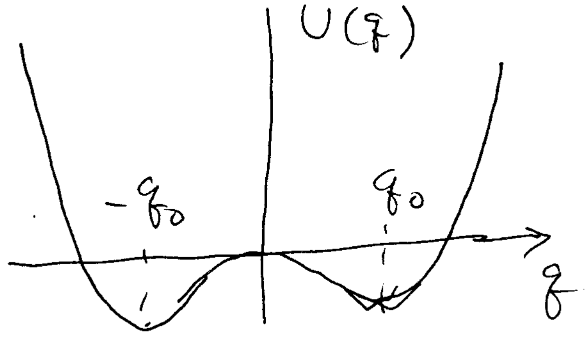
Instantons and Solitons

(Kinks, vortices and monopoles)

We will now discuss the role of topological non-trivial configurations in QFT. There are many reasons for looking at this problem. One motivation is the study of mechanism for quantum disorder, i.e. of the physical origin of phases of QFT's exhibiting confinement and/or lack of long range order. This is also related to the problem of understanding tunneling processes in QFT. Also, this leads to an understanding of the existence of textured or topological excitations of interest both in QFT and Statistical Physics.

We will begin by defining what do we mean by topological excitations. Consider first a very simple problem, a double-well anharmonic oscillator. The potential for the coordinate q has the form



$U(q)$ has two degenerate minima at $\pm q_0$. Suppose we want to compute the tunneling amplitude from $+q_0$ to $-q_0$ (in imaginary time)

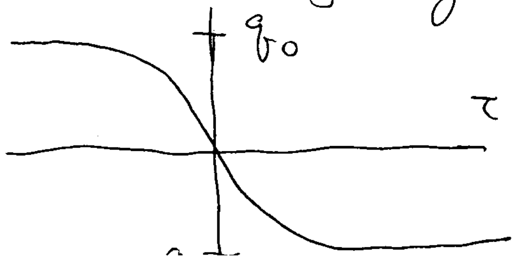
$$\langle q_0 | e^{-H\tau} | -q_0 \rangle$$

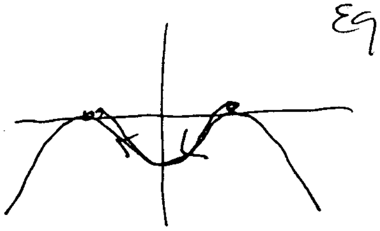
In path integral language this corresponds to a classical trajectory (in imaginary time). The (Euclidean) Lagrangian is

$$L = \frac{1}{2} m \dot{q}^2 + U(q) \qquad \text{(i.e. the energy)}$$

$$\dot{q} = \frac{\partial q}{\partial \tau}$$

The trajectory looks like this





Eq. Newton problem

It is easy to show (see below in these notes) that the Classical Euclidean Action is finite. Such ~~[illegible]~~ finite action Euclidean trajectories (or configurations) are called instantons. It has non-trivial topology in that ~~it~~ interpolates between to inequivalent ground states (or vacua) at $\tau = \pm \infty$, and that it cannot be smoothly deformed to the trivial configurations $q = q_0$ or $q = -q_0$. In this case we see that we can either get an instanton or an anti-instanton which executes the opposite trajectory.

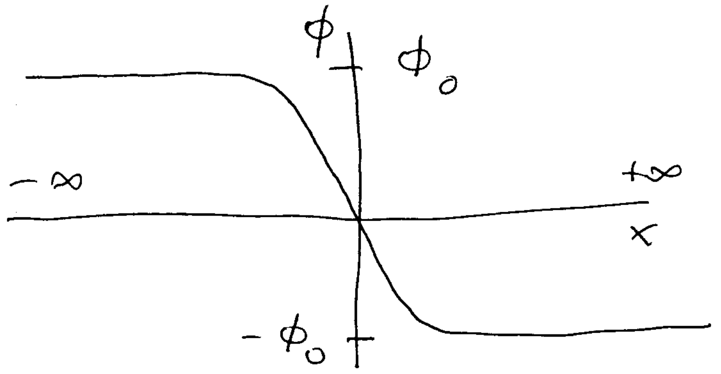
Similarly we can look at a ϕ^4 field theory in 1+1 Minkowsky space-time, which has the Lagrangian

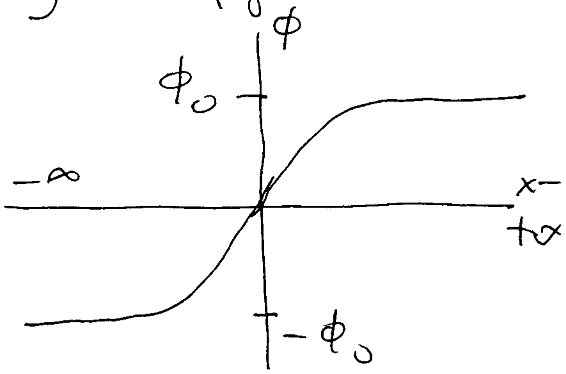
$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - U(\phi)$$

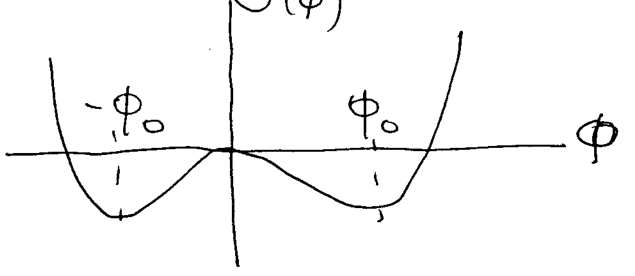
and a Hamiltonian density

$$\mathcal{H} = \frac{\Pi^2}{2} + \frac{1}{2}(\partial_x\phi)^2 + U(\phi)$$

We can find now finite energy (static) configurations of $\mathcal{H}$. These static configurations obey the same equations in space x as the Euclidean anharmonic (double well) oscillator (we just discussed) does in imaginary time. The finite energy configuration is





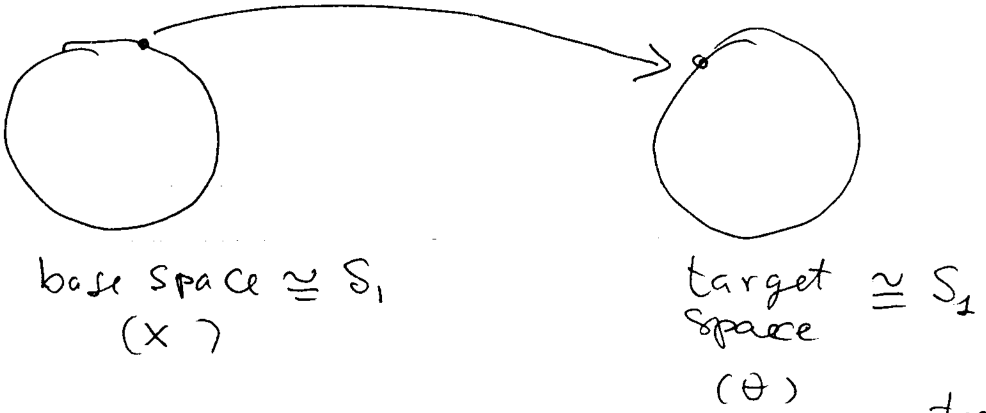


These finite energy states are (topological) solitons are called kinks or domain walls.

