Instantons and Solitons

(Kinks, vortices and monopoles)

We will now discuss the role of topological non-trivial configurations in QFT. There are many reasons for looking at this problem. One motivation is the study of mechanism for quantum disorder, i.e., of the physical origin of phases of QFT's exhibiting confinement and/or lack of long range order. This is also related to the problem of understanding tunneling processes in QFT.

Also, this leads to an understanding of the existence of textured or topological excitations of interest both in QFT and Statistical Physics.

We will begin by defining what do we mean by topological excitations. Consider first a very simple problem, a double-well
anharmonic oscillator. The potential $U(q)$ has the form

\[ U(q) \]

$U(q)$ has two degenerate minima at $\pm q_0$. Suppose we want to compute the tunneling amplitude from $+q_0$ to $-q_0$ (in imaginary time):

\[ \langle q_0 | e^{-\frac{Ht}{i}} | -q_0 \rangle \]

In path integral language this corresponds to a classical trajectory (in imaginary time). The (Euclidean) Lagrangian is

\[ L = \frac{1}{2} m \dot{q}^2 + U(q) \]

(i.e., the energy)

\[ \frac{\dot{q}}{\dot{q}} = \frac{d\dot{q}}{dt} \]

The trajectory looks like this
It is easy to show (see below in these notes) that the classical Euclidean action is finite. Such finite action Euclidean trajectories (or configurations) are called instantons. It has non-trivial topology in that it interpolates between two inequivalent ground states (vacua) at \( t = \pm \infty \), and that it cannot be smoothly deformed to the trivial configuration \( \tilde{\phi} = \tilde{\phi}_0 \) or \( \tilde{\phi} = -\tilde{\phi}_0 \). In this case we see that we can either get an instanton or an anti-instanton which executes the opposite trajectory.

Similarly we can look at a \( \phi^4 \) field theory in 1+1 Minkowski space-time, which has the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left( \partial \phi \right)^2 - \frac{1}{2} \left( \Delta \phi \right)^2 - U(\phi)
\]
and a Hamiltonian density

\[ H = \frac{\pi^2}{2} + \frac{1}{2} (\partial \phi)^2 + U(\phi) \]

we can find now finite energy (static) configurations of \( H \). These static configurations obey the same equations in space \( x \) as the Euclidean action (double well) oscillators (we just discussed) does in imaginary time. The finite energy configuration is

These finite energy states are (topological) solitons are called \( \phi \)-kinks or \( \phi \)-domain walls.

The classical solutions are

\[ \phi = \pm \phi_0 \tanh \left( \frac{x-x_0}{\sqrt{2} \xi} \right) \]

\((\xi = 1 \text{m} \text{m}^{-1})\)
Let us consider now a case with a continuous symmetry. For simplicity we will begin with \( U(1) \). In this case the matter field is a complex scalar field and we will be interested in the regime in which \( \phi = \phi_0 e^{i\theta} \), where \( \phi_0 \) is the vacuum expectation value, and \( \theta \) is a phase, with \( 0 \leq \theta < 2\pi \). Let us consider first the case of a solution in \( D=1+1 \) dimensions. We will work with periodic boundary conditions in space and seek time-independent finite energy states. Since we have imposed PBC's the line has been effectively wrapped into a circumference of radius \( R = \frac{L}{2\pi} \). A configuration of this type is then an assignment (or mapping) of a point of the base space (which topologically is a circle \( S^1 \)) to a value of the phase \( \theta \).
However, the phase is also topologically another circumference, \( S_1 \), which we will call the target space. Pictorially, a configuration is a mapping

\[
S_1 \rightarrow S_1
\]

A constant field configuration is a point in the target space \( S_1 \)

\[
\Phi = \Phi_0 e^{i\theta_0}
\]

There are many configurations which can be obtained by a smooth, \( \Theta \) (continuous) deformation of the constant field configuration. We will represent these configurations as curves on the circumference. They are
closed due to the PBC's

$\theta_0$

This is a trivial configuration which can be deformed smoothly to the constant field configuration. It is a "contractible loop".

