{a_1<\cdots<a_N} da_1 \cdots da_N \prod_{j=1}^{N} \operatorname{sgn}(\tau - a_j)\right]}{\sum_{N=0}^{\infty} C^N e^{-2\sqrt{2}\frac{\mu^3 N}{\lambda}} \int_{a_1<a_2<\cdots<a_N} da_1 \cdots da_N}$$

$$C = \frac{AK\, 2\tau}{\tanh\left(\frac{\mu\tau}{\sqrt{2}}\right)} \approx 2AK\tau \qquad (\tau \to \infty)$$

$$\int_{a_1<\cdots<a_N} da_1 \cdots da_N = \frac{T^N}{N!}$$

$$\Rightarrow \langle x(0)\, x(\tau) \rangle = \frac{\mu^2}{\lambda}\, e^{-(\Delta E)\cdot \tau}$$

$$\Delta E \cong 2AK\, e^{-\frac{2\sqrt{2}\mu^3}{\lambda}}$$

$\Rightarrow$ The ground state and the 1st excited state are split by an amount

$$\sim 2AK\, e^{-A}$$

$$A = \frac{2\sqrt{2}\,\mu^3}{\lambda}$$

which has an essential singularity in λ.

References:

- Aspects of Symmetry (Coleman)
  Solitons and Instantons (Rajaraman)
  Gauge Fields and Strings (Polyakov)

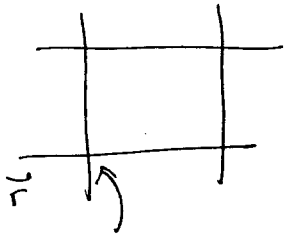
Topological Excitations and Phase Transitions

i.e. Vortices and Disorder
Monopoles and Confinement

We will now discuss two specific examples in which topological excitations (i.e. instantons) drive the phase transition. We will discussed the U(1) non-linear σ-model (i.e. a compactified or periodic scalar) and U(1) compact electrodynamics.

(A) Vortices and phase transitions (Kosterlitz and Thouless)

Consider the following problem in classical D=2 Statistical Mechanics.



$0 \leq \theta(\vec{r}) < 2\pi$

On every site of a lattice we define a two-component unit vector $\vec{n} = (\cos\theta, \sin\theta$

The interaction energy is

$$E = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \vec{n}(\vec{r}) \cdot \vec{n}(\vec{r}') = -J \sum_{\langle \vec{r}, \vec{r}' \rangle} \cos(\theta(\vec{r}) - \theta($$

nearest

$$E \equiv -J \sum_{\vec{r},\mu} \cos(\Delta_\mu \theta(\vec{r}))$$

$\mu = 1, 2$

Note: the naive continuum limit is

$$E \propto \int d^2x \, (\vec{\nabla}\theta)^2$$

which would be a free (Gaussian) field if it were not for the periodicity (symmetry)

$$\theta(\vec{r}) \to \theta(\vec{r}) + 2\pi N(\vec{r})$$

where $N(\vec{r}) \in \mathbb{Z}$. This <u>local</u> symmetry requires that all observables must obey the periodicity condition.

We saw before that in this case the theory must have vortices. We will now that it does and we'll find out what role they play. We will investigate this question using a method called (lattice) <u>duality</u> first introduced by Kramers and Wannier for the 2D

Ising model and closely related to topology

The P.F. is

$$Z = \prod_{\vec{r}} \int_0^{2\pi} \frac{d\theta(\vec{r})}{2\pi} \; e^{-E[\theta]/T}$$

$$= \prod_{\vec{r}} \int_0^{2\pi} \frac{d\theta(\vec{r})}{2\pi} \; e^{+\beta \sum_{(\vec{r},\mu)} \cos(\Delta_\mu \theta(\vec{r}))}$$

$$\beta = \frac{J}{T}$$

Consider an expression of the form $e^{V(\theta)}$, ~~$e^{V(\theta)}$~~ which is <u>periodic</u> under $\theta \to \theta + 2\pi$

