L.6 The Loop expansion

The perturbative expansion ~~expansion~~ developed above seems to be limited to ~~...~~ situation in ~~...~~ which (a) λ is small and (b) the vacuum is symmetric ($\langle\phi\rangle=0$). There are however other expansion schemes.

One such scheme is WKB in Q.M. WKB is however difficult to generalize to a field thy. as it stands. There is another way to proceed.

$$Z[J]=\int \mathcal{D}\phi\, e^{-S[\phi]/a} \qquad a = \text{"}\hbar\text{"}$$

$$Z[J] = \int \mathcal{D}\phi\; e^{-\frac{1}{a}\int d^dx\, \mathcal{H}_{int}\left[\frac{\delta}{\delta J}\right]} \int \mathcal{D}\phi\; e^{-\frac{1}{a}\int d^dx\, \mathcal{H}_0[\phi] + \frac{1}{a}\int J\cdot\phi}$$

$$= \int \mathcal{D}\phi\; e^{-\frac{1}{a}\int d^dx\, \mathcal{H}_{int}\left(\frac{\delta}{\delta J}\right)} \int \mathcal{D}\phi\; e^{-\int \frac{1}{2a}\phi G^{-1}\phi + J\phi}$$

$$= \int \mathcal{D}\phi\; e^{-\frac{1}{a}\int d^dx\, \mathcal{H}_{int}\left(\frac{\delta}{\delta J}\right)} \int e^{\frac{a}{2}\int J G_0 J}$$

Thus the rules are the same but: every vertex acquires a weight $\frac{1}{a}$

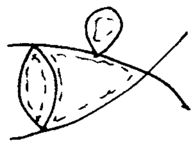
every prop " " a.

A graph with N external ~~lines~~ points, I internal lines will have a weight (to order n) a^{I-n}

How many mom. int.? There are n δ-functions but 1 expresses overall mom. cons. $\Rightarrow n-1 \Rightarrow$ the # L of indep. mom. integrations is $L = I - (n-1)$

$\Rightarrow$ the weight $a^{I-n} = a^{L-1}$

Thus we really are expanding in powers of the # of indep. integra(tions) or loops



$a^{3-1} = a^2$

$N = 4$
$n = 4$
$I = 6$
$L = 3 = 6-(4-1) = 3$

in this case, though, it coincides with an expansion in powers of λ.

If we had several couplings the sit. is totally different.

(1) The tree approx. (no loops) $\Rightarrow$

Let's compute Γ to this order.

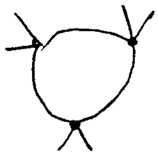
(i) $\Gamma^{(2)}(k_1,k_2) = (2\pi)^d\,\delta^d(k_1+k_2)(k_1^2+m_0^2)$ $\quad$ [$\Gamma^{(2)} = G_0^{-1} - \Sigma$ and $\Sigma = 0$ at the tree le(vel)

since all 1PI graphs contain at least one loop.

(ii) $\Gamma^{(3)} = 0$

(iii) $\Gamma^{(4)}(k_1,\dots,k_4) =$ ~~(2π)~~  $= (2\pi)^d\,\delta^d(k_1+\dots+k_4)\,\lambda$

all higher vertices contain, at least, one loop; e.g. (in ϕ^4 thy.)

$\Gamma^{(6)} =$  $+ \dots$

$$\Rightarrow \Gamma\{\bar\phi\} = \sum_{N=1}^{\infty} \frac{1}{N!} \int \frac{d^dq_1}{(2\pi)^d}\cdots\frac{d^dq_N}{(2\pi)^d}\, \Gamma^{(N)}(q_1,\dots,q_N)\, \bar\phi(-q_1)\cdots\bar\phi(-q_N)$$

$$\equiv \frac{1}{2!}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}(2\pi)^d\delta^d(q_1+q_2)(k_1^2+m_0^2)\,\bar\phi(-q_1)\,\bar\phi(-q_2)$$