**Def:** we say that two configurations (or mappings) are homotopic to each other if they can be deformed continuously into each other.

Clearly configurations can be added since their phases can also be added. Under addition smooth these mappings, which we will call homotopies, form a group. However there are configurations that cannot be deformed smoothly to the identity. One can
is a configuration which winds around the circumference just once. But there are as many configurations which wind once around the circle and they can all be deformed smoothly into each other but not onto the identity.

a topologically non-trivial configuration with winding # = 1

another topologically non-trivial configuration with winding # = 1

they are homotopic to each other!
Hence, homotopy can be classified by a winding number which labels an equivalence class of homotopies and hence is a topological invariant. In this case, the homotopy classes are labelled by the integers and hence the set of homotopy classes of $S_1 \rightarrow S_1$ is isomorphic to $\mathbb{Z}$. We state this conclusion by the symbol $\pi_1(S_1) \cong \mathbb{Z}$.

Note: This means that homotopy classes for groups, usually called homotopy group $\pi_1$.

Winding Number $N = \frac{1}{2\pi i} \int_0^L dx \oint_{\partial \gamma} \partial x \theta = \frac{(\Delta \theta)_{\gamma}}{2\pi i}$

or $N = \frac{1}{2\pi i} \int_0^L dx e^{i\theta(x)} - i dx e^{-i\theta(x)}$

Q: Does this always work? NO. To see this consider the same problem but for an O(3)
Since the target space of a non-linear 0-model is the 2-sphere $S_2$, we now have mappings from $S_1 \rightarrow S_2$.

Again, a constant field can fix a point $\vec{n}_0$ on $S_2$. An arbitrary smooth configuration is represented by a closed loop $\gamma$ on $S_2$ (because $S_1$ is closed).

Clearly, all smooth loops on $S_2$ are contractible and can be deformed (shrunk) continuously to a point. Thus they are topologically trivial. We express this by the statement that

$$\pi_1(S_2) = 0$$
Q: Does this mean that everything is trivial for $D > 1$? NO!

Consider as an example a complex scalar field in $D=2$ (this, once again, is either $D+1$-dimensional Euclidean space for instantons or $D=2$ space for solitons). Since the action has terms of the form $|D^2 \phi|^2$, it is clear that to avoid divergences we must seek solutions which approach zero at a constant amplitude $\phi_0$

$$\phi \to \phi_0 e^{i \theta}$$

but perhaps an variable phase. Consider now a large circle of radius $R$, large enough so that $|\phi| \approx \phi_0$. Then the configurations on the circle are smooth maps of the circle $S^1$ onto
Q: Does this mean that everything is trivial in higher dimensions? \textbf{No}, again!

Consider a field with a $U(1)$ symmetry in $D=2$ Euclidean space (either a 1+1-Dim Euclidean space-time for instantons or a $D=2$ space for solitons).

Let $\phi(x) = \rho(\vec{r}) e^{i \theta(\vec{r})}$

where $\rho(\vec{r}) \rightarrow \rho_0 \quad r \rightarrow \infty$

of radius $R$

and consider a large circle on the plane for $R$ large enough. $\rho(R) \sim \rho_0$ but $\theta(r) \propto$ can wind. Indeed the action (or energy, depending on the case)

$S = \int d^2x \left[ \frac{1}{2} |\nabla \phi|^2 + U(\phi) \right]$  

$\Rightarrow$ we can minimize it by letting $\rho_0$

$\nabla \phi \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty$
the phase of the field $\Phi$, which is also another circle $S_1 \Rightarrow$ These configurations are also homotopies with homotopy classes $\pi_1(S_1) \cong \mathbb{Z}$ and are classified by a winding number $N$

$$N = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{i\Phi(\varphi)}}{i\varphi} e^{-i\Theta(\varphi)}$$

where $N = \frac{\Delta \theta}{2\pi}$

Clearly $\Phi(r)$ must have a singularity somewhere in the region inside $\Gamma$. Let that point be the origin when $\rho(r) \to 0$ fast enough for $S < \infty$
Such a configuration is called a vortex and the real two component field \((\phi_1, \phi_2)\)
\[
\phi = \phi_1 + i \phi_2
\]
is tangent to the circle of radius \(r\).