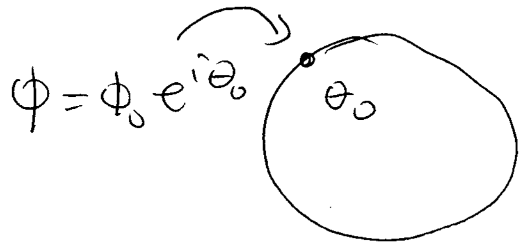
The classical solutions are $\phi = \pm \phi_0 \tanh\left(\frac{x-x_0}{\sqrt{2}\xi}\right)$
$(\xi = |m_0|^{-1})$

Let us consider now a case with a continuous symmetry. For simplicity we will begin with $U(1)$. In this case the matter field is a complex scalar field and we will be interested in the regime in which $\phi = \phi_0 e^{i\theta}$, where ϕ_0 is the vacuum expectation value, and θ is a phase, with $0 \leq \theta < 2\pi$. Let us consider first the case of a <u>soliton</u> in $D=1+1$ dimensions. We will work with periodic boundary conditions in space and we seek time-independent finite energy states. Since we have imposed PBC's the line has been effectively wrapped onto a circumference of radius $R = \frac{L}{2\pi}$. A configuration of this type is then an <u>assignment</u> (or <u>mapping</u>) of a point of the <u>base space</u> (which topologically is a circumference S_1) to a value of the phase θ.

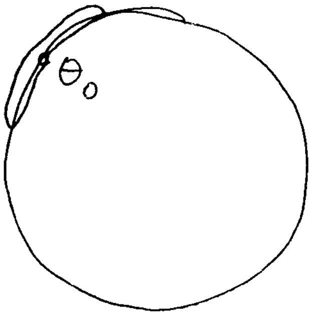
However, the phase is also topologically another circumference, S_1, which we will call the target space. Pictorially a configuration is a mapping $S_1 \longrightarrow S_1$



A constant field configuration is represented by a point in the target space S_1



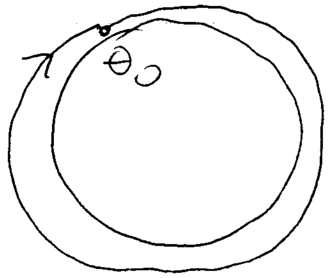
There are many configurations which can be obtained by a smooth, ~~diff~~ (continuous) deformation of the constant field config. We will represent these configurations as closed curves on the circumference. They are closed due to the PBC's



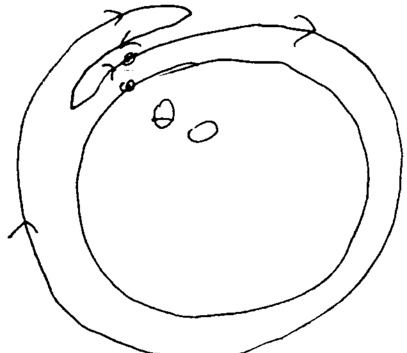
This is a trivial configuration which can be deformed smoothly to the constant field configuration. It is a "contractible loop".

<u>Def.</u> we say that two configurations (or mappings) are <u>homotopic</u> to each other if they can be deformed continuously into each other.

Clearly configurations can be added since their phases can also be added. Under addition these smooth mappings, which we will call <u>homotopies</u>, form a <u>group</u>. However there are configurations that cannot be ~~defo~~ deformed smoothly to the identity. One ca is a configuration which *winds* around the circumference just ~~one~~ *once*. But there are ∞ many configs which wind once around the circle and they can all be deformed smoothly into *each other* but *not* onto the identity.



a topologically non-trivial config. with winding # = 1

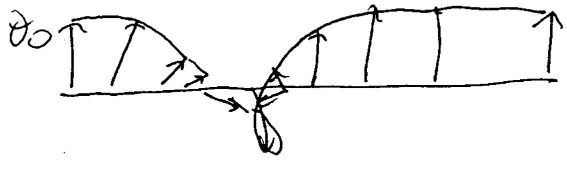


another topologically non-trivial config. with winding # = 1

⟷

they are homotopic to each other!

$n=0$



$n=1$

Hence, homotopies can be classified by a winding number which labels an equivalence class of homotopies and hence is a topological invariant.

In this case, the homotopy classes are labelled by the integers and hence the set of homotopy classes of $S_1 \to S_1$ is isomorphic to $\mathbb{Z}$. We state this conclusion by the symbol

$$\boxed{\pi_{①}(S_1) \cong \mathbb{Z}}$$

target space

homotopy classes of mappings from $\boxed{S_1} \to S_1$

Note: this means that homotopy classes for groups; usually called homotopy group

$$\text{Winding Number} = N = \frac{1}{2\pi}\int_0^L dx\, \partial_x \theta = \frac{(\Delta\theta)_L}{2\pi} \in \mathbb{Z}$$

or
$$N = \frac{1}{2\pi}\int_0^L dx \; e^{i\theta(x)}\, i\partial_x e^{-i\theta(x)}$$

Q: Does this always work? NO. To see this consider the same problem but for an O(3)

Since the target space of ~~a~~ this non-linear σ-model is the 2-sphere S_2, ~~we~~ now have mappings from

$$\underset{\overset{\uparrow}{\text{base}}}{S_1} \longrightarrow \underset{\overset{\uparrow}{\text{target}}}{S_2}$$

Again, a constant field config. is represented by a point $\vec{n}_0$ on S_2. An arbitrary smooth configuration is represented by a closed loop Γ on S_2 (because S_1 is closed)

Clearly all smooth loops on S_2 ~~are~~ are contractible ~~contractable~~ and can be deformed (shrunk) continuously ~~[illegible]~~ to a point. ⇒ they are topologically trivial. We express this by the statement that

$$\pi_1(S_2) = 0$$

Q: Does this mean that everything is trivial for $D \geq 1$? NO!