$\Rightarrow$ $e^{V(\theta)}$ can be expanded in Fourier series

$$e^{V(\theta)} = \sum_{l \in \mathbb{Z}} e^{il\theta} V_l$$

Note: $e^{i\theta} \in U(1)$ and $\frac{1}{\sqrt{2\pi}} e^{il\theta}$ are vectors in the l-th irred. representation of $U(1)$

$e^{i\theta_1} \in U(1)$ and $e^{i\theta_2} \in U(1) \Rightarrow e^{i(\theta_1+\theta_2)} \in U(1)$

but $e^{il_1\theta} \in U(1)$ and $e^{il_2\theta} \in U(1)$

$$\Rightarrow e^{il_1\theta}\, e^{il_2\theta} = e^{i(l_1+l_2)\theta} \in U(1)$$

$\Rightarrow$ The representations of $U(1)$ form a group under the addition. The representations of $U(1)$ are labelled by integers $\Rightarrow$ the representations are the group $\mathbb{Z}$. ("dual group")

In particular

$$e^{\beta\cos\theta} = \sum_{l=-\infty}^{+\infty} I_l(\beta)\, e^{il\theta}$$

$$I_l(\beta) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{\beta\cos\theta}\, e^{-il\theta}$$

is the modified Bessel function.

For β large $\quad I_\beta(l) \approx \frac{e^\beta}{\sqrt{2\pi}}\, e^{-\frac{l^2}{2\beta}}$

$$\Rightarrow Z = \# \int \prod_r \frac{d\theta(r)}{2\pi} \sum_{\{l_\mu(\vec{r})\}} e^{-\sum_{(r,\mu)} \frac{l_\mu^2(\vec{r})}{2\beta} + i\sum_{(r,\mu)} l_\mu(r)\Delta_\mu\theta}$$

i.e. we assign one $l_\mu(r) \in \mathbb{Z}$ to each link $(\vec{r},\mu)$

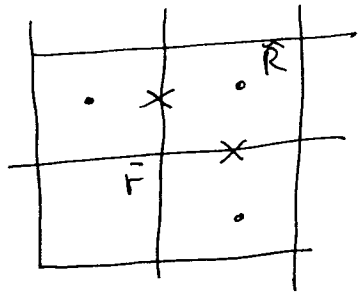
Since $\int_0^{2\pi} \frac{d\theta(r)}{2\pi} e^{i\theta(\vec{r})\Delta_\mu l_\mu(\vec{r})} = \delta(\Delta_\mu l_\mu(r))$

$$\Delta_\mu l_\mu = \sum_{\mu=1,2} [l_\mu(r) - l_\mu(r-e_\mu)]$$

$$Z = \# \sum_{\{l_\mu(\vec{r})\}} \prod_{\vec{r}} \delta(\Delta_\mu l_\mu(\vec{r})) \, e^{-\sum_{(r,\mu)} \frac{l_\mu^2(\vec{r})}{2\beta}}$$

Solve the constraint by going to the dual lattice

$\Delta_\mu l_\mu = 0 \Rightarrow l_\mu = \varepsilon_{\mu\nu} \Delta_\nu S$



where $\underline{S \in \mathbb{Z}}$

defined on the sites of the <u>dual</u> <u>lattice</u> $\vec{R}$

$$\Rightarrow Z = \# \sum_{\{S(\vec{R})\}} e^{-\frac{1}{2\beta} \sum_{(R,\mu)} (\Delta_\mu S(R))^2}$$

Note: $\beta \longleftrightarrow \frac{1}{\beta}$ (strong coupling $\longleftrightarrow$ weak coupling)

$U(1) \longleftrightarrow \mathbb{Z}$

direct lattice $\longleftrightarrow$ dual lattice.