Consider for example the case of a complex scalar field coupled to a \(U(1)\) Maxwell gauge field. The (Euclidean) Lagrangian is
\[
\mathcal{L} = \frac{1}{4} |D \phi|^2 - \frac{m^2}{2} |\phi|^2 + \frac{\lambda}{4} |\phi|^4 + \frac{1}{4} F_{\mu \nu}^2
\]
\[
\Rightarrow \quad j^\mu = \frac{i}{2} \phi^* D^\mu \phi \quad \text{is the current}
\]
\[
D^\mu = \partial^\mu + i \frac{g}{\hbar c} A^\mu
\]
and the equations of motion are \(\partial_\mu F_{\mu \nu} = j_\nu + LG \text{ eqn for } \phi\).

\[
\Rightarrow \quad |D_\mu \phi| \to 0 \quad \text{as } r \to \infty \Rightarrow
\]
\[
\phi \to \phi_0 e^{i \theta}
\]
\[ D_\mu \phi \rightarrow \phi_o e^{i(\omega \mu \theta + \frac{q}{\hbar c} A_\mu)} \]  
\[ \rightarrow |D_\mu \phi|^2 \rightarrow 0 \Leftrightarrow \partial_\mu \theta + \frac{q}{\hbar c} A_\mu \rightarrow 0 \]

\[ \Rightarrow \frac{(\Delta \theta)^2}{2\pi} = \frac{1}{2\pi} \oint_{\Sigma} d\vec{r} \cdot \vec{A} \cdot \frac{q}{\hbar c} \]

but \[ \oint_{\Sigma} d\vec{r} \cdot \vec{A} = \oint_{\Sigma} \vec{B} \cdot d\vec{s} = \Phi \text{ flux} \]

For this to hold

\[ A_\theta \rightarrow \frac{\Phi}{2\pi r} \quad \text{as} \quad r \rightarrow \infty \]

(pure gauge \( \Rightarrow B(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \))

\[ \Rightarrow N = \frac{1}{2\pi} \frac{q}{\hbar c} \Phi = \frac{q}{\hbar c} \Phi \]

\[ \phi_o = \frac{\hbar c}{e} \text{ flux quantum} \]

\[ \Rightarrow N = \left( \frac{q}{e} \right) \frac{\Phi}{\phi_o} \]

\[ \Rightarrow \text{the winding } \# \ N \ (\text{vorticity}) \text{ is determined by the } \# \text{ of flux quanta} \Rightarrow \text{Abrikosov vortex} \]

(also called Nielsen-Olesen)
In this case $B(r) \to 0$ as $r \to \infty$
and $\phi(r) \to 0$ as $r \to \infty$ but $\lambda \neq 0$.

\begin{align*}
\phi(r) &\quad \text{scale} \\
\text{or} & \\
B(r) &\quad \text{scale}
\end{align*}

$\frac{\lambda}{1 \text{[m]}}$

In general $\lambda > 5$.

These are finite action (or energy) solutions. But for $\Phi = 0$ vortices have logarithmically divergent self-energy.