Consider as an example a complex scalar field in $D=2$ (this, once again, is either $D=1+1$-dimensional Euclidean space for instantons or $D=2$ space for solitons). Since the action has terms of the form $|D_\mu \phi|^2$ it is clear that to avoid divergences we must seek solutions which approach at most a constant amplitude ϕ_0

$$\phi \xrightarrow[r \to \infty]{} \phi_0 \, e^{i\theta}$$

but perhaps an variable phase. Consider now a large circle of radius R, large enough so that $|\phi| \approx \phi_0$. Then the configurations on the circle are smooth maps of the circle (S_1) onto

Q: Does this mean that everything is trivial in higher dimensions? <u>NO</u>, again!

Consider a field with a U(1) symmetry in D = 2 Euclidean space (either a 1+1-Dim Euclidean space-time for instantons or a D = 2 space for solitons)

Let $\phi(x) = \rho(\vec{r})\, e^{i\theta(\vec{r})}$

where $\rho(\vec{r}) \xrightarrow[r\to\infty]{} \phi_0$

and consider a large circle of radius R on the plane ⇒ For R large enough $\rho(R) \simeq \phi_0$

but $\theta(r)$ can wind. Indeed the action (or energy, depending of the case)

$$S = \int d^2x \left[\frac{1}{2} |D_\mu \phi|^2 + U(\phi) \right]$$

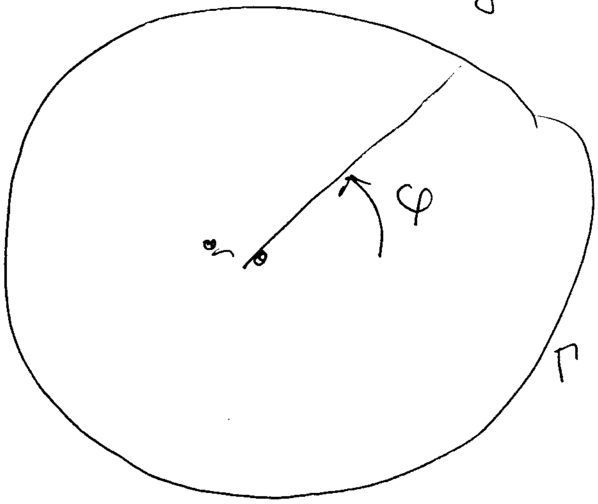
⇒ we can minimize it by letting

$$\partial_\mu \phi \to 0 \quad \text{as} \quad R \to \infty$$

the phase of the field ϕ, which is also another circle $S_1 \Rightarrow$ These configurations are also homotopies with homotopy classes $\pi_1(S_1) \cong \mathbb{Z}$ and are classified by a <u>winding number</u> N

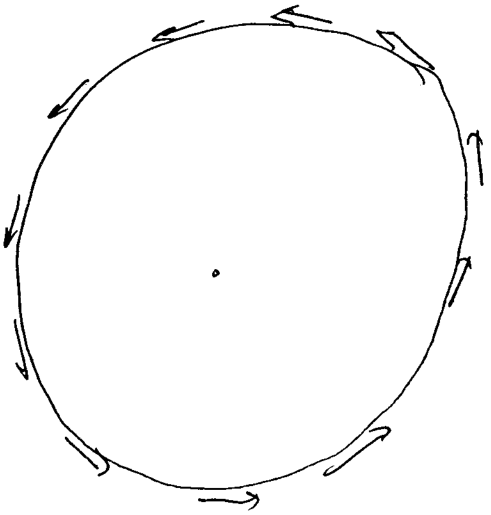
$$N = \cancel{\int_0 \frac{d\theta}{2\pi}} \int_0^{2\pi} \frac{d\varphi}{2\pi} \; e^{i\theta(\varphi)} \; i\partial_\varphi \; e^{-i\theta(\varphi)}$$

azymuthal angle $\Rightarrow N \doteq \frac{(\Delta\theta)_\Gamma}{2\pi}$



Clearly $\phi(\bar{r})$ must have a singularity somewhere ~~inside~~ in the region inside Γ. Let that point be the origin where $\rho(r) \xrightarrow[r \to 0]{} 0$ fast enough for $S < \infty$

Such a configuration is called a vortex



and the real two component field (ϕ_1, ϕ_2)

$$\phi = \phi_1 + i\phi_2$$

is tangent to the circle of radius R.

Consider for example the case of a complex scalar field coupled to a U(1) Maxwell gauge field. The (Euclidean) Lagrangian is

$$\mathcal{L} = |D\phi|^2 - m_0^2|\phi|^2 + \lambda|\phi|^4 + \frac{1}{4}F_{\mu\nu}^2$$

$$\Rightarrow \quad j^\mu = \frac{i}{2}\phi^* D^\mu \phi \quad \text{is the current}$$

$$D_\mu = \partial_\mu + i\frac{q}{\hbar c}A_\mu$$

and the Eqns of motion are $\partial_\mu F_{\mu\nu} = j_\nu$

+ LG eqn for ϕ.

$$\Rightarrow |D_\mu \phi| \to 0 \quad \text{as} \quad r \to \infty \Rightarrow$$

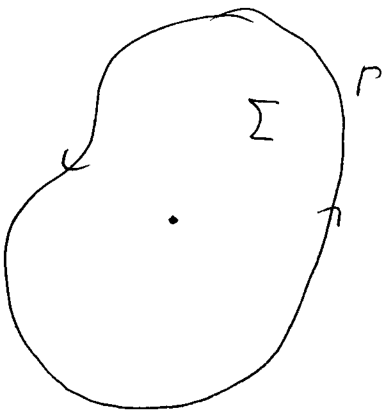
$$\phi \to \phi_0 e^{i\theta}$$

$$D_\mu \phi \to \phi_0 \, i \left(\partial_\mu \theta + \frac{q}{\hbar c} A_\mu \right) e^{i\theta}$$

$$\Rightarrow |D_\mu \phi|^2 \to 0 \Longleftrightarrow \partial_\mu \theta + \frac{q}{\hbar c} A_\mu \to 0$$

$$\Rightarrow \frac{(\Delta\theta)_\Gamma}{2\pi} = \frac{1}{2\pi} \oint_\Gamma d\vec{r} \cdot \vec{A} \quad \frac{q}{\hbar c}$$

but $\displaystyle\oint_\Gamma d\vec{r} \cdot \vec{A} = \iint_\Sigma \vec{B} \cdot d\vec{S} = \Phi$ flux