$$+\frac{1}{4!}\lambda\int\frac{d^dq_1}{(2\pi)^d}\cdots\frac{d^dq_4}{(2\pi)^d}(2\pi)^d\,\delta^d(q_1+\cdots+q_4)\,\bar\phi(-q_1)\cdots\bar\phi(-q_4)+0$$

$$\Gamma_0(\bar\phi) = \int d^dx\left[\frac{1}{2}(\nabla\bar\phi)^2+\frac{m_0^2}{2}\bar\phi^2+\frac{\lambda}{4!}\left(\bar\phi^2\right)^2\right]$$ it is just the Landau theory.

$\Rightarrow m_0^2>0 \Rightarrow \bar\Phi=0 \;\wedge\; m_0^2<0 \Rightarrow \bar\Phi^2=-6\frac{m_0^2}{\lambda}$

(b) <u>One loop corrections:</u>

$L=1 \Rightarrow I=n$ (i.e. # of internal lines = order of P.T.)

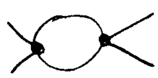
In ϕ^4 thy. a graph with N external points, I internal lines and of order n satisfies

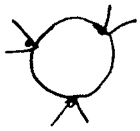
$$4n = N+2I$$

(i.e. the # of lines emerging from all vertices must either be tied up together or attached to an external point)

$\Rightarrow L=1 \Rightarrow I=n \Rightarrow n=\frac{N}{2}$

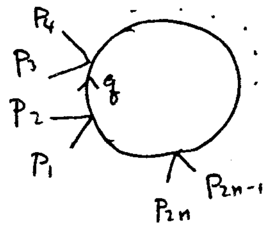
Thus $\Gamma^{(2)} =$ 

$\Gamma^{(4)} =$  one loop graphs.

$\Gamma^{(6)} =$ 

a contribution to $\Gamma^{(N)}$ has the form

$(N=2n)$



$$\bar{\Gamma}_1^{N}(0,\dots,0) = -\left(-\frac{\lambda}{4!}\right)^n \frac{1}{n!}\, S_n \int \frac{d^dq}{(2\pi)^d}\left[\frac{1}{q^2+m_0^2}\right]^n \times (N-1)!$$

" # of ways of attaching p's to the external ver

where $S_n = (4\times 3)^n \times n!$

" # of ways of reordering the vertices.

$$\Rightarrow \Gamma_1[\Phi] = \cancel{(2\pi)^d}\sum_{N=1}^{\infty} \frac{1}{N!}\,\Phi^N\, \bar{\Gamma}_1^{N}(0,\dots,0)\,(2\pi)^d\,\delta^d(0)$$

$$U_1[\Phi] = -\sum_{n=1}^{\infty}\frac{1}{(2n)!}\,\Phi^{2n}\left(-\frac{\lambda}{4!}\right)^n \frac{(4\times 3)^n\, n!}{n!}\,(2n-1)!\int_q\left[\frac{1}{q^2+m_0^2}\right]^n$$

$$U_1[\Phi] = -\sum_{n=1}^{\infty}\frac{1}{2n}\int_q\left[-\frac{\lambda\Phi^2}{2}\,\frac{1}{q^2+m_0^2}\right]^n$$

$$\ln(1+x) = -\sum_{n=1}^{\infty}\frac{(-x)^n}{n}$$

$$U_1[\Phi] = \frac{1}{2}\int_q \ln\left[1+\frac{\lambda\Phi^2/2}{q^2+m_0^2}\right]$$

$$\Rightarrow U[\Phi] = \frac{m_0^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 + \cancel{\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\ln[q^2}$$

$$+\ \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\ln\left[q^2+m_0^2+\frac{\lambda\Phi^2}{2}\right] \underbrace{-\ \frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\ln(q^2+m_0^2)}_{\text{const-energy shift.}}$$