Let us consider now the case in which there is no electromagnetic field and assume that the potential is steep enough so as to constrain the field to be of fixed length $\Rightarrow \phi = \phi_0 e^{i\theta}$. 
This is a problem at the core of the vortex. But even ignoring this we can see what happens. In this case
\[ \mathcal{G}_\mu \phi = \phi_0 \mathcal{G}_\mu \theta \mathcal{G}_\theta \]
\[ \Rightarrow |\mathcal{G}_\mu \phi|^2 = \phi_0^2 \mathcal{G}_\mu \theta^2 \]
and the energy (or Euclidean action) is
\[ E = \int d^3x \frac{\phi_0^2}{2} \mathcal{G}_\mu \theta^2 \]
Let us assume that we have vortices at \( \{ x_i \} \) with vorticities \( \{ n_i \} \Rightarrow \theta \) untwist
\[ \mathcal{G} \theta = \rho(x) = \sum n_i \mathcal{G}_i \delta(x - x_i) \]
\[ \theta(x) = \sum n_i \mathcal{G}_i \text{Im} \log (z - z_i) \quad (z \in \mathbb{C}) \]
\[ \Rightarrow \mathcal{G} \theta = \theta \quad \text{except at singular points} \]
\[ \text{(the vortex cores)} \]
\[ \Rightarrow E = \frac{\phi_0^2}{2} \int d^3x (\mathcal{G}_\mu \theta)^2 \]
\[ = -\frac{\phi_0^2}{2} \int d^3x \theta \mathcal{G}_\mu \theta \]
\[ = + \frac{\phi_0^2}{2} \int d^3x \rho(x) \theta(x) \]
\[ \text{(same as in 2D electrostatics!)} \]
Let us solve
\[- \nabla^2 \Theta = \rho(x)\]
\[\Theta(x) = \int d^2 x \ G(x-y) \ \rho(y)\]
where \(G(x-y)\) is the 2D Green function.
\[- \nabla^2 G(x-y) = \delta^2(x-y)\]

\[\Rightarrow E = \frac{\phi_0^2}{2} \int d^2 x \ \rho(x) \ \Theta(x)\]
\[= \frac{\phi_0^2}{2} \int d^2 x \ \int d^2 y \ \rho(x) \ G(x-y) \ \rho(y)\]
\[= \frac{\phi_0^2}{2} \sum_{i,j} n_i \ n_j \ G(x_i-x_j)\]
\[= \frac{\phi_0^2}{2} \sum_i n_i^2 \ G(0) + \frac{\phi_0^2}{2} \sum_{i>j} n_i \ n_j \ G(x_i-x_j)\]
\[= \frac{\phi_0^2}{2} \left( \sum_i n_i \right)^2 \ G(0) + \phi_0^2 \sum_{i>j} n_i \ n_j \left[ G(x_i-x_j) - G(0) \right]\]

I will assume a configuration with vanishing total vorticity \(\sum_i n_i = 0\) (I need this since \(G(0)\) is singular!)

\[\Rightarrow E[\{n_i\}] = \phi_0^2 \sum_{i>j} n_i \ n_j \left[ G(x_i-x_j) - G(0) \right]\]
I will define

$$G(0) = \lim_{a \to 0} G(\vec{a} \vec{l})$$

and use that the Green function in D dimension is

$$G(x) = \int \frac{dp}{(2\pi)^D} e^{i \vec{p} \cdot \vec{x}} = \frac{\Gamma(D/2 - 1)}{4\pi^{D/2} \Gamma(D/2)} \frac{1}{p^{D-2}}$$

$$= \frac{\Gamma(D/2 - 1)}{4\pi^{D/2} \Gamma(D/2)} \left( \frac{1}{(x_1)^{D-2}} - \frac{1}{a^{D-2}} \right) \to \frac{1}{2\pi} \ln \left( \frac{a}{(x_1)} \right)$$

when \( a = |\vec{a} \vec{l}| \) is a short distance cutoff.

$$\Rightarrow E[n] = \frac{\phi_0^2}{2\pi} \sum_{i > j} n_i n_j \ln \left( \frac{a}{|x_i - x_j|} \right)$$

a log interaction!

For a pair \( n_1 = -n_2 = 1 \) \((R >> a)\)

$$E(+,-; R) = -\frac{\phi_0^2}{2\pi} \ln \left( \frac{a}{R} \right) = \frac{\phi_0^2}{2\pi} \ln \left( \frac{R}{a} \right)$$

distance E
The vortex configurations we have just discussed have the (important!) peculiarity that they are singular and that the energy of a single isolated vortex diverges logarithmically at both long and short distances. In this sense there are not finite action solutions.

