Jordan-Wigner Transformation

A naïve look at the Hamiltonian leaves us with a puzzle. We have been raised to think that quadratic Hamiltonians are trivial. So, why the fuzz? isn't $\hat{H}$ bilinear in σ's? The problem is that, in spite of the fact that H is a bilinear form in σ's, it is <u>not</u> a free theory. It is trivial to check that the equation of motion for $\hat{\sigma}_3(r)$ is <u>not</u> linear since

$$i\,\partial_t \hat{\sigma}_3(r) = [\hat{\sigma}_3(r), \hat{H}] = -2i\,\hat{\sigma}_2(r) = -2\,\hat{\sigma}_3(r)\,\hat{\sigma}_1(\vec{r})$$

and

$$i\,\partial_t \hat{\sigma}_1(r) = [\hat{\sigma}_1(r), \hat{H}] = +\lambda\,\hat{\sigma}_3(r)\hat{\sigma}_1(r)\sum_{j=1,\dots,d}\left(\hat{\sigma}_3(r+e_j) + \hat{\sigma}_3(r-e_j)\right)$$

The reason behind this problem is the fact that the (equal-time) commutation relations

$$\left[\hat{\sigma}_3(\vec{R}) , \hat{\sigma}_3(\vec{R}') \right] = \left[\hat{\sigma}_1(\vec{R}) , \hat{\sigma}_1(\vec{R}') \right] = 0$$

$$\left[\hat{\sigma}_3(\vec{R}) , \hat{\sigma}_1(\vec{R}') \right] = 0 \qquad (\vec{R} \neq \vec{R}')$$

$$\left\{ \hat{\sigma}_3(\vec{R}) , \hat{\sigma}_1(\vec{R}) \right\} = 0$$

(and $\hat{\sigma}_3^2 = \hat{\sigma}_1^2 = I$)

which <u>are not canonical</u>. They seem to describe objects which are <u>bosons</u> on <u>different sites</u> but <u>fermions</u> on the <u>same</u> site. Alternatively they can be regarded as <u>bosons</u> with <u>hard cores</u>. Indeed the <u>raising</u> and <u>lowering</u> operators

$$\hat{\sigma}^{\pm} = \frac{1}{2} (\hat{\sigma}_1 + i \hat{\sigma}_2)$$

have the property

$$\hat{\sigma}^{+} |\uparrow\rangle = 0 \qquad \hat{\sigma}^{+} |\downarrow\rangle = |\uparrow\rangle$$

$$\hat{\sigma}^{-} |\uparrow\rangle = |\downarrow\rangle \qquad \hat{\sigma}^{-} |\downarrow\rangle = 0$$

and can be regarded as the creation and annihilation operators of some oscillator but with the hard core constraint that the boson occupation number

$\hat{n} \equiv \hat{\sigma}^+ \hat{\sigma}^-$ should only have eigenvalues 0 and 1

since $\hat{\sigma}^+ \hat{\sigma}^- |\uparrow\rangle = |\uparrow\rangle$

$\hat{\sigma}^+ \hat{\sigma}^- |\downarrow\rangle = 0$

If $D = 2$, the quantum problem has $d = D - 1 = 1$. It turns out that there is a very neat and useful transformation which will enable us to deal with this problem. This is the Jordan-Wigner transformation. The key idea behind this transformation is that, in one-dimension only, hard-core bosons are equivalent to fermions! Qualitatively this is easy to understand. If the particles live on a line, the bosons cannot ~~get~~ get into exchanged positions by purely dynamical effects since the hard-core condition forbids that possibility. Similarly one-dimensional fermions cannot change their relative ordering as a result of their dynamics (Pauli pple.).

Thus, the strategy is to show that our problem is secretely a <u>fermion problem</u>. From now on we will restrict our discussion to $d=1$ $(D=2)$

Let us consider the operator $\hat{K}(n)$

$$\hat{K}(n) = \prod_{j=-\frac{L}{2}+1}^{n} \left(-\hat{\sigma}_1(j)\right)$$

This operator flips all of the spins <u>to the left</u> of site $n+1$.