For this to hold

$$A_\theta \to \frac{\Phi}{2\pi r} \quad \text{as } r \to \infty$$

(pure gauge $\Rightarrow B(r) \to 0$ as $r \to \infty$)

$$\Rightarrow N = \frac{1}{2\pi} \frac{q}{\hbar c} \Phi \equiv \frac{q}{hc} \Phi$$

$\phi_0 = \dfrac{hc}{e}$ flux quantum

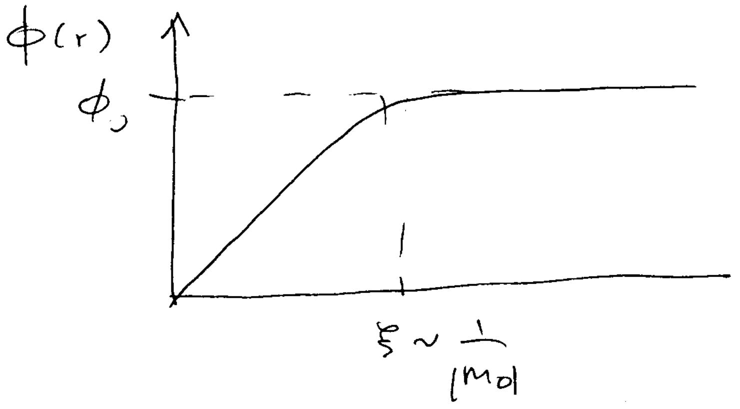
$$\Rightarrow N = \left(\frac{q}{e}\right) \Phi / \phi_0$$

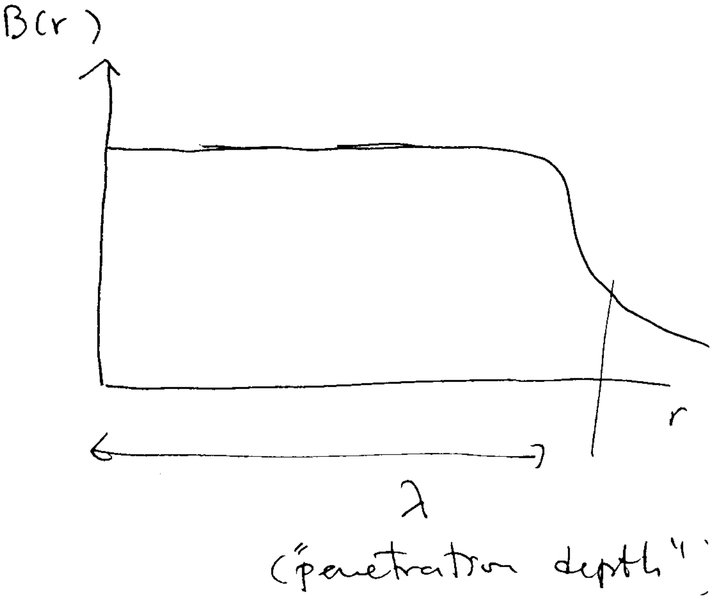
$\Rightarrow$ the winding # N (vorticity) is determined by the # of flux quanta. $\Rightarrow$ Abrikosov vortex (also called Nielsen-Olsen

In this case $B(r) \to 0$ as $r \to \infty$

and $\phi(r) \to 0$ as $r \to 0$ but on $\neq$

scales





In general $\lambda > \xi$

These are finite action (or energy) solutions. But for $\Phi = 0$ vortices have logarithmically divergent selfenergies.

Let us consider now the case ~~[illegible]~~ in which there is no ~~[illegible]~~ electromagnetic field and assume that the potential is steep enough so as to constrain the field to be of fixed length $\Rightarrow$ $\phi = \phi_0 \, e^{i\theta}$

This is a problem at the core of the vortex. But even ignoring this we can see what happens. In this case

$$\partial_\mu \phi = \phi_0 \, i \partial_\mu \theta \, e^{i\theta}$$

$$\Rightarrow |\partial_\mu \phi|^2 = \phi_0^2 (\partial_\mu \theta)^2$$

and the energy (or Euclidean action) is

$$E = \int d^2x \, \frac{\phi_0^2}{2} (\partial_\mu \theta)^2$$

Let us assume that we have vortices at $\{\vec{x}_i\}$ ~~ϕ~~ with vorticities $\{n_i\}$ $\Rightarrow$ θ must satisfy

$$-\nabla^2 \theta = \rho(x) = \sum_i n_i \delta^2(\vec{x}-\vec{x}_i)$$

$$\theta(x) = \Sigma_i n_i \, \mathrm{Im} \log (z - z_i) \qquad (z \in \mathbb{C})$$

i.e. $\nabla^2 \theta = 0$ except at singular points (the "vortex cores")

note: $\theta(x)$ is <u>multivalued</u>, with a branch cut ending at each $\vec{x}_i$

$$\Rightarrow E = \frac{\phi_0^2}{2} \int d^2x \, (\partial_\mu \theta)^2 =$$

(same as in 2D electrostatics!)

$$= -\frac{\phi_0^2}{2} \int d^2x \, \theta \, \partial_\mu^2 \theta$$

$$= + \frac{\phi_0^2}{2} \int d^2x \, \rho(x) \, \theta(x)$$

Let us solve

$$-\nabla^2 \Theta = \rho(x)$$

$$\Theta(x) = \int d^2x \; G(x-y) \; \rho(y)$$

where $G(x-y)$ is the 2D Green function.

$$-\nabla^2 G(x-y) = \delta^2(x-y)$$

$$\Rightarrow \quad E = \frac{\phi_0^2}{2} \int d^2x \; \rho(x) \, \Theta(x)$$

$$= \frac{\phi_0^2}{2} \int d^2x \int d^2y \; \rho(x) \; G(x-y) \, \rho(y)$$

$$= \frac{\phi_0^2}{2} \sum_{i,j} n_i \, n_j \; G(x_i - x_j)$$

$$= \frac{\phi_0^2}{2} \sum_i n_i^2 \, G(0) + \cancel{2} \frac{\phi_0^2}{\cancel{2}} \sum_{i>j} n_i \, n_j \, G(x_i \text{-}$$

$$= \frac{\phi_0^2}{2} \Big(\sum_i n_i\Big)^2 G(0) + \phi_0^2 \sum_{i>j} n_i n_j \big[G(x_i - x_j) - G$$