There are other theories with instantons that have finite action solutions. Let us begin with the $O(3)$ non-linear $\sigma$-model in $D=2$.

$$S = \int d^2 x \; \frac{1}{2\xi} \| \mathbf{n} \|^2 \quad \mathbf{n}^2 = 1$$

Let us implement the constraints with field $\lambda(x)$.

$$S = \int d^2 x \; \frac{1}{2\xi} \left[ (\nabla \mathbf{n})^2 + \lambda(x) (\mathbf{n}^2 - 1) \right]$$
\[ \frac{\delta S}{\delta n_a(x)} - \nabla \cdot \frac{\delta S}{\delta \nabla n_a(x)} = 0 \quad (1) \]

\[ \frac{\delta S}{\delta \lambda(x)} = 0 \quad (2) \]

(2) \implies \quad \overline{n}^2(x) = 1 \quad \text{constraint}

(1) \implies \quad \lambda \ n^a(x) = - \nabla^2 n^a(x)

\implies \quad \lambda n_a n^a = - \overline{n} \cdot \nabla^2 \overline{n}

\lambda(x) = - \overline{n} \cdot \nabla^2 \overline{n}

\implies \quad (1) \text{ becomes: Trivial}

\[ \nabla^2 n^a(x) = n^a(x) \cdot \overline{n}(x) \cdot \nabla^2 \overline{n}(x) \]

(Non-trivial!)

To minimize the Euclidean action with a smooth configuration \( n^a(x) \), it must satisfy \( \overline{n}^2(x) = 1 \) (everywhere) and

\[ r \ll \| \nabla \overline{n}(x) \|^2 = 0 \quad \text{(finite action)} \]
Hence $n^a(x) \to n^a_0$ for some fixed unit vector (say $(0,0,1)$).

Then the allowed configurations must approach the same vector as $r \to \infty$.

This is equivalent to saying that the points at spatial infinity are identified and the plane $\mathbb{R}^2$ has been warped into a sphere $S_2$ (at least topologically).

Example: Stereographic projection.

Since the manifold of the $n^a(x)$ fields is also an $S_2$ sphere, the configurations are maps from the $S_2$ (base space) to an $S_2$ (target space).

The topological classes are $\pi_2(S_2) \cong \mathbb{Z}_2$. 

\[ I21 \]
What is the winding number (or topological charge $Q$) in this case?

Let $Q$ be defined by

$$Q = \frac{1}{8\pi^2} \int_{S_2} d\Omega \, \varepsilon_{\mu \nu} \bar{n} \cdot \left( \nabla_\mu \bar{n} \times \nabla_\nu \bar{n} \right)$$

$$= \frac{1}{8\pi^2} \int_{S_2} d\Omega \, \varepsilon_{\mu \nu} \varepsilon_{abc} \, n^a \nabla_\mu n^b \nabla_\nu n^c$$

We will show that this is a topological invariant.

Let $\xi_1$ and $\xi_2$ be two Euler angles in the target space (since the target space is $S_2$ it must be possible to define two Euler angles, analogous to $\theta$ and $\varphi$).

$$dS_{\text{target}} = \frac{d^2\xi}{\sin^2 \xi} \left[ \frac{1}{2} \varepsilon_{rs} \varepsilon_{abc} \frac{\partial \varepsilon_{nb}}{\partial \xi_r} \frac{\partial \varepsilon_{nc}}{\partial \xi_s} \right]$$
or, what is the same,

\[ d S_{\text{target}} = \frac{1}{2} \left( \frac{\partial n_1 \times \partial n_2}{\partial \xi_1 \partial \xi_2} - \frac{\partial n_1 \times \partial n_2}{\partial \xi_2 \partial \xi_1} \right) d^2 \xi \]

\[ \Rightarrow \text{ We can write } \]

\[ Q = \frac{1}{8\pi} \int_{S_{\text{base}}} d^2 x \varepsilon_{\mu \nu} \varepsilon_{ab} \varepsilon_{ce} n_a \frac{\partial n_b}{\partial x^\mu} \frac{\partial n_c}{\partial x^\nu} \]

\[ = \frac{1}{8\pi} \int_{S_{\text{base}}} d^2 x \varepsilon_{\mu \nu} \varepsilon_{ab} \varepsilon_{ce} n_a \frac{\partial n_b}{\partial \xi^r} \frac{\partial n_c}{\partial \xi^s} d^2 \xi \]