We saw before that the actual mass μ^2 is s.t.

$$\mu^2 = \Gamma^{(2)}(0)$$

$$\Rightarrow \Gamma^{(2)}(0) = m_0^2 + \frac{\lambda}{2}\int_q \frac{1}{q^2+m_0^2}$$

$$\Rightarrow \mu^2 = m_0^2 + \frac{\lambda}{2}\int_q \frac{1}{q^2+m_0^2}$$

or to order one loop:

$$\mu^2 = m_0^2 + \frac{\lambda}{2}\int_q \frac{1}{q^2+\mu^2}$$

For $\lambda=0$, T_0 was s.t. $m_0^2=0 \Rightarrow T_c$ s.t. $\mu^2=0$

$$\boxed{\mu^2 = \chi^{-1}}$$

$$0 = T_0 - T_c + \frac{\lambda}{2}\int_q \frac{1}{q^2}$$

$$\Rightarrow T_c = T_0 - \frac{\lambda}{2}\int_q \frac{1}{q^2} < T_0 \qquad \Rightarrow m_c^2 = -\frac{\lambda}{2}\int_q \frac{1}{q^2}$$

$$m_0^2 = T - T_0 = T - T_c + T_c - T_0 = m_c^2 + \delta T$$

$$\mu^2 = m_c^2 + \delta T + \frac{\lambda}{2}\int_q \frac{1}{q^2+\mu^2}$$

$$\mu^2 = \delta T + \frac{\lambda}{2}\int_q \left[\frac{1}{q^2+\mu^2} - \frac{1}{q^2}\right]$$

when does p.t. become important?

$$\frac{\lambda}{2}\int_q \left[\frac{1}{q^2+\mu^2} - \frac{1}{q^2}\right] \simeq 1 \qquad \text{(Ginzburg crit.)}$$

$$\frac{\lambda\,\delta T}{2}\int_q \frac{1}{q^2(q^2+\delta T)} \cong 1$$

$d \leq 4$ it diverges as $\delta T \to 0$ (IR divergence)

If $\delta T \to 0 \Rightarrow \int_q \left(\frac{1}{q^2}\right)^2$ diverges as $\delta T \to 0$ if $d<4$

This type of divergence ($q \simeq 0$) is known as an infrared divergence and will appear even if a short-distance (large mom.) cutoff is present provided $\delta T \to 0$ and $d<4$. Then the Landau thy. breaks down near T_c for $d<4$. But so does the loop expansion.

$\boxed{L.7}$ Mass and Coupling Constant Renormalization.

If we look back at either μ^2 or $U(\Phi)$ we observe that both diverge (if $d>2$) when the momentum cutoff is removed: $\Lambda \to \infty$

We showed above that

$$\mu^2 = \delta T + \frac{\lambda \mu^2}{2} \int_q \frac{1}{q^2(q^2+\mu^2)} \quad ; \quad \mu^2 = m_0^2 + \frac{\lambda}{2} \int_q \frac{1}{q^2+\mu^2}$$

$\Rightarrow$ up to higher order corrections we can write

$$U(\Phi) = \frac{\mu^2 \Phi^2}{2} + \frac{\lambda}{4!}\Phi^4 + \frac{1}{2}\int_{q,\Lambda} \ln\left(q^2+\mu^2+\frac{\lambda}{2}\Phi^2\right) - \frac{\lambda \Phi^2}{2}\int_q \frac{1}{q^2+\mu^2}$$

if $d<4$ $U(\Phi)$ is now finite as $\Lambda \to \infty$. The price paid is to introduce an arbitrary parameter, the <u>renormalized mass</u> μ

$$\Rightarrow m_0^2 = \mu^2 - \frac{\lambda}{2}\int_{q,\Lambda} \frac{1}{q^2+\mu^2}$$

$\Rightarrow$ up to higher order $\Gamma^{(2)}(p) = p^2 + \mu^2$ and it is finite.