<u>Alternative procedure</u> (Villain) (equivalent)

We can also make the replacement

$$e^{\beta\cos\theta} \to \# \sum_{\ell} e^{-\frac{\beta}{2}(\theta - 2\pi\ell)^2}$$

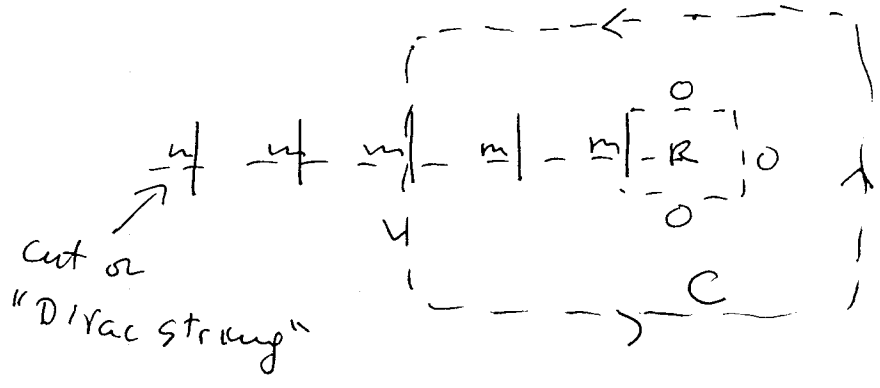
and we recover periodicity if

$$\theta \to \theta + 2\pi k$$
$$\ell \to \ell + k$$

$\Rightarrow$ the integers ℓ are associated with vortices.

$$\Rightarrow \int \mathcal{D}\theta\, e^{\beta \sum_{r,\mu} \cos(\Delta_\mu \theta)} \to \sum_{\{\ell_\mu(r)\}} \int \frac{d\theta(r)}{2\pi}\, e^{-\frac{\beta}{2}(\Delta_\mu \theta - 2\pi\ell_\mu)^2}$$

θ: has no vorticity. Consider a config. of ℓ_μ:



$$\sum_{C} \ell_\mu = m \quad \text{at } R$$

$\Rightarrow$ θ must jump by $2\pi m$ across the cut

The dual theory has degrees of freedom which are integers. It can be regarded as a "height" of some column of atoms

$\Rightarrow \{S(R)\}$ is a discretized surface

(surface roughening or discrete gaussian)

Suppose we wanted to compute

$$\langle e^{i\theta(\vec{r})} \; e^{-i\theta(\vec{r}\,')} \rangle$$

The dual of this expectation value requires that we now solve the constraint

$$\Delta_\mu \ell_\mu = n(x)$$

$$n(\vec{x}) = \delta(\vec{x}-\vec{r}) - \delta(\vec{x}-\vec{r}\,')$$

The solution is

$$\ell_\mu = \varepsilon_{\mu\nu}(\Delta_\nu S + B_\nu)$$

where $B_\mu(R)$ is a vector field on the links of the dual lattice /

$$\varepsilon_{\mu\nu}\Delta_\mu B_\nu = n(x) \qquad \text{(flux!)}$$

Let us work on the Discrete Gaussian Model

$$Z = \sum_{\{S(R)\}} e^{-\frac{1}{2\beta}\sum_{(R,\mu)} (\Delta_\mu S(R))^2}$$

Poisson Summation Formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{y\to 0} \int_{-\infty}^{+\infty} d\phi \sum_{m=-\infty}^{+\infty} e^{i2\pi m\phi - ym^2} f(\phi)$$

$$\Rightarrow Z = \sum_{\{m(R)\}} \prod_R \int d\phi(R)\; e^{-\frac{1}{2\beta}\sum_{(R,\mu)}(\Delta_\mu\phi)^2 + i\sum_R 2\pi m(R)\phi}\; e^{-y\sum_R m^2(R)}$$