Clearly on the "high-temperature" ground state $|+\rangle$

$$\hat{K}(n)\,|+\rangle = (-1)^n\,|+\rangle$$

(i.e. it is a counting operator in the phase with $\lambda \ll 1$)

but, for $\lambda \gg 1$, we get

$$\hat{K}(n)\,|\uparrow \cdots\cdots\cdots \uparrow\rangle = |\downarrow \cdots \overset{n}{\downarrow}\overset{n+1}{\uparrow} \cdots \uparrow\rangle$$

This state is called a <u>kink</u> (or <u>topological soliton</u>). Clearly $\hat{K}(n)$ disturbs the boundary conditions (for $\lambda > \lambda_c$) but it does not for $\lambda < \lambda_c$ where

$$\langle \Psi_0 |\, \hat{K}(n)\, | \Psi_0 \rangle \neq 0 \qquad (\lambda < \lambda_c)$$

i.e. the high-temperature phase (disordered) is a <u>condensate of kinks</u>. It is also known as a <u>disorder operator</u> (Kadanoff and Ceva).

(*) They are called Majorana fermions

We will now see that a clever combination of order (i.e. $\hat\sigma_3$) and disorder ($\hat K$) operators yields a <u>fermi field</u>. Let us consider the operators

$$\hat\chi_1(j) = \hat K(j-1)\,\hat\sigma_3(j) \qquad (\text{Jordan - Wigner})$$

$$\hat\chi_2(j) = i\,\hat K(j)\,\hat\sigma_3(j)$$

(with $\hat\chi_1(-\frac{L}{2}+1) \equiv \hat\sigma_1(-\frac{L}{2}+1)$; $\hat\chi_2 \equiv -\hat\sigma_2(-\frac{L}{2}+1)$)

These operators obey the property $\hat\chi_1^2 = \hat\chi_2^2 = 1 \quad (\forall j)$

(since they are products of Pauli matrices) but

$$\{\chi_1(j), \chi_1(j')\} = 2\delta_{jj'}$$

$$\{\chi_1(j), \chi_2(j')\} = \{\chi_1(j), \chi_1(j')\} = 0 \quad (\text{if } j \neq j')$$

These are <u>almost fermions</u>!(*) Let us define $\hat\psi$ and $\hat\psi^\dagger$ through ($\hat\sigma^\pm = \frac{1}{2}(\hat\sigma_3 \mp i\hat\sigma_2)$)

$$\hat\psi^\dagger(j) \equiv \hat K(j-1)\,\hat\sigma^+(j)$$

$$\hat\psi(j) \equiv \hat K(j-1)\,\hat\sigma^-(j)$$

(i.e. $\hat\psi = \frac{1}{2}(\chi_1 + i\chi_2)$, $\hat\psi^\dagger = \frac{1}{2}(\chi_1 - i\chi_2)$)

It is straightforward to check that

$$\{\hat\psi(j), \hat\psi^\dagger(j')\} = \delta_{jj'} \;\; ; \;\; \{\hat\psi(j), \hat\psi(j')\} = 0$$

It is also easy to invert the transformation.

Let us observe that the fermion number operator $\psi^\dagger(j)\psi(j)$ is

$$\psi^\dagger(j)\psi(j) = \hat{K}(j-1)\hat{\sigma}^+(j)\hat{K}(j-1)\sigma^-(j)$$

Since $[\hat{K}(j-1), \hat{\sigma}^+(j)] = 0$ and $\hat{K}^2 = 1$, we get

$$\psi^\dagger(j)\psi(j) = \hat{\sigma}^+(j)\hat{\sigma}^-(j) = \frac{1}{4}(\hat{\sigma}_3(j) - i\hat{\sigma}_2(j))(\hat{\sigma}_3(j) + i\hat{\sigma}_2(j))$$

$$= \frac{1}{2}(1 + \hat{\sigma}_1(j))$$

Hence

$$-\hat{\sigma}_1(j) = -2\hat{\psi}^\dagger(j)\hat{\psi}(j) + 1 \equiv 1 - 2\hat{n}(j)$$

Since the operator $\hat{n}(j)$ ~~has~~ has eigenvalues $0, 1$ we can also write

$$-\hat{\sigma}_1(j) = e^{i\pi\hat{n}(j)} = e^{i\pi\hat{\psi}^\dagger(j)\psi(j)}$$

Hence, the inverse J-W transformation is

$$\hat{\sigma}^+(j) = e^{i\pi\sum_{k<j}\hat{\psi}^\dagger(k)\hat{\psi}(k)}\hat{\psi}^\dagger(j)$$

$$\hat{\sigma}^-(j) = e^{i\pi\sum_{k<j}\hat{\psi}^\dagger(k)\hat{\psi}(k)}\hat{\psi}(j)$$

and

$$\hat{\sigma}_3(j) = e^{i\pi\sum_{k<j}\hat{\psi}^\dagger(k)\hat{\psi}(k)}(\hat{\psi}^\dagger(j) + \psi(j))$$

$$\hat{\sigma}_2(j) = e^{i\pi\sum_{k<j}\hat{\psi}^\dagger(k)\hat{\psi}(k)}\frac{1}{i}(-\psi^\dagger(j) + \psi(j))$$