I will assume a configuration with vanishing total vorticity $\sum_i n_i = 0$ (I need this since $G(0)$ is singular!)

$$\Rightarrow \; E[n_i] = \phi_0^2 \sum_{i>i} n_i \, n_j \, \big[G(x_i - x_j) - G(0)\big]$$

I will define

$$G(0) = \lim_{a\to 0} G(|\vec{a}|)$$

and use that the Green function in D dimensions is

$$G(\vec{x}) = \int \frac{d\vec{p}^{\,D}}{(2\pi)^D} \frac{e^{i\vec{p}\cdot\vec{x}}}{p^2} = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}|x|^{D-2}}$$

$$\Rightarrow \quad G(|x|) - G(|\vec{a}|) =$$

$$= \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}} \left(\frac{1}{|x|^{D-2}} - \frac{1}{a^{D-2}}\right) \xrightarrow[D\to 2]{} \frac{1}{2\pi}\ln\left(\frac{a}{|x|}\right)$$

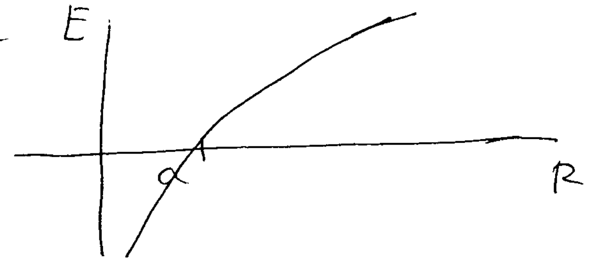
when $a = |\vec{a}|$ is a short distance cutoff.

$$\Rightarrow \quad E[n] = \frac{\phi_0^2}{2\pi}\sum_{i>j} n_i n_j \ln\left(\frac{a}{|x_i - x_j|}\right)$$

a log interaction!

For a pair $n_1 = -n_2 = 1$ $(R \gg a)$

$$E(+,-;\underset{\substack{\uparrow\\ \text{distance}}}{R}) = -\frac{\phi_0^2}{2\pi}\ln\left(\frac{a}{R}\right) = \frac{\phi_0^2}{2\pi}\ln\left(\frac{R}{a}\right)$$



The vortex configurations we have just discussed have the (important!) peculiarity that they are singular and that the energy of a single isolated vortex diverges logarithmically at both long and short distances. In this sense these are not finite action solutions.

There are other theories with instantons that have finite action solutions. Let us begin with the $O(3)$ non-linear σ-model in $D=2$.

$$S = \int d^2x \, \frac{1}{2g} \| \nabla \vec{n} \|^2 \qquad \vec{n}^2 = 1$$

Let us implement the constraints with a Lagrange multiplier field $\lambda(x)$

$$S = \int d^2x \, \frac{1}{2g} \left[(\nabla \vec{n})^2 + \lambda(x) (\vec{n}^2 - 1) \right]$$

~~$\frac{\delta S}{\delta \vec{n}} = \nabla^2 \vec{n} + \lambda(x) \vec{n} = 0$~~

$$\frac{\delta S}{\delta n_a(x)} - \nabla_\mu \frac{\delta S}{\delta \nabla_\mu n_a(x)} = 0 \qquad (1)$$

$$\frac{\delta S}{\delta \lambda(x)} = 0 \qquad (2)$$

(2) $\Rightarrow$ $\vec{n}^{\,2}(x) = 1$ constraint

① $\Rightarrow$ $\lambda\, n^a(x) = -\nabla^2 n^a(x)$

$$\Rightarrow \quad \lambda \underset{\substack{\| \\ 1}}{n_a n^a} = -\vec{n}\cdot\nabla^2\vec{n}$$

$$\lambda(x) = -\vec{n}\cdot\nabla^2\vec{n}$$

$\Rightarrow$ ① becomes:

$$\boxed{\nabla^2 n^a(x) = n^a(x)\ \vec{n}(x)\cdot\nabla^2\vec{n}(x)}$$

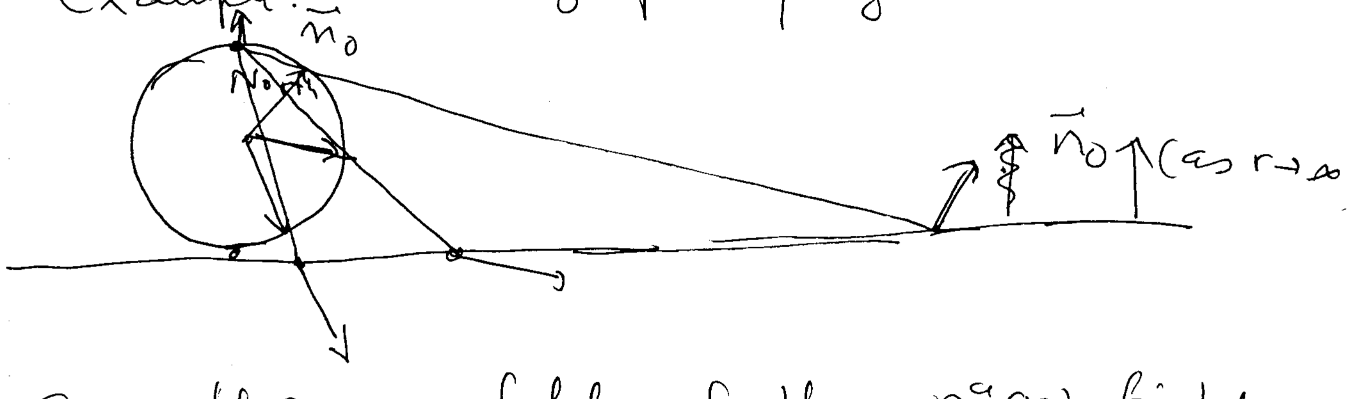
(non-trivial!)