\[ \text{(Jacobian:)} \varepsilon_{rs} d^2 \xi = \varepsilon_{\mu \nu} \frac{\partial \xi^r}{\partial x^\mu} \frac{\partial \xi^s}{\partial x^\nu} d^2 x \]

\[ Q = \frac{1}{8\pi} \int_{S_{\text{target}}} d^2 \xi \varepsilon_{rs} \varepsilon_{ab} \varepsilon_{ce} n_a \frac{\partial n_b}{\partial \xi^r} \frac{\partial n_c}{\partial \xi^s} \]

\[ \equiv \frac{1}{4\pi} \int_{S_{\text{target}}} dS_{\text{target}} n_a \equiv \frac{1}{4\pi} \int_{S_{\text{target}}} dS_{\text{target}} \]

\[ (\text{with } dS_{\text{target}} \parallel n_a) \text{. But the area of } \]

\[ \text{target} \]

\[ \text{is } dS_{\text{target}} \parallel n_a \text{. But the area of} \]

\[ \text{target} \]
the 2-sphere $S^2_{\text{target}}$, \( \int_{S^2_{\text{target}}} \), is \( \frac{4\pi}{\text{area}} \)

\[ \Rightarrow \] \[ \Omega \] counts how many times is the 2-sphere $S^2_{\text{target}}$ swept as we span the compactified plane $S^2_{\text{base}}$.

Clearly $\Omega$ does not change if we deform the configuration smoothly.

\[ \pi_2(S^2) \cong \mathbb{Z} \]

and $\Omega$ classifies the homotopy classes.

Let us use the identity (trivial)

\[ (\partial \mu \, n_a \pm \epsilon_{\mu \nu} \epsilon_{abc} \, n_b \, \partial \nu \, n_c) \geq 0 \]

since

\[ \epsilon_{\mu \nu} \epsilon_{abc} \, n_b \, \partial \nu \, n_c \leq \epsilon_{\mu \nu} \epsilon_{abc} \, n_b \, \partial \nu \, n_c \]

and

\[ \epsilon_{\mu \nu} \, \epsilon_{\mu \nu} = \delta_{\nu \sigma} \]

\[ \Rightarrow \]

\[ (\epsilon_{\mu \nu} \epsilon_{abc} \, n_b \, \partial \nu \, n_c)^2 = \delta_{\nu \sigma} (\partial \nu \partial \sigma) \]
\[
= (\hat{n} \times \partial_\nu \hat{n}) \cdot (\hat{n} \times \partial_\nu \hat{n}) \\
\sum_a (\hat{A} \times \hat{B})^2 = \hat{A} \cdot \hat{B}^2 - (\hat{A} \cdot \hat{B})^2 \\
= n^2 (\hat{A} \cdot \hat{B})^2 - (\hat{n} \cdot \partial_\nu \hat{n})^2 = (\hat{A} \cdot \hat{B})^2 \\
\text{such that } \hat{n}^2 = 1 \quad \text{and} \quad \hat{n} \cdot \partial_\nu \hat{n} = 0 \\
\Rightarrow (\partial_\mu \hat{n} \pm \varepsilon_{\mu \nu} \hat{n} \times \partial_\nu \hat{n})^2 = \\
= 2 (\partial_\mu \hat{n})^2 \pm 2 \varepsilon_{\mu \nu} \partial_\mu \hat{n} \cdot (\hat{n} \times \partial_\nu \hat{n}) \geq 0 \\
\Rightarrow (\partial_\mu \hat{n})^2 \geq \varepsilon_{\mu \nu} \hat{n} \cdot \partial_\mu \hat{n} \times \partial_\nu \hat{n}
\]

and
\[
S(\hat{n}) = \frac{1}{2g} \int d^2 x (\partial_\mu \hat{n})^2 \geq \frac{1}{2g} \int d^2 x \varepsilon_{\mu \nu} \hat{n} \cdot \partial_\mu \hat{n} \times \partial_\nu \hat{n}
\]

\[
S(\hat{n}) \geq \frac{4\pi Q}{\hat{A}}
\]
is a lower bound on

\text{charges with topological charge } Q
We will now look for configurations that saturate the bound, i.e.
\[ S[\tilde{n}] = \frac{4\pi Q}{\tilde{e}} \]