Boundary Conditions: If $\hat{\sigma}_3(\frac{L}{2}+1) = \eta\,\hat{\sigma}_3(-\frac{L}{2}+1)$ $(\eta = \pm 1)$

$$\Rightarrow \hat{Q}\,\hat{\psi}(\tfrac{L}{2}+1) = \eta\,\hat{\psi}(-\tfrac{L}{2}+1)$$

or $\hat{\psi}(\frac{L}{2}+1) = \hat{Q}\eta\,\hat{\psi}(-\frac{L}{2}+1)$ (where $[Q, H] = 0$)

We can use these results to find a simple version of the Hamiltonian for the equivalent fermi system: (L even)

$$\hat{H} = -N_s + \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} 2\hat{\psi}^\dagger(j)\hat{\psi}(j) + \lambda \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} (\psi^\dagger(j)-\psi(j))(\psi^\dagger(j+1)+\psi(j+1))$$

+ boundary term

Here $N_s = L$. The boundary term is given by

$$-\lambda\eta\, \hat{\sigma}_3(\tfrac{L}{2})\hat{\sigma}_3(-\tfrac{L}{2}+1) = -\lambda\eta\, \hat{Q}(\psi(\tfrac{L}{2})-\psi^\dagger(\tfrac{L}{2}))(\psi^\dagger(-\tfrac{L}{2}+1)+\psi(-\tfrac{L}{2}+1))$$

where

$$\hat{Q} = \hat{R} = \prod_{j=-\frac{L}{2}+1}^{L/2} (-\hat{\sigma}_1(j)) = e^{i\pi\hat{N}}$$

and $\hat{N}$ is the total # of fermions. Notice that $\hat{N}$ does not commute with $\hat{H}$ but $[e^{i\pi\hat{N}}, \hat{H}] = 0$. Thus we can only tell if the fermion # is even or odd. Also notice that PBC's for the spins ($\eta = +1$) implies that the fermions obey

$$\hat{\psi}^\dagger(\tfrac{L}{2}+1) = \hat{Q}\; \hat{\psi}(-\tfrac{L}{2}+1)$$

Thus for $\hat{N}$ even, the fermions obey PBC's but for $\hat{N}$ odd, ~~they~~ they obey APBC's.

In practice, it turns out that $E_0^- > E_0^+ \Rightarrow$ we can work in the even sector.

Diagonalization

The fermion hamiltonian is bilinear in fermion fields. Thus, it should be diagonalizable by a suitable canonical transformation. Since fermion number is not conserved, this transformation is not just a Fourier transform.

In the even sector

$$\hat{Q}\,|\Psi\rangle = |\Psi\rangle$$

the F.T. is (L even)

$$\psi(j) = \frac{1}{L} \sum_{k=-\frac{L}{2}+1}^{\frac{L}{2}} e^{i 2\pi \frac{kj}{L}} \, \tilde{a}(k)$$

$$\tilde{a}(k) = \sum_{j=-\frac{L}{2}+1}^{L/2} e^{-i 2\pi kj/L} \, \psi(j)$$

such that

$$\{ \tilde{a}(k), \tilde{a}^{\dagger}(k') \} = L\,\delta_{k,k'}$$

$$\{ \tilde{a}(k), \tilde{a}(k') \} = \{ \tilde{a}^{\dagger}(k), \tilde{a}^{\dagger}(k') \} = 0$$

Also

$$\frac{1}{L} \sum_{k=-\frac{L}{2}+1}^{\frac{L}{2}} e^{-i 2\pi \frac{kj}{L}} = \delta_{j,0}$$

$$\frac{1}{L} \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} e^{i 2\pi \frac{kj}{L}} = \delta_{k,0}$$

In the thermodynamic limit ($L \to \infty$) we get

$$\lim_{L\to\infty} \frac{L}{2\pi}\, \delta_{k,0} \equiv \delta(k) \qquad \text{(Dirac's } \delta\text{-function)}$$

and k fills up uniformly the interval $[-\pi, \pi]$ (this is the first Brillouin zone). Thus

$$\delta_{j,0} \underset{L\to\infty}{=} \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{-ikj} \qquad k \equiv \frac{2\pi k}{L}$$

$$2\pi\,\delta(k) = \lim_{L\to\infty} \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} e^{ikj}$$