<u>Version 1</u>

For z small, $z = e^{-y}$, we can approximate

$$\sum_{m(R)\in\mathbb{Z}} e^{-y\,m^2(R) + i\,2\pi\, m(R)\phi(R)} =$$

$$= 1 + 2e^{-y}\cos(2\pi\phi(R)) + O(e^{-4y})$$

$$\approx e^{2z\cos(2\pi\phi(R))} + O(z^2)$$

$$\Rightarrow Z = \int \prod_R d\phi(R) \; e^{-\frac{1}{2\beta}\sum_{(R,\mu)} (\Delta_\mu \phi)^2 + 2z \sum_R \cos(2\pi\phi_R)}$$

$$\underset{a\to 0}{\approx} \int \mathcal{D}\phi \; e^{-S[\phi]}$$

$(a \to 0)$ $$S[\phi] = \frac{K}{2}\int d^2x \, (\partial_\mu \phi)^2 + \frac{g}{2a^2}\int d^2x \, \cos(2\pi\phi)$$

$$\frac{K}{\cancel{\beta}} = \frac{1}{\beta} = \frac{T}{J} \qquad 2z = \frac{g}{2} \Rightarrow g = 4z$$

$$\sqrt{K}\,\phi = \varphi$$

different β!

$$S[\phi] = \int d^2x \, \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{g}{2a^2}\cos(\beta\varphi)$$

Sine-Gordon Field Theory!

$$\sqrt{K}\,\beta = 2\pi$$

$$\beta = \frac{2\pi}{\sqrt{K}}$$

$$\Rightarrow \beta = 2\pi\sqrt{\frac{J}{T}}$$

Version 2: Integrate ϕ out:

$$\int \prod_R d\phi(R)\; e^{-\frac{1}{2\beta}\sum_{R,\mu}(\Delta_\mu\phi)^2 + i\sum_R 2\pi m(R)\phi(R) - \sum_R y m^2}$$

$$= e^{-\sum_R y\, m^2(R)}\; e^{-\frac{1}{2}\sum_{R,R'}(2\pi)^2 m(R)\, G(R-R')\, m(R')}$$

$$\times \left(\int \prod_R d\phi(R)\; e^{-\frac{1}{2\beta}\sum_{R,\mu}(\Delta_\mu\phi)^2}\right)$$

where $-\Delta^2 G = \beta\,\delta_{R,R'}$

lattice Laplacian

For $|R-R'| \gg a$

$$G(R-R') \approx \frac{\beta}{2\pi}\ln\left(\frac{\vec{R}-\vec{R}'}{a}\right)$$

lattice spacing

$$\Rightarrow Z = \sum_{\{m(R)\}}{}' \; e^{-E[\{m(R)\}]}$$

$$= \sum_{\{m(R)\}}{}' \; e^{-\frac{1}{2}\sum_{R,R'}(2\pi)^2 \frac{\beta}{2\pi} m(R)\ln\left(\frac{|R-R'|}{a}\right) m(R')}$$

$$\times\; e^{-\sum_R y\, m^2(R)}$$

$\sum_R m(R) = 0$

core energy

$$Z = \sum_{\{m(R)\}}{}' \; e^{-\frac{\pi}{2T}\sum_{R,R'} m(R)\ln\left(\frac{|R-R'|}{a}\right) m(R')}$$

This is a $D=2$ (logarithmic) Coulomb gas. We saw before that this precisely the contribution of vortices.

$$Z \propto \sum_{\{m(R)\}}{}' \; e^{-\frac{1}{2}\sum_{(R,R')} m(R)\, m(R')\, U(R-R')}$$

$$U(R-R') = \frac{1}{2\pi} \log \left| \frac{R-R'}{a} \right|$$

$$t = \frac{T}{4\pi^2} \quad , \quad y = \frac{\mu}{t}$$

What is the self energy (or action) of one vortex?

$$E = \mu + U(L)$$

where L is the linear size of the system.

in 3D $\quad U(R) \sim \frac{1}{R} \xrightarrow[R\to\infty]{} 0$

in 2D $\quad U(R) \sim \frac{1}{2\pi} \ln\left(\frac{R}{a}\right) \longrightarrow$ diverges as $R \to \infty$

($\Rightarrow$ this forces the neutrality condition)

What is the entropy for 1 vortex?