In QFT we need to remove all strong cutoff dependence as $\Lambda \to \infty$. Thus one needs to find a procedure ~~of~~ defining the C.C

As $d \to 4$ another divergence appears, this time in $\Gamma^{(4)}$

Let g be the renormalized coupling constant.

$$\Rightarrow \Gamma^{4}(0) = g = \lambda - \frac{3\lambda^2}{2} \int_q \frac{1}{(q^2+\mu^2)^2}$$

or $$g = \lambda - \frac{3\lambda^2}{2} \int_q \left(\frac{1}{q^2+\mu^2}\right)^2$$

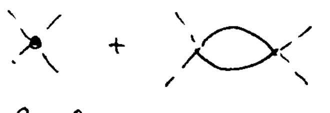
$$\Rightarrow \lambda = g + \frac{3}{2} g^2 \int_q \frac{1}{(q^2+\mu^2)^2} + \cdots$$

$$U(\Phi) = \frac{\mu^2 \Phi^2}{2} + \frac{g}{4!}\Phi^4 + \frac{1}{2}\int_q \ln\left(q^2+\mu^2+\frac{g\Phi^2}{2}\right) - \frac{g\Phi^2}{2}\int \frac{1}{q^2+\mu^2}$$
$$+ \frac{g}{16}\Phi^4 \int \left(\frac{1}{q^2+\mu^2}\right)^4$$

$\Rightarrow$ we can get a finite $U(\Phi)$, $\Gamma^{(2)}(k)$, $\Gamma^{(4)}(k_i)$ provided the mass and coupling constants are renormalized.

$$m_0^2 = \mu^2 - \frac{g}{2}\int_q \frac{1}{q^2+\mu^2}$$

$$\lambda = g + \frac{3g^2}{2}\int_q \left(\frac{1}{q^2+\mu^2}\right)^2$$

and $\Gamma^{(4)} =$  + 2 permut =

$$\Gamma^{(4)}(k_i) = \lambda - \frac{\lambda^2}{2}\int_q \frac{1}{(q^2+\mu^2)((k_1+k_2-q)^2+\mu^2)} + 2 \text{ perm}.$$

becomes $$\Gamma_R^{(4)}(k_i) = g - \frac{g^2}{2}\int_q \left[\frac{1}{(q^2+\mu^2)((k_1+k_2-q)^2+\mu^2)} - \left(\frac{1}{q^2+\mu^2}\right)^2\right] + 2 \text{ pe}$$

with these def. $\Gamma^{(4)}$ is finite if $d<6$

$$\Rightarrow \quad \mu^2 = \bar{\Gamma}^{(2)}(k=0) = \left.\frac{\partial^2 U}{\partial \Phi^2}\right|_{\Phi=0}$$

$$g = \bar{\Gamma}^{(4)}(k_i=0) = \left.\frac{\partial^4 U}{\partial \Phi^4}\right|_{\Phi=0}$$

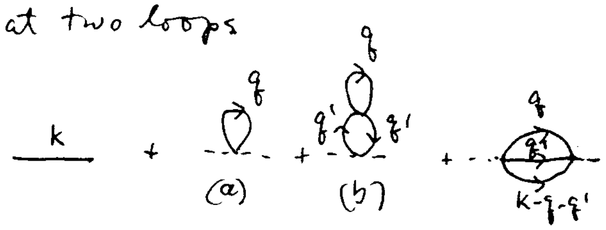
Of course we could choose to define our effective (or renormalized) coupling const. at arbitrary momenta. Indeed if we want to study the massless ($\mu^2=0$ or critical) theory we'll be forced to do so.

Two loops

I've just shown, that, to order of one-loop, we can get a finite theory provided the mass and the coupling constant are renormalized. Is this true to all orders?