Notice that $2\pi\,\delta(0) = L \Rightarrow \delta(0) = \frac{L}{2\pi}$

Similarly: $\tilde{a}_k = \hat{a}(k) \Rightarrow \{\hat{a}(k), \hat{a}^\dagger(k')\} = 2\pi\,\delta(k-k')$

These definitions can be used to get

$$\sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \psi^\dagger(j)\,\psi(j) = \frac{1}{L^2} \sum_j \sum_{k,k'} e^{-i2\pi\frac{kj}{L}} e^{i2\pi\frac{k'j}{L}}\, \tilde{a}^\dagger(k)\,\tilde{a}(k')$$

$$\underset{L\to\infty}{=} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi}\, 2\pi\,\delta(k'-k)\, \hat{a}^\dagger(k)\,\hat{a}(k')$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, \hat{a}^\dagger(k)\,\hat{a}(k)$$

$$\sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \psi^\dagger(j)\,\psi(j\pm1) \underset{L\to\infty}{=} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi}\, e^{\pm ik'}\, 2\pi\,\delta(k'-k)\, \hat{a}^\dagger(k)\,\hat{a}(k')$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{\pm ik}\, \hat{a}^\dagger(k)\,\hat{a}(k)$$

$$\lim_{L\to\infty} \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \psi^\dagger(j)\,\psi^\dagger(j+1) = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\int_{-\pi}^{\pi}\frac{dk'}{2\pi}\, 2\pi\,\delta(k+k')\, e^{-ik'}\, \hat{a}^\dagger(k)\,\hat{a}^\dagger(k')$$

$$= \int_{-\pi}^{\pi}\frac{dk}{2\pi}\, e^{ik}\, \hat{a}^\dagger(k)\,\hat{a}^\dagger(-k)$$

and

$$\lim_{L\to\infty} \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \psi(j)\,\psi(j+1) = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\int_{-\pi}^{\pi}\frac{dk'}{2\pi}\, 2\pi\,\delta(k+k')\, e^{ik'}\,\hat{a}(k)\,\hat{a}(k')$$

$$= \int_{-\pi}^{\pi}\frac{dk}{2\pi}\, e^{-ik}\, \hat{a}(k)\,\hat{a}(-k)$$

By collecting terms, we find

$$H = -L + 2\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(1+\lambda\cos k)\,\hat{a}^\dagger(k)\,\hat{a}(k)$$

$$+ \lambda\int_{-\pi}^{\pi}\frac{dk}{2\pi}\left(e^{ik}\,\hat{a}^\dagger(k)\,\hat{a}^\dagger(-k) - e^{-ik}\,\hat{a}(k)\,\hat{a}(-k)\right)$$

This Hamiltonian is reminiscent of the <u>pairing Hamiltonian</u> of the BCS theory of superconductivity. Notice that we can rewrite H in the form

$$\cancel{H = -L + \int_{-\pi}^{\pi}\frac{dk}{2\pi}\Big\{(1+\lambda\cos k)\left[\hat{a}^\dagger(k)\,\hat{a}(k) + \hat{a}^\dagger(-k)\,\hat{a}(-k)\right] +}$$

$$\cancel{+\,\lambda\left(e^{ik}\,\hat{a}^\dagger(k)\,\hat{a}^\dagger(-k) - e^{-ik}\,\hat{a}(k)\,\hat{a}(-k)\right)\Big\}}$$

$$= -L + 2\int_{0}^{\pi}\frac{dk}{2\pi}\,(1+\lambda\cos k)\left(\hat{a}^\dagger(k)\,\hat{a}(k) + \hat{a}^\dagger(-k)\,\hat{a}(-k)\right) +$$

$$+ \int_0^{\pi} \frac{dk}{2\pi} \; 2\lambda \sin k \;\; i\left(\hat a^{\dagger}(k)\, \hat a^{\dagger}(-k) + \hat a(k)\, \hat a(-k)\right)$$

It is possible to write $\hat H$ in terms of the spinor field $\hat\Psi(k)$

$$\hat\Psi(k) = \begin{pmatrix} \hat a^{\dagger}(k) \\ \hat a(-k) \end{pmatrix}$$

$$\hat\Psi^{\dagger}(k) = \left(\hat a(k) \quad \hat a^{\dagger}(-k) \right)$$

Notice that the two components of $\hat\Psi$ <u>are not independent</u>. Indeed if we ~~we~~ denote by $\hat C$ the matrix

$$\hat C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we get

$$\hat\Psi^{\dagger}(k) = \left[\hat C\, \hat\Psi(-k)\right]^{T} = \hat\Psi^{T}(-k)\, \hat C$$