Entropy $= \ln L^2$

# of available state.

$$\Rightarrow \left(\text{Free energy}\right)_{1\text{ vortex}} = E - tS$$

$$= \left(\mu + \frac{1}{2\pi}\ln\left(\frac{L}{a}\right)\right) - t\,2\ln\left(\frac{L}{a}\right)^2$$

$F < 0$ if $t > t_c$

$$2t_c = \frac{1}{2\pi} \Rightarrow \text{for } \boxed{t > \frac{1}{4\pi}}$$

For $t > t_c = \frac{1}{4\pi}$ free vortices proliferate (or "condense $\Rightarrow$ vortex plasma

For $t < t_c$ free vortices are suppressed and exist only in vortex–antivortex pairs.

$\Rightarrow$ "dielectric"

We can ask a similar question in the sine-Gordon language. For what values of β is the $\cos\beta\varphi$ relevant

We will see in a few lectures that the

Scaling dimension of $\cos\beta\varphi$:

We will show below that

$$\langle e^{i\beta\varphi(x)} e^{-i\beta\varphi(y)} \rangle \propto \frac{1}{|x-y|^{2\Delta}}$$

$\Delta = \frac{\beta^2}{4\pi}$ is the scaling dimension.

Since we are in $D=2$ dimensions, $\cos\beta\varphi$ is <u>irrelevant</u> if $\Delta > 2$

$$\Rightarrow \frac{\beta^2}{4\pi} > 2 \Rightarrow \beta^2 > \beta_c^2 = 8\pi$$

Since $\beta^2 = \frac{4\pi^2}{T} \Rightarrow \frac{4\pi^2}{T} > 8\pi$

$$T < T_c = T_{KT} = \frac{\pi}{2}$$

Kosterlitz-Thouless

Hence for $T < T_c$, $\cos\beta\varphi$ is irrelevant and we can work with the free field model $\Rightarrow$ no unbound vortices. For $T > T_c$ $\cos\beta\varphi$ is relevant, vortices proliferate $\Rightarrow$ φ gets pinned at minima $\frac{2\pi}{\beta} n$, $n \in \mathbb{Z}$

(B) Monopoles (Polyakov)

We will now discuss compact QED. We saw before that this theory has monopoles which we will see play a role analogous to the vortices of the 2D system.

We define compact QED on a lattice (as we did before) in terms of an angle-valued vector field $0 < A_\mu(\vec{r}) < 2\pi$ defined on the links $(\vec{r}, \mu)$ of a cubic lattice.

The partition function is

$$Z = \int_0^{2\pi} \prod_{(r,\mu)} \frac{dA_\mu(r)}{2\pi} \, e^{-S[A]}$$

$$-S(A) = \frac{1}{2g^2} \sum_{(r,\mu\nu)} \cos\left(F_{\mu\nu}(r)\right)$$

plaquette

$$F_{\mu\nu}(r) = \sum_{\text{plaquette}} A_\mu \equiv \Delta_\mu A_\nu - \Delta_\nu A_\mu$$

<u>Note</u>: due to the periodicity $A_\mu \to A_\mu + 2\pi l_\mu$ the Bianchi Identity for $F_{\mu\nu}$ cannot be satisfied but it is violated modulo 2π

$\Rightarrow$ We can have monopoles with a quantized magnetic charge. We can use an analogous procedure to what we did for vortices

$$-S = \frac{1}{2g^2} \sum_{(\vec{r},\mu\nu)} \cos F_{\mu\nu}$$

plaquette

$$\int \mathcal{D}A_\mu \, e^{-S} \equiv \int \prod_{(\vec{x},\mu\nu)} \frac{dA_\mu(x)}{2\pi} \sum_{\{m_{\mu\nu}\}} e^{-\frac{1}{4g^2}\left(F_{\mu\nu} + 2\pi m_{\mu\nu}\right)^2}$$

on plaquettes!