Let's try at two loops

$\Gamma^{(2)}(k) =$



$$\Gamma^{(2)}(0) = \mu_1^2 = m_0^2 + \underbrace{\frac{\lambda}{2} D_1(m_0^2,\Lambda)}_{(a)} - \underbrace{\frac{\lambda^2}{4} D_2(m_0^2,\Lambda)\, D_1(m_0^2,\Lambda)}_{(b)} - \underbrace{\frac{\lambda^2}{6} D_3(0,m_0^2,\Lambda)}_{(c)}$$

$$D_1(m_0^2,\Lambda) = \int_\Lambda \frac{d^dq}{(2\pi)^d} \frac{1}{q^2+m_0^2} \; ; \quad D_2(m_0^2,\Lambda) = \int_q^\Lambda \left(\frac{1}{q^2+m_0^2}\right)^2$$

$$D_3(k,m_0^2,\Lambda) = \int_{q_1 q_2}^\Lambda \frac{1}{(q_1^2+m_0^2)(q_2^2+m_0^2)((k-q_1-q_2)^2+m_0^2)}$$

$$D_1(m_0^2;\Lambda) = \int_q^\Lambda \frac{1}{q^2+m_0^2} \cong \int_q^\Lambda \frac{1}{q^2+\mu_1^2 - \frac{\lambda}{2} D_1(\mu_1^2;\Lambda)}$$ (up to higher orders)

$$\cong \int_q^\Lambda \frac{1}{q^2+\mu_1^2} + \int_q \left(\frac{1}{q^2+\mu_1^2}\right)^2 \frac{\lambda}{2} D_1(\mu_1^2,\Lambda) + \cdots$$

$$\simeq D_1(\mu_1^2,\Lambda) + \frac{\lambda}{2} D_2(\mu_1^2,\Lambda)\, D_1(\mu_1^2,\Lambda) + \cdots$$

$$\Rightarrow \mu_1^2 = m_0^2 + \frac{\lambda}{2}\left[D_1(\mu_1^2,\Lambda) + \frac{\lambda}{2} D_2(\mu_1^2,\Lambda)\, D_1(\mu_1^2,\Lambda)\right]$$

$$-\frac{\lambda^2}{4} D_2(\mu_1^2,\Lambda)\, D_1(\mu_1^2,\Lambda) - \frac{\lambda^2}{6} D_3(0,\mu_1^2,\Lambda)$$

$$\underset{\mu_1^2}{m_0^2} = m_1^2 - \frac{\lambda}{2} \underset{\sim\Lambda^2}{D_1(\mu_1^2,\Lambda)} + \frac{\lambda^2}{6} \underset{\sim\Lambda^2}{D_3(0,\mu_1^2,\Lambda)}$$

note that the one-loop renorm. has partially cancelled the 2loop contrib. This is a general feature.

$$\Rightarrow \Gamma^{(2)}(k) = k^2 + \mu_1^2 - \frac{\lambda^2}{6}\left[D_3(k,\mu_1^2,\Lambda) - D_3(0,\mu_1^2,\Lambda)\right]$$

note that this subtraction reduces the degree of div. of D_3

$$O(D_1) \approx \frac{\Lambda^d}{\Lambda^2} = \Lambda^{d-2} \xrightarrow[d\to 4]{} \Lambda^2$$

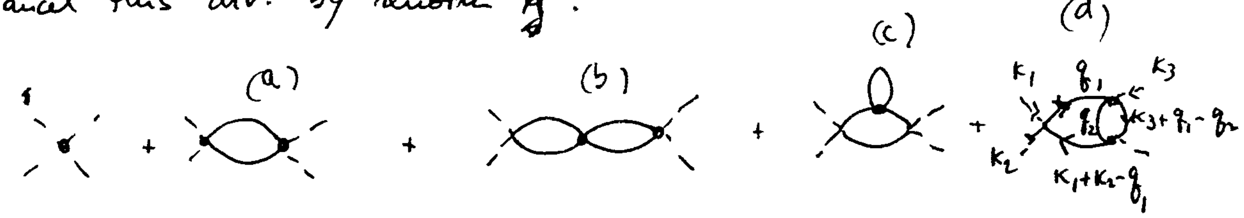
$$O(D_2) \approx \frac{\Lambda^d}{\Lambda^4} = \Lambda^{d-4} \xrightarrow[d\to 4]{} \ln\Lambda$$