To minimize the (Euclidean) action with a smooth configuration $n^a(x)$ it must satisfy $\vec{n}^{\,2}(x) = 1$ (everywhere) and $\lim_{r\to\infty} r\,\|\nabla_\mu \vec{n}(x)\|^2 = 0$ (finite action)

Hence ⇒ $n^a(x) \xrightarrow[r\to\infty]{} n^a_0$ some fixed unit vectr (say (0,0,1))

⇒ the allowed configurations must approach the same vector as $r \to \infty$
This is equivalent to say that the points at spacial infinity are identified and the plane $\mathbb{R}^2$ has been warped onto a sphere S_2 (at least topologicall

Example: stereographic projection



Since the manifold of the $n^a(x)$ fields is also an S_2 sphere, the configuration are maps from the S_2 (base space) to an S_2 (target space)
The topological classes are $\pi_2(S_2) \equiv \mathbb{Z}_2$ as we shall

What is the winding number (or <u>topological charge</u> Q) in this case?

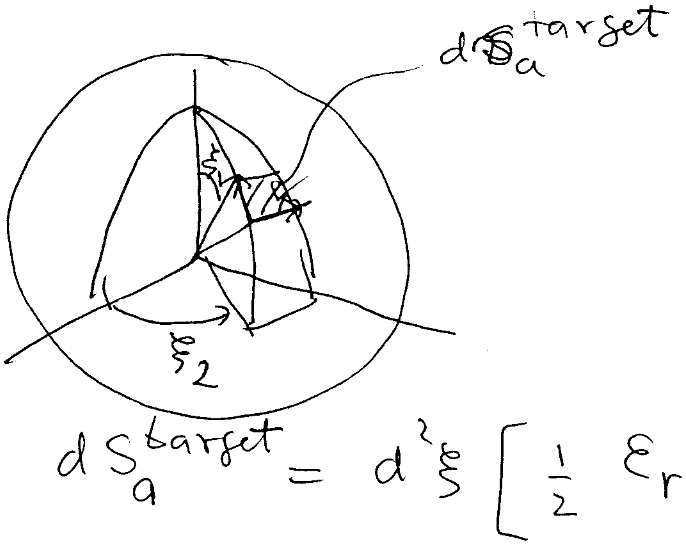
Let Q be defined by

$$Q = \frac{1}{8\pi}\int_{S_2} d^2x \; \varepsilon_{\mu\nu}\, \vec{n}\cdot\left(\nabla_\mu \vec{n} \times \nabla_\nu \vec{n}\right)$$

$$\equiv \frac{1}{8\pi}\int_{S_2} d^2x \; \varepsilon_{\mu\nu}\, \varepsilon_{abc}\, n^a \nabla_\mu n^b \nabla_\nu n^c$$

We will show that this is a topological invariant.

Let ξ_1 and ξ_2 be two Euler angles in the target space (since the target space is S_2 it must be possible to define two Euler angles, analogous to θ and φ)



infinitesimal oriented area ~~of~~ the target sphere

$$dS_a^{target} = d^2\xi \left[\frac{1}{2}\,\varepsilon_{rs}\,\varepsilon_{abc}\,\frac{\partial n_b}{\partial \xi_r}\frac{\partial n_c}{\partial \xi_s}\right]$$

or, what is the same,

$$d\vec{S}_{target} = \frac{1}{2}\left(\frac{\partial\vec{n}}{\partial\xi_1}\times\frac{\partial\vec{n}}{\partial\xi_2} - \frac{\partial\vec{n}}{\partial\xi_2}\times\frac{\partial\vec{n}}{\partial\xi_1}\right)d^2\xi$$

⇒ We can write

$$Q = \frac{1}{8\pi}\int_{S_{base}} d^2x \; \varepsilon_{\mu\nu}\,\varepsilon_{abc}\, n_a \frac{\partial n_b}{\partial x_\mu}\frac{\partial n_c}{\partial x_\nu}$$

$$= \frac{1}{8\pi}\int_{S_{base}} d^2x \; \varepsilon_{\mu\nu}\,\varepsilon_{abc}\, n_a \frac{\partial n_b}{\partial \xi_r}\frac{\partial \xi_r}{\partial x_\mu}\frac{\partial n_c}{\partial \xi_s}\frac{\partial \xi_s}{\partial x_\nu}$$

(Jacobian: of the mapping $S_2^{base} \to S_2^{target}$) : $\varepsilon_{rs}\, d^2\xi = \varepsilon_{\mu\nu}\frac{\partial \xi_r}{\partial x_\mu}\frac{\partial \xi_s}{\partial x_\nu}\, d^2x$

$$Q = \frac{1}{8\pi}\int_{S_2^{target}} d^2\xi \; \varepsilon_{rs}\,\varepsilon_{abc}\, n_a \frac{\partial n_b}{\partial \xi_r}\frac{\partial n_c}{\partial \xi_s}$$

$$\equiv \frac{1}{4\pi}\int_{S_2^{target}} dS_a^{target}\, n_a \equiv \frac{1}{4\pi}\int_{S_2^{target}} dS^{target}$$

(since $dS_a^{target} \parallel n_a$). But the area of the 2-sphere S_2^{target}, $\int_{S_2^{target}} dS^{target}$, is $\underline{\underline{4\pi}}$

$\Rightarrow$ Q counts how many times is the 2-sphere S_2^{target} swept as we span the compactified plane S_2^{base} $\Rightarrow$ $\boxed{Q = N\,\mathcal{E}}$

Clearly Q does not change if we deform the configuration smoothly $\Rightarrow$

$$\pi_2(S_2) \cong \mathbb{Z}$$

and Q classifies the homotopy classes

Q: topological charge.

Let us use the identity (trivial)

$$\left(\partial_\mu n_a \pm \varepsilon_{\mu\nu}\varepsilon_{abc} n_b \partial_\nu n_c\right)^2 \geq 0$$

since

$$\left(\varepsilon_{\mu\nu}\varepsilon_{abc} n_b \partial_\nu n_c\right)^2 = \left(\varepsilon_{\mu\nu}\, \vec{n}\times\partial_\nu \vec{n}\right)\cdot\left(\varepsilon_{\mu\sigma}\, \vec{n}\times\partial_\sigma \vec{n}\right)$$

and $\varepsilon_{\mu\nu}\,\varepsilon_{\mu\sigma} = \delta_{\nu\sigma}$

$$\Rightarrow \left(\varepsilon_{\mu\nu}\varepsilon_{abc} n_b \partial_\nu n_c\right)^2 = \delta_{\nu\sigma}\left(\vec{n}\times\partial_\nu\vec{n}\right)\cdot\left(\vec{n}\times\partial_\sigma\vec{n}\right.$$

$$= (\vec{n} \times \partial_\nu \vec{n}) \cdot (\vec{n} \times \partial_\nu \vec{n})$$