The must obey
\[ (\partial_\mu \tilde{n} \pm \epsilon_{\mu\nu} \tilde{n} \times \partial_\nu \tilde{n})^2 = 0 \]

\[ \Rightarrow \text{self-dual and anti-self-dual solution} \]
\[ \partial_\mu \tilde{n} = \pm \epsilon_{\mu\nu} \tilde{n} \times \partial_\nu \tilde{n} \]

\( \text{with} \quad \tilde{n}^2 = 1 \)

\[ \text{and} \quad S[\tilde{n}] = 4\pi Q \]

We solve these equations using a stereographic projection of the points on the 2-sphere \( S^2 \) target onto a plane with coordinates \( \omega \), and \( \omega_2 \):

\[ \omega_1 = \frac{2n_1}{1-n_3} \quad \omega_2 = \frac{2n_2}{1-n_3} \]

\[ \omega = \omega_1 + i\omega_2 = \frac{2n_1 + i n_2}{1} \]
\[ n = n_1 + i n_2 \]

\[ \frac{\partial n}{\partial x_1} = \frac{2}{(1-n_3)^2} (\partial_1 n + n \overleftarrow{\partial_1} n_3) \]

\[ \Rightarrow \text{Self Dual Eqns: } \partial_1 n = \mp i n \overrightarrow{\partial_2} n_3 \]
\[ \partial_2 n = \pm i n \overleftarrow{\partial_1} n_3 \]

\[ \Rightarrow \partial_1 \omega = \pm i \partial_2 \omega \]

\[ \text{or } \frac{\partial \omega_1}{\partial x_1} = \pm \frac{\partial \omega_2}{\partial x_2} \quad \text{Cauchy - Riemann} \]

\[ \frac{\partial \omega_1}{\partial x_1} = \pm \frac{\partial \omega_2}{\partial x_2} \]

\[ \Rightarrow \omega \text{ must be an analytic function of } z = x + iy \]

(But not entire)
Notice: Poles are allowed but not cuts!
we can rewrite the action \( \omega \) in the form
\[
S = \int d^2x \left( \frac{d\omega}{dz} \right) \left( 1 + \frac{1}{2} \omega^2 \right)^{-1}
\]
with \( |\omega| = \frac{S}{4\pi} \)

A solution is:
\[
\omega(z) = \text{const} \left( \frac{z - z_0}{\lambda} \right)^\chi \quad \lambda \in \mathbb{R} \\
\quad z_0 \in \mathbb{C} \\
\quad \chi = n \in \mathbb{Z}^+ 
\]

\( \lambda \) and \( z_0 \) are the so-called zero-modes of the solution, representing the location \( z_0 \) of the instanton and the scale \( \lambda \).

In general:
\[
\omega(z) = \text{const} \prod_i \left( \frac{z - z_i}{\lambda} \right)^{m_i} \prod_j \left( \frac{\lambda}{z - z_j^*} \right)^{n_j}
\]

\[ Q = \sum_i (m_i - n_i) \]
\[ \uparrow \quad \text{instanton, anti-instanton} \]
In coordinate space, for $Q=1$, the instanton looks like

Scale invariance

$\lambda$ arbitrary

Translation invariance

$\tilde{Q}$ arb.

In 2+1 dimensions there are finite energy solutions, called Skyrmions.

$\tilde{Q} = 0 \quad \omega (z) = \frac{\tilde{Q}}{\lambda}$

$Q=1$

$\Rightarrow \quad n_3 = \frac{x^2 - 4\lambda}{x^2 + 4\lambda}$

$\nu = 1, 2 \quad m_\nu = \frac{4\lambda n_\nu}{x^2 + 4\lambda}$
This is all very nice but there are a number of problems. In the case of matrices, we will show that it is possible to compute the partition function in terms of these variables. In fact in that case, in the absence of vertices, $Z$ is trivial. But in the case of the non-linear $O$-model it turns out to be very hard to compute $Z$ in terms of instanton configuration.