$$O(D_3) \simeq \frac{\Lambda^{2d}}{\Lambda^6} = \Lambda^{2(d-3)} \xrightarrow[d\to 4]{} \Lambda^2 \rightarrow \ln\Lambda$$ by subtraction.

$\Rightarrow \Gamma^{(2)}$ diverges like $\ln \Lambda$ as $d \to 4$

Can we cancel this div. by renorm λ ?

$\Gamma^{(4)} =$



$$\Gamma^{(4)}(k_i) = \lambda - \frac{\lambda^2}{2}\left[I(k_1+k_2, m_0^2, \Lambda) + 2\text{ permut.}\right] \qquad (a)$$

$$+ \frac{\lambda^3}{4}\left[I^2(k_1+k_2, m_0^2, \Lambda) + 2\text{ permut.}\right] \qquad (b)$$

$$+ \frac{\lambda^3}{2}\left[I_3(k_1+k_2, m_0^2, \Lambda)\, D_1(m_0^2, \Lambda) + 2\text{ permut.}\right] \qquad (c)$$

$$+ \frac{\lambda^3}{2}\left[I_4(k_i, m_0^2, \Lambda) + 6\text{ permut.}\right] \qquad (d)$$

$$I(k, m_0^2, \Lambda) = \int_q^\Lambda \frac{1}{(q^2+m_0^2)((k-q)^2+m_0^2)}$$

$$I_3(k, m_0^2, \Lambda) = \int_q^\Lambda \frac{1}{(q^2+m_0^2)^2((k-q)^2+m_0^2)}$$

$$I_4(k_i, m_0^2, \Lambda) = \int_{q_1 q_2}^\Lambda \frac{1}{(q_1^2+m_0^2)(q_2^2+m_0^2)((k_1+k_2-q_1)^2+m_0^2)((k_3+q_1-q_2)^2+m_0^2)}$$

$$O(I) \cong \Lambda^{d-4} \xrightarrow[d\to 4]{} \ln\Lambda$$

$$O(I_3) \simeq \Lambda^{d-6} \xrightarrow[d\to 4]{} \Lambda^{-2}$$

$$O(I_4) \simeq \frac{\Lambda^{2d}}{\Lambda^8} = \Lambda^{2(d-4)} \xrightarrow[d\to 4]{} \ln^2\Lambda$$

mass renorm $\Rightarrow$ $I(K, m_0^2, \Lambda) = I\left(K, \mu_1^2 - \frac{\lambda}{2} D_1(\mu_1^2, \Lambda), \Lambda\right) \approx$

$$= \int_q \frac{1}{\left(q^2 + \mu_1^2 - \frac{\lambda}{2} D_1\right)\left[(K-q)^2 + \mu_1^2 - \frac{\lambda}{2} D_1\right]} \approx$$

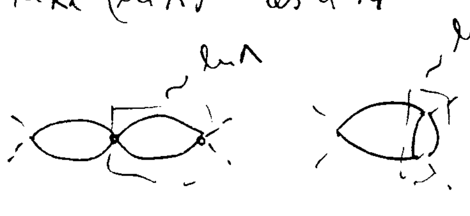
$$\cong I(K, \mu_1^2, \Lambda) + \frac{\lambda}{2} D_1(K, \mu_1^2, \Lambda)\, I_3(K, \mu_1^2, \Lambda) \times 2$$

$$\Rightarrow \Gamma^{(4)}(K_i, \mu_1^2, \Lambda) = \lambda - \frac{\lambda^2}{2}\left[I(K_1+K_2, \mu_1^2, \Lambda) + 2 \text{ perm.}\right] \quad (a)$$

$$+ \frac{\lambda^3}{2}\left[I_4(k_i, \mu_1^2, \Lambda) + 5 \text{ perm.}\right] \quad (d)$$

$$+ \frac{\lambda^3}{4}\left[I^2(K_1+K_2, \mu_1^2, \Lambda) + 2 \text{ perm.}\right] \quad (b)$$