This is a <u>real</u> spinor field (or <u>Majorana</u> fermion) (not <u>Dirac</u> which is <u>complex</u>). In terms of $\hat\Psi(k)$ we can write H

$$H = +L\left(-1 + \int_0^{\pi} \frac{dk}{2\pi}\, 2(1+\lambda\cos k)\right) - \int_0^{\pi} \frac{dk}{2\pi}\, \hat\Psi^{\dagger}(k) \begin{bmatrix} +2(1+\lambda\cos k) & (-2i\lambda\sin k) \\ (2i\lambda\sin k) & -2(1+\lambda\cos k) \end{bmatrix} \hat\Psi(k)$$

Bogoliubov Transformation:

$$a(k) = u(k)\,\eta(k) - i\,v(k)\,\eta^\dagger(-k)$$

$$a(-k) = u(k)\,\eta(-k) + i\,v(k)\,\eta^\dagger(k)$$

with $u(k)$ and $v(k)$ real functions of k (to be determined below)

and

$$\eta(k) = u(k)\,a(k) + i\,v(k)\,a^\dagger(-k)$$

$$\eta(-k) = u(k)\,a(-k) - i\,v(k)\,a^\dagger(k)$$

This transformation is canonical, i.e.

$$\{a(k), a^\dagger(k')\} = 2\pi\,\delta(k-k') \Rightarrow \{\eta(k), \eta^\dagger(k')\} = 2\pi\,\delta(k-k')$$

iff

$$u^2(k) + v^2(k) = 1$$

i.e.

$$u(k) = \cos\theta(k)$$

$$v(k) = \sin\theta(k)$$

We will determine $\theta(k)$ and $\theta(-k)$ by demanding that the fermion number non-conserving term cancel out (in terms of the η's!)

$$a^\dagger(k)\,a(k) + a^\dagger(-k)\,a(-k) =$$

$$= u^2(k)\left(\eta^\dagger(k)\,\eta(k) + \eta^\dagger(-k)\,\eta(-k)\right) +$$

$$+ v^2(k)\left(\eta(-k)\,\eta^\dagger(-k) + \eta(k)\,\eta^\dagger(k)\right)$$

$$+ i\,u(k)\,v(k)\left[-\eta^\dagger(k)\,\eta^\dagger(-k) + \eta(-k)\,\eta(k) + \eta^\dagger(-k)\,\eta^\dagger(k) - \eta(k)\,\eta(-k\right.$$

$$a^\dagger(k)\,a^\dagger(-k) - a(-k)\,a(k) =$$

$$= u^2(k)\left[\eta^\dagger(k)\,\eta^\dagger(-k) + \eta(k)\,\eta(-k)\right] +$$

$$+ v^2(k)\left[\eta(-k)\,\eta(k) + \eta^\dagger(-k)\,\eta^\dagger(k)\right]$$

$$+ i\,u(k)\,v(k)\left[\eta(-k)\,\eta^\dagger(-k) - \eta^\dagger(k)\,\eta(k) + \eta(k)\,\eta^\dagger(k) - \right.$$

$$\left. - \eta^\dagger(-k)\,\eta(-k)\right]$$

with

$$\alpha(k) = 2(1 + \lambda\cos k)$$

$$\beta(k) = 2\lambda\sin k$$

Hence

$$\alpha(k)\left(a^\dagger(k)\,a(k) + a^\dagger(-k)\,a(-k)\right) + i\beta(k)\left[a^\dagger(k)\,a^\dagger(-k) - a(-k)\,a(\right.$$

$$= \left[\alpha(k)\left(u^2(k) - v^2(k)\right) + 2\beta(k)\,u(k)\,v(k)\right]\left(\eta^\dagger(k)\,\eta(k) + \eta^\dagger(-k)\,\eta(-k)\right)$$

$$+ \left[-2u(k)\,v(k)\,\alpha(k) + \beta(k)\left(u^2(k) - v^2(k)\right)\right] i\left(\eta^\dagger(k)\,\eta^\dagger(-k) + \eta(k)\,\eta(-k)\right)$$

$$+ 2L\left(v^2(k)\,\alpha(k) - u(k)\,v(k)\,\beta(k)\right)$$

The condition is:

$$-2\alpha(k)\,u(k)v(k) + \beta(k)\,\left(u^2(k) - v^2(k)\right) = 0$$

$u(k) = \cos\theta(k)$ and $v(k) = \sin\theta(k) \Rightarrow$

$$-\alpha(k)\,\sin 2\theta(k) + \beta(k)\cos 2\theta(k) = 0$$

and

$$\tan 2\theta(k) = +\,\frac{\beta(k)}{\alpha(k)}$$

$$\Rightarrow \boxed{\tan(2\theta(k)) = +\,\frac{\lambda\sin k}{1+\lambda\cos k}}$$