since $(\vec{A} \times \vec{B})^2 = \vec{A}^2 \vec{B}^2 - (\vec{A} \cdot \vec{B})^2$

$$= \vec{n}^2 (\partial_\nu \vec{n})^2 - (\vec{n} \cdot \partial_\nu \vec{n})^2 \equiv (\partial_\nu \vec{n})^2$$

since $\vec{n}^2 = 1$ and $\vec{n} \cdot \partial_\nu \vec{n} = 0$

$$\Rightarrow (\partial_\mu \vec{n} \pm \varepsilon_{\mu\nu} \vec{n} \times \partial_\nu \vec{n})^2 =$$

$$= 2 (\partial_\mu \vec{n})^2 \pm 2 \varepsilon_{\mu\nu} \partial_\mu \vec{n} \cdot (\vec{n} \times \partial_\nu \vec{n}) \geq$$

$$\Rightarrow (\partial_\mu \vec{n})^2 \geq \varepsilon_{\mu\nu} \vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}$$

and

$$S[\vec{n}] = \frac{1}{2g} \int d^2x \, (\partial_\mu \vec{n})^2 \geq \frac{1}{2g} \int d^2x \, \varepsilon_{\mu\nu} \vec{n} \cdot \partial_\mu \vec{n} \times$$

$$\boxed{S[\vec{n}] \geq \frac{4\pi Q}{\cancel{g}}}$$

is a lower bound for configs. with topological charge Q

We will now look for configurations that saturate the bound, i.e.

$$S[\vec{n}] = \frac{4\pi Q}{}$$

The must obey

$$\left(\nabla_\mu \vec{n} \pm \epsilon_{\mu\nu} \vec{n} \times \nabla_\nu \vec{n}\right)^2 = 0$$

$\Rightarrow$ self-dual and anti-self dual soluti

$$\nabla_\mu \vec{n} = \pm \epsilon_{\mu\nu} \vec{n} \times \nabla_\nu \vec{n}$$

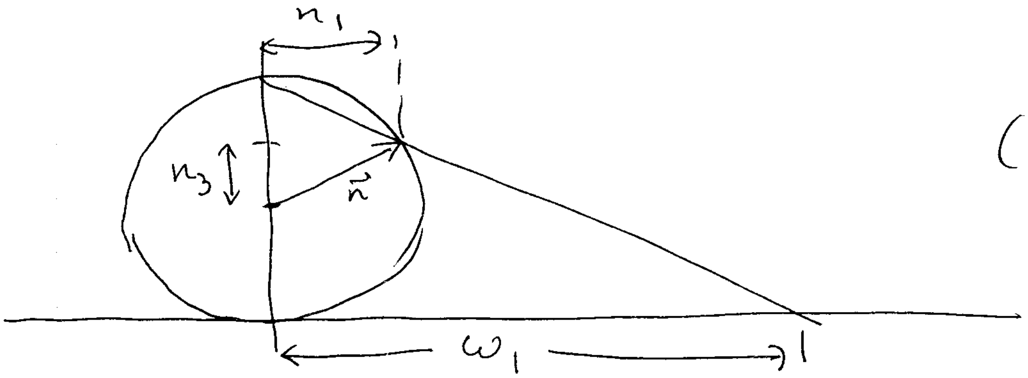
with $\vec{n}^2 = 1$

and $S[n] = 4\pi Q$

We solve those equations using a stereographic projection of the points on the 2-sphere S_2^{target} onto a plane with coordinates ω_1 and ω_2,

$$\omega_1 = \frac{2n_1}{1-n_3}, \qquad \omega_2 = \frac{2n_2}{1-n_3}$$

$$\omega = \omega_1 + i\omega_2 = 2\,\frac{n_1 + i n_2}{}$$



(same with n_2)

$$n = n_1 + i\, n_2$$

$$\partial_1 \omega \equiv \frac{\partial \omega}{\partial x_1} = \frac{2}{(1-n_3)^2} \left(\partial_1 n + n \overleftrightarrow{\partial_1} n_3 \right)$$

$\Rightarrow$ Self Dual Eqns: $\partial_1 n = \mp i n \overleftrightarrow{\partial_2} n_3$

$$\partial_2 n = \pm i n \overleftrightarrow{\partial_1} n_3$$

$$\Rightarrow \quad \partial_1 \omega = \pm i \, \partial_2 \omega$$

or $\frac{\partial_{\cancel{1}} \omega_1}{\partial x_1} = \pm \frac{\partial \omega_2}{\partial x_2}$ Cauchy-Riemann

$$\frac{\partial \omega_1}{\partial x_2} = \mp \frac{\partial \omega_2}{\partial x_1}$$

$\Rightarrow$ ω must be an analytic

(but <u>not entire</u>) function of $z = x_1 + i x_2$

Notice: Poles are allowed but not cuts!

We can rewrite the action in the form

$$\Rightarrow \quad S = \int d^2x \; \frac{\left|\frac{d\omega}{dz}\right|}{\left(1+\frac{|\omega|^2}{4}\right)^2}$$

with $|Q| = \frac{S}{4\pi}$

① A solution is

$$\omega(z) = \text{const}\left(\frac{z-z_0}{\lambda}\right)^n \qquad \begin{array}{l}\lambda \in \mathbb{R}\\ z_0 \in \mathbb{C}\\ n \in \mathbb{Z}^+\end{array}$$

λ and z_0 are the so called zero-modes of the solution, representing the location z_0 of the instanton and the scale λ

In general

$$\omega(z) = \text{const}\prod_i\left(\frac{z-z_i}{\lambda}\right)^{m_i}\prod_j\left(\frac{\lambda}{z-z_j}\right)^{n_j}$$