$A_\mu \to A_\mu + 2\pi l_\mu$ "discrete" one-form

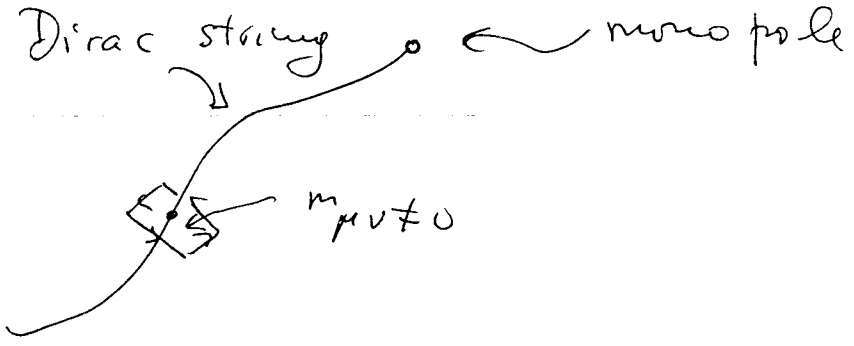
$F_{\mu\nu} \to F_{\mu\nu} + 2\pi\left(\Delta_\mu l_\nu - \Delta_\nu l_\mu\right)$

$m_{\mu\nu} \to \left(\Delta_\mu l_\nu - \Delta_\nu l_\mu\right)$ to enforce periodicity

"discrete" 2-form

$\Rightarrow$ $F_{\mu\nu}$ may satisfy Bianchi but $m_{\mu\nu}$ <u>does not</u>

In fact $\frac{1}{2}\varepsilon_{\mu\nu\lambda}\Delta_\mu m_{\nu\lambda} = \Phi$ measures the violation of the Bianchi identity. Notice that $\Phi \in \mathbb{Z}$. The integers $\{m_{\mu\nu}\}$ live on plaquettes and represent Dirac strings.



$$Z = \sum_{\{m_{\mu\nu}\}} \cancel{e^{-\frac{g^4}{4} m_{\mu\nu}^2}} \; e^{-\frac{g^2}{4}\sum_{(r,\mu\nu)} m_{\mu\nu}^2(r)}$$

$$\int \mathcal{D}A \times e^{-\frac{i}{2}\sum_{(r,\mu\nu)} m_{\mu\nu} F_{\mu\nu}}$$

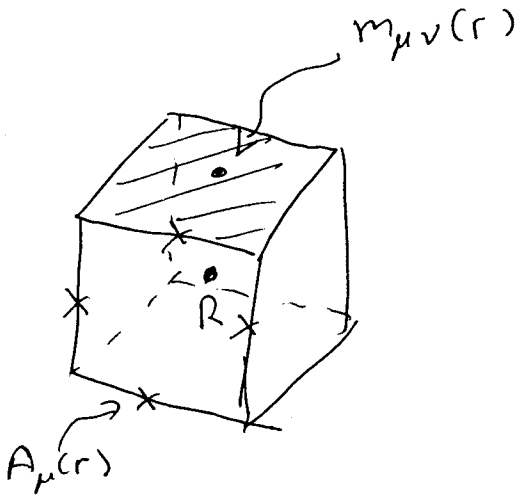
$$\int \mathcal{D}A_\mu \; e^{+\frac{i}{2}\sum_{(r,\mu\nu)} A_\mu \Delta_\nu m_{\mu\nu}} = \delta(\Delta_\nu m_{\mu\nu})$$

is the constraint on every link

Solution

$$\Delta_\nu m_{\mu\nu} = 0 \Rightarrow m_{\mu\nu} = \varepsilon_{\mu\nu\lambda}\Delta_\lambda S \quad , \; S(R)\in\mathbb{Z}$$

dual cubic s.t.



$$\Rightarrow \quad Z = \sum_{\{m_{\mu\nu}(r)\}} e^{-\sum_{(r,\mu\nu)} \frac{g^2}{4} (m_{\mu\nu}(r))^2} \prod_{(r,\mu)} \delta(\Delta_\nu m_{\mu\nu})$$

$$= \sum_{\{S(R)\}} e^{-\frac{g^2}{2} \sum_{(R,\mu)} (\Delta_\mu S(R))^2}$$

it is again a Discrete Gaussian Model!

but in $D=3$ dimensions!