With this choice, H becomes

$$H = \int_0^{\pi}\frac{dk}{2\pi}\;\omega(k)\left(\eta^\dagger(k)\eta(k) + \eta^\dagger(-k)\eta(-k)\right) + \mathcal{E}_0 L$$

where

$$\mathcal{E}_0(\lambda) = -1 + 2\int_0^{\pi}\frac{dk}{2\pi}\left(v^2(k)\,2\lambda\sin k - u(k)v(k)\,2(1+\lambda\cos k)\right)$$

$$= -1 + \int_0^{\pi}\frac{dk}{2\pi}\left[4\lambda\sin k\sin^2\theta(k) - 2\sin 2\theta(k)\,(1+\lambda\cos k)\right]$$

$$\omega(k) = \alpha(k)\cos 2\theta(k) + \beta(k)\sin 2\theta(k)$$

$$= 2\left[(1+\lambda\cos k)\cos(2\theta(k)) + \lambda\sin k\,\sin(2\theta(k))\right]$$

I will choose the signs of $\cos 2\theta(k)$ and $\sin 2\theta(k)$ in such a way that $\omega(k) \geq 0$.

$$\omega(k) = \alpha(k)\cos 2\theta(k) + \beta(k)\sin 2\theta(k)$$

Choose

$$\text{sign} \cos 2\theta(k) = \text{sign}\, \alpha(k)$$

$$\text{sign} \sin 2\theta(k) = \text{sign}\, \beta(k)$$

$$\omega(k) = |\alpha(k)|\, |\cos 2\theta(k)| + |\beta(k)|\, |\sin 2\theta(k)|$$

But

$$|\cos 2\theta(k)| = \frac{1}{\sqrt{1+\tan^2 2\theta(k)}} = \frac{1}{\sqrt{1+\left(\frac{\beta(k)}{\alpha(k)}\right)^2}}$$

$$|\sin 2\theta(k)| = \frac{|\tan 2\theta(k)|}{\sqrt{1+\tan^2 2\theta(k)}} = \frac{\left|\frac{\beta(k)}{\alpha(k)}\right|}{\sqrt{1+\left(\frac{\beta(k)}{\alpha(k)}\right)^2}}$$

$$\Rightarrow \quad \omega(k) = \frac{|\alpha(k)|}{\sqrt{1+\left(\frac{\beta(k)}{\alpha(k)}\right)^2}} + \frac{|\beta(k)|\left|\frac{\beta(k)}{\alpha(k)}\right|}{\sqrt{1+\left(\frac{\beta(k)}{\alpha(k)}\right)^2}} = \frac{|\alpha(k)|\left(1+\left(\frac{\beta(k)}{\alpha(k)}\right)^2\right)}{\sqrt{1+\left(\frac{\beta(k)}{\alpha(k)}\right)^2}}$$

$\Rightarrow$

$$\omega(k) = |\alpha(k)|\sqrt{1+\left|\frac{\beta(k)}{\alpha(k)}\right|^2} \Rightarrow \omega(k) = \sqrt{(\alpha(k))^2 + (\beta(k))^2}$$

$$\Rightarrow \boxed{\omega(k) = 2\sqrt{(1+\lambda\cos k)^2 + \lambda^2 \sin^2 k}}$$

After some algebra we get

$$\omega(k) = 2\sqrt{1+\lambda^2+2\lambda\cos k}$$

which is clearly positive. With these choices the ground state $|G\rangle$ is simply the state anihilated by all the ~~op~~ destruction operators.

$$\eta(k)\,|0\rangle = 0$$

$$\eta(-k)\,|0\rangle = 0$$

The ground state energy (density) is thus $\mathcal{E}_0$

$$\mathcal{E}_0(\lambda) = -1 + \int_0^{\pi} \frac{dk}{2\pi}\Big[4\lambda \sin k \sin^2 2\theta(k) - 2\sin 2\theta(k)\,(1+\lambda\cos k$$

$$\Rightarrow \mathcal{E}_0(\lambda) = -\int_0^{\pi}\frac{dk}{2\pi}\,\omega(k) = -\frac{1}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\omega(k) < 0$$

First Excited State:

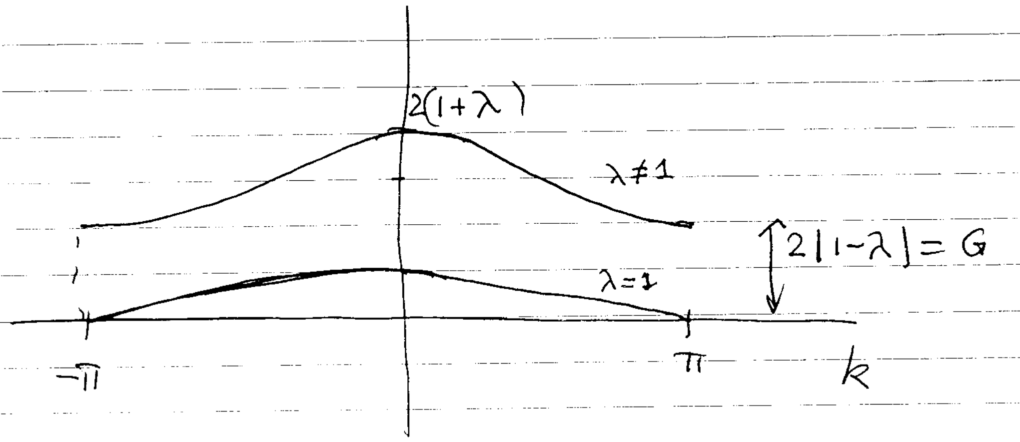
It is a fermion state

$$|k\rangle \equiv \eta^{\dagger}(k)\,|0\rangle$$

$$H\,|k\rangle = \big(E_0 + \omega(k)\big)\,|k\rangle$$

Excitation energy $\quad \mathcal{E}(k) = \omega(k) > 0$

Energy gap $\quad G(\lambda) = \min \mathcal{E}(k)$



$$\omega(k) = 2\sqrt{1+\lambda^2+2\lambda\cos k} = 2\sqrt{(1-\lambda)^2+4\lambda\cos^2\left(\tfrac{k}{2}\right)}$$

Notice that exactly at $\lambda=1$ the gap goes to zero!

This is the phase transition point!

Indeed

$$Z = \operatorname{tr} T^N \underset{\beta\to\infty}{\sim} \operatorname{tr} e^{-\beta H}$$

$$Z = e^{-NL\frac{f}{T}}$$

$$\Rightarrow \quad f \approx \mathcal{E}_0(\lambda)$$

(f = free energy density of the 2D IM)

$$\& \quad \mathcal{E}_0(\lambda) = -2\int_0^{\pi}\frac{dk}{2\pi}\sqrt{(1+\lambda)^2-4\lambda\sin^2\left(\tfrac{k}{2}\right)}$$

$$= -\frac{2|1+\lambda|}{\pi}\int_0^{\pi/2} dx\,\sqrt{1-(1-\gamma^2)\sin^2 x} = -\frac{2|1+\lambda|}{\pi} E\left(\tfrac{\pi}{2},\sqrt{1-\gamma^2}\right.$$

elliptic functi

$$1-\gamma^2 = \frac{4\lambda}{(1+\lambda)^2} \Rightarrow |\gamma| = \left|\frac{1-\lambda}{1+\lambda}\right|$$

If $\lambda \to 1 \Rightarrow \gamma \to 0$ and

$$E\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right) \underset{\gamma \to 0}{\approx} 1 + \frac{\gamma^2}{4}\left(\ln \frac{16}{\gamma^2} - 1\right) + O(\gamma^4)$$

$$\varepsilon_0(\lambda) \approx -2\left(\frac{1+\lambda}{\pi}\right)\left\{1 + \frac{\gamma^2}{4}\left(\ln \frac{16}{\gamma^2} - 1\right) + \cdots\right\}$$

$$= \varepsilon_0^{sing} + \varepsilon_0^{reg}$$

singular term

$$\varepsilon_0^{sing} = -\frac{4}{\pi}\left[1 + \frac{(1-\lambda)^2}{16}\left(\ln\left(\frac{64}{1-\lambda}\right)^2 - 1\right) + \cdots\right]$$