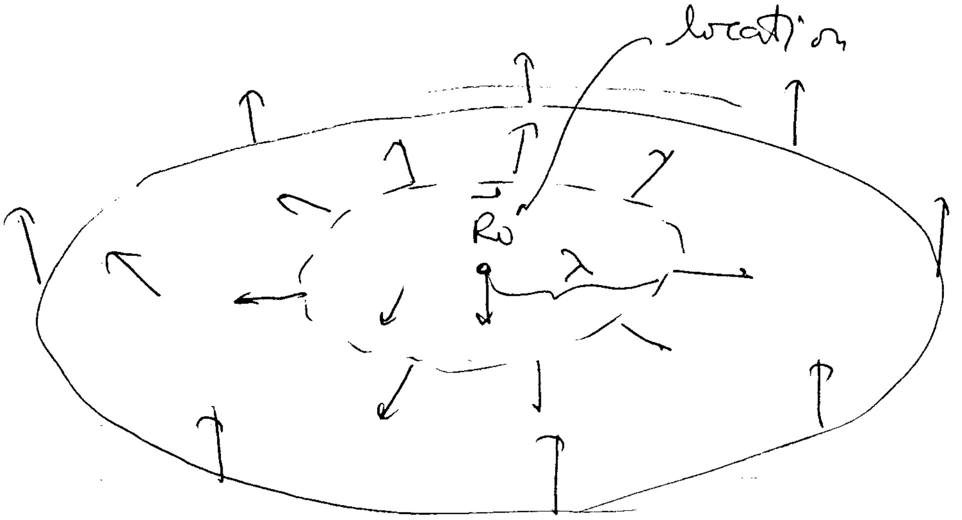
$$Q = \sum_i (\underset{\substack{\uparrow\\ \text{instantons}}}{m_i} - \underset{\substack{\uparrow\\ \text{anti-instantons.}}}{n_i})$$

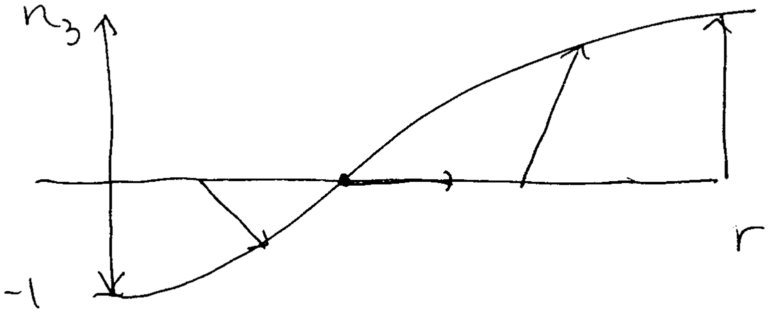
In coordinate space, for $Q=1$, the instanton looks like

scale invariance
$\Downarrow$
λ arbitrary

---

translation invariance
$\Downarrow$
z_0 arb.





In 2+1 dimensions these are finite energy solitons, called Skyrmions.

$z_0 = 0 \qquad \omega(z) = \frac{z}{\lambda}$

<u>$Q=1$</u>

$$\Rightarrow \qquad n_3 = \frac{\vec{x}^2 - 4\lambda^2}{\vec{x}^2 + 4\lambda^2}$$

$$i=1,2 \qquad n_i = \frac{4\lambda\, n_i}{\vec{x}^2 + 4\lambda^2}$$

This is all very nice but there are a number of problems. In the case of vortices we will ~~[illegible]~~ show that it is possible to ~~compl~~ compute the partition function in terms of these variables. In fact in that case in the absence of vortices Z is trivial. But in the case of the non-linear σ-model it turns out to be very hard to compute Z in terms of ~~summ~~ over ~~[illegible]~~ instanton configuration One of the problems originates from the scale invariance of the classical action which *requires* that we include instantons (and anti-instantons) of *all* sizes (not only in all locations) ⇒ one finds infrared problems and also the fact that the excitations are no longer well defined if they overlap considerably.

Also, for $N \geq 4$, non-linear σ-models with target space $O(N)$ *do not* have instantons since the maps of $S_2 \longrightarrow S_N$ are trivial (for $N > 3$), $\Rightarrow \pi_2(S_N) =$
for $N > 3$

However, there are theories which have instantons "for all N". These are chiral models (or non-linear σ-models) with target spaces of the form G/H where G is simply connected (and trivial $\pi_2(G) = 0$) and $H \supseteq U(1)$
(at least one $U(1)$ subgroup of G

Since G is simply connected, the configurations $g(x) \in G$ are *contractible* (i.e. can be deformed continuously to the identity) $\Rightarrow$ non-linear σ-models on G do not have instantons

But chiral theories on the coset G/H do. To see that we write the fields in the form (see Polyakov)

$$\varphi_a(x) = g_{ab}(x)\,\varphi_b^{(0)} \qquad x \in S^2_{\text{base}}$$

where $g \in G$ and $\varphi_b^{(0)}$ is a constant field, invariant under the subgroup H

$$h_{ab}\,\varphi_b^{(0)} = \varphi_a^{(0)} \qquad \forall\ h \in H$$

Notice that $g_{ab}(x)$ does not have to be continuous. Let us consider a set of matrices $g^{(N)}(x)$ defined on the northern hemisphere of S^2_{base} and $g^{(S)}(x)$ a set of matrices defined on the southern hemisphere of S^2_{base}

If at the Equator, which is isomorphic to the circle S^1, we have

$$g^{(N)}(x) = g^{(S)}(x)\cdot h(x) \qquad x \in \text{Equator} = S$$

and $h \in H$.

$\Rightarrow$ $g(x)$ is discontinuous at the Equator.

However $\varphi_a(x) \equiv g_{ab}(x)\,\varphi_b^{(0)}$ is continuous since $h_{ab}\,\varphi_b^0 = \varphi_a^0 \quad \forall\, h(x) \in$

$$\Rightarrow \quad \varphi_a^{(N)}(x) = g_{ab}^{(N)}(x)\,\varphi_b^{(0)}$$

$$\varphi_a^{(S)}(x) = g_{ab}^{(S)}\,\varphi_b^{(0)}$$

$\Rightarrow$ on the Equator we set

$$g_{ab}^{(N)}(x)\,\varphi_b^0 = g^{(S)}(x)\cdot h(x)\,\varphi_b^0$$
$$\equiv g_{ac}^{(S)}(x)\,h_{cb}(x)\,\varphi_b^0$$
$$= g_{ac}^{(S)}\,\varphi_c^0$$

it is continuous!

$\Rightarrow$ even though $g(x)$ is discontinuous at the Equator, $\varphi_a(x)$ is continuous and it defines a map $S_2 \longrightarrow G/H$ which can be classified according to the maps from the Equator $S_1 \longrightarrow H$

If $H = U(1) \times$ something trivial $\Rightarrow$ we can use the winding # of maps $S_1 \rightarrow U(1)$

What we showed is that

$$\pi_2(G/H) = \pi_1(H)$$

For $H \cong U(1)$, $\pi_1(U(1)) = \mathbb{Z}$

One example is the $\mathbb{CP}^{N-1}$ model we discussed before whose target space is the coset

$$\frac{SU(N)}{SU(N-1) \otimes U(1)}$$

which have instantons for <u>all</u> N.