Note that (c) has been cancelled by mass renorm. (b), (d) diverge like $(\ln \Lambda)^2$ as $d \to 4$



$$g_1 = \Gamma^{(4)}(0) = \lambda - \frac{3}{2}\lambda^2 D_2(\mu_1^2, \Lambda) + \frac{3}{4}\lambda^3 \left[D_2(\mu_1^2, \Lambda)\right]^2 + 3\lambda^3 I_4(k_i = 0, \mu_1^2,$$

Inverting

$$\boxed{\lambda = g_1 + \frac{3}{2} g_1^2\, D_2(\mu^2, \Lambda) + \frac{15}{4} g_1^3 \left[D_2(\mu_1^2, \Lambda)\right]^2 - 3 g_1^3 I_4(k_i = 0, \mu_1^2, \Lambda)}$$

$$\Rightarrow \Gamma^{(4)}(K_0) = g_1$$

$$- \frac{g_1^2}{2}\left\{ \left(I(K_1+K_2, \mu_1^2, \Lambda) - D_2(\mu_1^2, \Lambda)\right) + 2\text{ Permut.}\right\}$$

$$+ \frac{g_1^3}{4}\left\{ \left(I(K_1+K_2, \mu_1^2, \Lambda) - D_2(\mu_1^2, \Lambda)\right)^2 + 2\text{ permut}\right\}$$

$$+ \frac{g_1^3}{2}\left\{ \left[I_4(K_i; \mu_1^2, \Lambda) - I_4(K_i = 0; \mu_1^2, \Lambda)\right] - \right.$$

$$\left. - D_2(\mu_1^2, \Lambda)\left[I(K_i+K_2, \mu_1^2, \Lambda) - D_2(\mu_1^2, \Lambda)\right] + 5\text{ permut.}\right\}$$

Likewise

$$\Gamma^{(2)}(K) = K^2 + \mu_1^2 - \frac{g_1^2}{6}\left[D_3(K, \mu_1^2, \Lambda) - D_3(0, \mu_1^2, \Lambda)\right]$$

and coupling constant renormalization does not cancel the extra divergence.

$$\Rightarrow \boxed{\begin{aligned} m_0^2 = \mu_1^2 &- \frac{g_1}{2} D_1(\mu_1^2, \Lambda) - \frac{3}{4} g_1^2 D_2(\mu_1^2, \Lambda) D_1(\mu_1^2, \Lambda) \\ &+ \frac{g_1^2}{6} D_3(0, \mu_1^2, \Lambda) \end{aligned}}$$

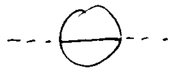
and $\Gamma^{(4)}$ is finite as $d \to 4$ $(\Lambda \to \infty)$ but $\Gamma^{(2)}$ still diverges like $\ln \Lambda$

Wave-Function Renormalization:

...8 So far we have succeeded in rendering the 4-point function finite. We still have to deal with the remaining div. in $\Gamma^{(2)}$

$$\Gamma^{(2)}(k) = k^2+\mu^2 - \frac{g_1^2}{6}\left[D_3(k,\mu^2,\Lambda) - D_3(0,\mu^2,\Lambda)\right]$$

Let's first observe that the divergence in $\Gamma^{(2)}$ comes from this graph



Moreover the divergent part is absent at ~~fast~~ zero k but present at finite k.