One of the problems originates from the scale invariance of the classical action which requires that we include instantons (and anti-instantons) of all sizes (not only in all locations) $\Rightarrow$ one finds infrared problems and also the fact that the excitations are no longer well defined if they overlap considerably.
Also, for \( N \geq 4 \), non-linear \( \sigma \)-models with target space \( O(N) \) do not have instantons. Since the maps of \( S_2 \rightarrow S_N \) are trivial (for \( N \geq 3 \)), \( \pi_2(S_N) = 0 \) for \( N \geq 3 \).

However, there are theories which have instantons "for all \( N \)". These are chiral models (or non-linear \( \sigma \)-models) with target space of the form \( G/H \) where \( G \) is simply connected (and trivial \( \pi_2(G) = 0 \)) and \( H \supseteq U(1) \) (at least an \( U(1) \) subgroup of \( G \)).

Since \( G \) is simply connected, the configurations \( g(x) \in G \) are contractible (i.e., can be deformed continuously to the identity) \( \Rightarrow \) non-linear \( \sigma \)-models on \( G \) do not have instantons.
But chiral theories on the coset $G/H$ are. To see that we write the fields in the form (see Polyakov)

$$\phi_a(x) = g_{ab}(x) \phi_b^{(0)}$$

where $g \in G$ and $\phi_b^{(0)}$ is a constant field, invariant under the subgroup $H$

$$h g_{ab}(x) h^{-1} = g_{ab}(x)$$

Notice that $g_{ab}(x)$ does not have to be continuous. Let us consider a set of matrices $g^{(a)}(x)$ defined on the northern hemisphere of $S^2_{base}$ and $g^{(b)}(x)$ a set of matrices defined on the southern hemisphere of $S^2_{base}$.

If at the equator, which is isomorphic to the circle $S^1$, we have

$$g^{(a)}(x) = g^{(b)}(x) \cong h(x)$$

and $h \in H$.

$$\Rightarrow \quad g(x) \text{ is discontinuous at the Equator.}$$
However, \( \psi_a^\bullet (x) \equiv g_{ab} \psi_b \) is continuous since \( \psi_b \equiv \psi_b^0 \) for all \( x \in \mathbb{R} \).

\[
\Rightarrow \quad \psi_a \equiv g_{ab} \psi_b \equiv g_{ab} \psi_b^0
\]

\[
\psi_a (x) = g_{ab} \psi_b \quad (\text{and similarly})
\]

\[
\Rightarrow \quad \text{we set}
\]

\[
g_{ab} (x) \psi_b^0 \equiv g_{ac} (x) \ h_c (x) \psi_b^0
\]

\[
= g_{ac} (x) \ h_c (x) \psi_b^0
\]

\[
= g_{ac} \psi_c^0
\]

It is continuous!

\[
\Rightarrow \quad \text{even though } g(x) \text{ is discontinuous at the Equator, } \psi_a (x) \text{ is continuous and it defines a map } S_2 \rightarrow G/H,
\]

which can be classified according to the maps from the Equator \( S_1 \rightarrow H \).

If \( H = U(1) \times \text{something trivial} \Rightarrow \)

we can use the winding \# of maps \( S_1 \rightarrow U(1) \).
What we showed is that
$$\pi_2(\mathbb{C}/\mathbb{H}) = \pi_1(\mathbb{H})$$

For $H \cong U(1)$, $\pi_1(U(1)) = \mathbb{Z}$

One example of the $\mathbb{C}P^{N-1}$ model we discussed before whose target space is the coset
$$\frac{SU(N)}{SU(N-1) \times U(1)}$$

which have instantons for all $N$.

Instantons in non-Abelian Gauge Theories

We now discuss what happens in the case of non-Abelian gauge theory.

Let us first construct the analog of $U(1)$ the vortex. It is the Dirac monopole.

The vortex was regarded as a single solution of the phase field $\Theta(x)$. These solutions were multi-valued, and had branch cuts. Dirac proposed to look at