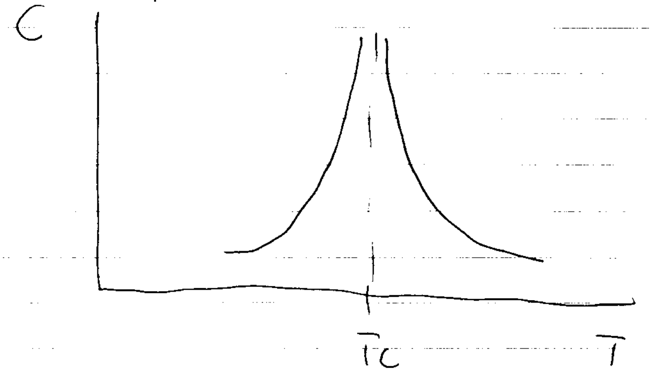
$$t = |1-\lambda|$$

$$\varepsilon_0^{sing}(t) \approx -\frac{4}{\pi}\left[1 + \frac{t^2}{8}\left(\ln\left(\frac{8}{|t|}\right) - \frac{1}{2}\right) + \cdots\right]$$

Specific heat: $C = C_{sing} + C_{reg}$

$$C_{sing} \approx -\frac{\partial^2 \varepsilon_0^{sing}}{\partial t^2} \approx +\frac{1}{2\pi}\ln\left(\frac{8}{|t|}\right) - \frac{3}{4\pi} + \cdots$$

This is the famous logarithmic divergence of the specific heat of the 2DIM



Also the gap $G(\lambda) \approx A\,|\lambda - \lambda_c|^{\nu_G}$

with $\lambda_c = 1$, $A = 1$ and $\nu = 1$

$\Rightarrow$ ξ = correlation length

$$\xi = \frac{v_s}{G}$$

But $v_s(\lambda=1) = +1 \Rightarrow \xi(\lambda) = \frac{1}{G(\lambda)} = \frac{1}{|\lambda-1|}$

Correlation length exponent $\nu_\xi = +1$

<u>Equations of Motion for $\lambda \approx 1$</u>

The (Majorana) fermions $\chi_1(j)$ and $\chi_2(j)$ have simple equations of motion. Indeed, one finds

$$i\,\partial_t \chi_1(j) = i\,\chi_2(j) - i\lambda\,\chi_2(j-1)$$

$$i\,\partial_t \chi_2(j) = -i\,\chi_1(j) + i\lambda\,\chi_1(j+1)$$

Restore a lattice constant $a_0 \neq 1$ and set $x_j = j a_0$.

$$\Rightarrow \chi_2(j-1) \approx \chi_2(x_j) - a_0\,\partial_x \chi_2(x_j)$$

$$\chi_1(j+1) \approx \chi_1(x_j) + a_0\,\partial_x \chi_1(x_j)$$

i.e.

$$\frac{1}{a_0\lambda}\, i\,\partial_t \chi_1 \cong i\,\frac{(1-\lambda)}{a_0\lambda}\,\chi_2 + i\,\partial_x \chi_2$$

$$\frac{1}{a_0\lambda}\, i\,\partial_t \chi_2 \cong -i\,\frac{(1-\lambda)}{a_0\lambda}\,\chi_1 + i\,\partial_x \chi_1$$

Rescaling time: $t \to x_0\,(a_0\lambda)$, $x \to x_1$ we get

$$i\partial_0 \chi_1 - i\partial_1 \chi_2 + i\left(\frac{1-\lambda}{a_0\lambda}\right)\chi_2 = 0$$

$$i\partial_0 \chi_2 - i\partial_1 \chi_1 - i\left(\frac{1-\lambda}{a_0\lambda}\right)\chi_1 = 0$$

~~If we define~~ $m = \lim_{\substack{a_0\to 0\\ \lambda\to 1}} \left(\frac{1-\lambda}{a_0\lambda}\right)$ (Scaling limit)

we obtain

$$i\partial_0 \chi_1 - i\partial_1 \chi_2 + im\,\chi_2 = 0$$

$$i\partial_0 \chi_2 - i\partial_1 \chi_1 - im\,\chi_1 = 0$$

With the notation: $\gamma_0 = -\sigma_2$ $\quad \gamma_1 = i\sigma_3$ $\quad \gamma_5 = \sigma_1$

we find that the spinor $\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}$ obeys the 1+1-dimensional Dirac equation

$$(i\not\partial - m)\chi = 0$$

Notice that χ is a <u>real</u> field (Majorana!)

$$\Rightarrow \quad \chi^+ = \chi^T$$

Thus, in the <u>scaling limit</u> of $\lambda \to 1$ (i.e. getting asymptotically close to the phase transition) and $a_0 \to 0$ ("continuum limit") the 2DIM is equivalent to (or <u>defines</u>!) the field theory of free Majorana fermions. Notice that this works only at distances <u>long</u> compared with the lattice constant (i.e. $a_0 \to 0$) but comparable to $\xi \approx \frac{1}{|m|}$.