Let's attempt to remove this divergence by rescaling the Green's function and define

$$\Gamma_R^{(2)}(k,\mu_R^2) = Z_\phi(g,\mu^2,\Lambda)\,\Gamma^{(2)}(k,\mu^2,\Lambda)$$

or

$$G_R^{(2)} = Z_\phi^{-1}\, G_c^{(2)}$$

(note that this is equivalent to a change in the scale of ϕ by $Z_\phi^{1/2}$ thus the name of W-F-R) where

$$Z_\phi = 1 + g\, z_1 + g^2 z_2 + \dots$$

Let's require

$$\left.\frac{\partial \Gamma_R^{(2)}}{\partial k^2}\right|_{k=0} = 1 \qquad (\text{i.e. } \Gamma^R = k^2+\mu_R^2)$$

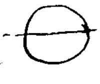
$$\Gamma_R^{(2)}(0) = \mu_R^2$$

Then we have:

$$1 = Z_\phi\left(1 - \frac{g_1^2}{6}\frac{\partial}{\partial k^2} D_3(k,\mu^2,\Lambda)\Big|_{k=0}\right)$$

$$\mu_R^2 = Z_\phi\,\mu^2$$

Solving for z_1, z_2 $\Rightarrow$ $z_1 = 0$

$$\boxed{z_2 = \frac{g_1^2}{6}\frac{\partial}{\partial k^2} D_3(k,\mu^2,\Lambda)\Big|_{k=0}}$$ $\equiv \frac{\partial}{\partial k^2}$  $\Big|_{k=0}$

and $\mu_R^2 = (1 + g^2 z_2)\mu^2$

Recall the connection between $\Gamma^{(4)}$ and $G_c^{(4)}$

$$G_c^{(4)} = - G_c^2 \cdots G_c^2 \, \Gamma^{(4)}$$

$$\Rightarrow G_c^{(4)} = - Z_\phi^{4} \, G_R^{(2)} \cdots G_R^{(2)} \, \Gamma^{(4)}$$

Define $\left.\begin{array}{l} \Gamma_R^{(4)} = Z_\phi^2 \, \Gamma^{(4)} \\ G_{cR}^{(4)} = Z_\phi^{-2} \, G_c^{(4)} \end{array}\right\} \Rightarrow \begin{array}{l} \Rightarrow \boxed{g_R = Z_\phi^2 g} \\ G_{cR}^{(4)} = - G_R^{(2)} \cdots G_R^{(2)} \, \Gamma_R^{(4)} \end{array}$

why do we do this? In higher orders we have insertions of the form  which are cancelled from  provided the internal propagators are renormalized. This overall rescaling cancels these contributions.

Likewise $\Gamma_R^{(N)} = Z_\phi^{N/2} \, \Gamma^{(N)}$

$$G_{cR}^{(N)} = Z_\phi^{-N/2} \, G_c^{(N)}$$

Let's recapitulate:

i) We calculated the bare functions ~~$G^{(2)}, G^{(4)}$~~ $\Gamma^{(2)}, \Gamma^{(4)}$ etc.

ii) We replaced m_0^2 first by $\mu^2 = \Gamma^{(2)}(0)$ and later by $\mu_R^2 = Z_\phi \mu^2$

iii) We replaced λ by $g = \Gamma^{(4)}(0)$ and later by $g_R = g Z_\phi^2$

iv) Z_ϕ was obtained by requiring $\left.\frac{\partial \Gamma_R^{(2)}}{\partial k^2}\right|_{k=0} = 1 = Z_\phi \left.\frac{\partial \Gamma^{(2)}}{\partial k^2}\right|_{k=0}$

v) $\Gamma_R^{(N)} = Z_\phi^{N/2} \Gamma^{(N)}$ and $G_{cR}^{(N)} = Z_\phi^{-N/2} \, G_R^{(N)}$

and the theory is finite provided the mass ~~and~~, coupling