The same line of argument shows that we can write this as a sum over monopole configurations (a Coulomb gas) or as a sine-Gordon theory. ξ

For the monopole gas we get (using the PSF)

$$Z = \sum_{\{m(R)\}}{}' \; e^{-\frac{1}{2}\sum_{R,R'}\left(\frac{2\pi}{g}\right)^2 m(R)\, G(R-R')\, m(R')}$$

(neutrality)

where $G(R-R') = \dfrac{1}{4\pi|R-R'|}$

In this case we have a ~~D~~ three-dimensional (neutral) Coulomb gas. It is well known that this gas is a plasma phase <u>at all temperatures</u> $t = \frac{g^2}{4\pi^2}$

$\Rightarrow$ energy $< \infty$ but entropy still $\sim \ln L \Rightarrow$ <u>entropy dominates</u> $\forall$

$\Rightarrow$ monopoles proliferate (or "condense") for <u>all</u> values of g! $\Rightarrow$ Polyakov showed that this leads to <u>an Area ~~Law~~ Law</u> for Wilson Loop operators $\Rightarrow$ Confinement

$\Rightarrow$ Monopole "condensation" $\Longrightarrow$ confinement.

Once again, the Coulomb Gas is equivalent to a sine Gordon theory in the same Euclidean dimension

$$S_D = \int dx^D \left[\frac{1}{2} (\partial \phi)^D - \frac{y}{a^D} \cos \beta \phi \right]$$

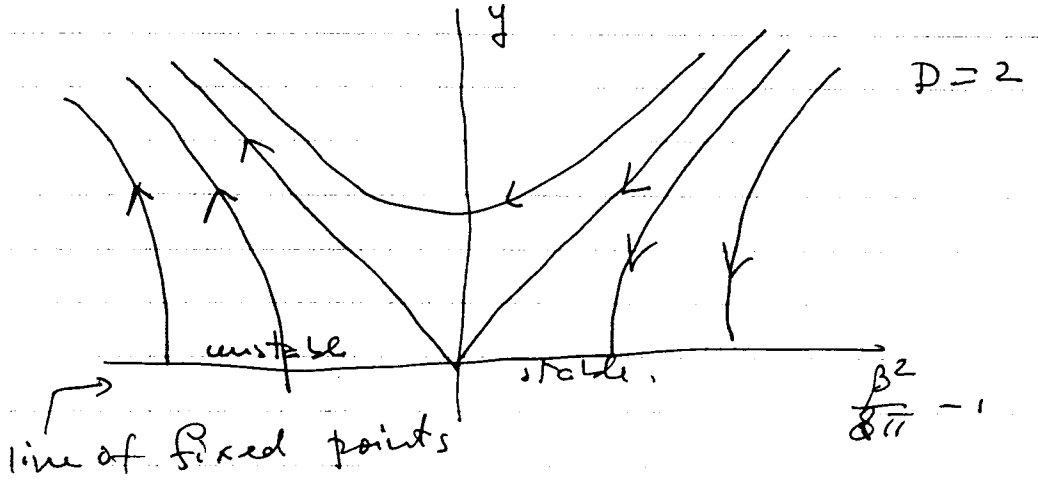
Monopole Gas: $\beta = \frac{2\pi}{g}$

Vortex Gas: $\beta = \frac{2\pi}{\sqrt{T}}$

y: fugacity.

Main Difference:

$D=2$ RG Flow $\Leftrightarrow$ Kosterlitz



$D > 2$ $\quad [\phi] \neq 1 \quad ([\phi] \sim L^{-(D-2)}) \Rightarrow$

β is not dimensionless $\Rightarrow$ no fixed line

$\Rightarrow y$ always grows $\Rightarrow$ plasma phase of monopoles

$\Rightarrow$ screening and screening length $\Leftrightarrow$ confinement