* Lecture 22 (3/14)

There are several other ways to formulate the ideas of the R.G. The ideas expressed above are useful at a qualitative level but become quite complicated to implement when ~~the fixed point~~ it is difficult to access the form of the F.P. Hamiltonian. In practice, unless one makes uncontrollable approximations, one has to resort to heavy computer simulations to carry out the program (MCRG). If the F.P. happen either at weak or strong coupling then some form of P.T. can be used. There are still other alternatives.

Let me introduce the same block-spin ideas but with a slightly ≠ flavor.

For the sake of definiteness consider a Heisenberg spin model on a hypercubic lattice (O(3))

$$\mathcal{H} = \frac{1}{T} \sum_{\langle \vec r, \vec r\,' \rangle} \vec S(\vec r)\cdot \vec S(\vec r\,') = -\frac{1}{2T} \sum_{\langle \vec r, \vec r\,' \rangle} \left(\vec S(\vec r) - \vec S(\vec r\,')\right)^2 + \text{const.}$$

with $S^2 = 1$ as a constraint.

Expanding in Taylor series

$$\vec S(r) - \vec S(\vec r\,') = a\, \vec\nabla_\mu \vec S(\vec r)\cdot \hat n_\mu + O(a^2)$$

$$O(a^2) = \frac{a^2}{2} \nabla_\nu \nabla_\mu \vec S(\vec r)\ \hat n_\mu\ \hat n_\nu$$

$$a^d \sum_{\vec r} f(\vec r) \equiv \int d^d x\, f(\vec x)$$

$$H = \frac{a^{2-d}}{2T} \int d^d x\, (\nabla \vec S)^2 + \frac{a^{4-d}}{8T} \int d^d x\, (\nabla_\mu^2 \vec S)^2 + \dots$$

If we are interested in the behavior in the continuum limit, it is then clear that the second term measures ~~..~~ fast variations of $\vec{S}$, at least compared with the 1^{st} term, and those should be important only for ~~..~~ $\langle \vec{S}(\vec{x}) \cdot \vec{S}(\vec{y}) \rangle$ ~~for~~ when $|\vec{x}-\vec{y}| \simeq a$ but not of $|\vec{x}-\vec{y}| >> a$. Only smooth variations should be important.

~~..~~ I will ignore such terms. I will ~~..~~ show later on that they are irrelevant (in the technical sense).

Thus we have ~~to~~ understand the prop. of

$$H = \int dx^d \; \frac{1}{2g} (\nabla_\mu \vec{S})^2 \qquad \text{with} \qquad S^2 = 1$$

$$\text{and} \quad \boxed{g = T\, a^{d-2}} \qquad (T \text{ dimensionless})$$

In general S_x, S_y, S_z have ~~for~~ Fourier components with momenta ~~too~~ small compared with the cutoff $\Lambda > \frac{1}{a}$ but also of the same order. We can use the constraint $S_1^2 + S_2^2 + S_3^2 = 1$ to require that one of the components, say S_3, to have only fast components and be small. We'll see that this is in fact consistent.

Let

$$S_1 = \sqrt{1 - S_3^2} \cos\theta$$
$$S_2 = \sqrt{1 - S_3^2} \sin\theta$$

|| satisfies the constraint.

$$\mathcal{H} = \frac{1}{2g} (\nabla_\mu \vec{S})^2 = \frac{1}{2g} (\nabla_\mu S_3)^2 + \frac{1}{2g} (1 - S_3^2)(\nabla_\mu \theta)^2 + \frac{1}{2g} \frac{(S_3 \nabla_\mu S_3)^2}{1 - S_3^2}$$

Let us consider a generalization of the σ-model with N components.

$$S^a(x) \qquad a = 1, \dots, N$$

$$\vec{S} \cdot \vec{S} = 1$$

The (Euclidean) Lagrangian is

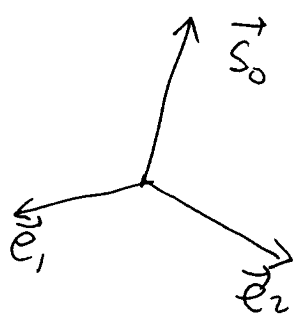
$$\mathcal{L} = \frac{1}{2g} \left(\nabla_\mu \vec{S} \right)^2$$

with the local constraint $\vec{S}^2 = 1$.

Consider now a configuration $\vec{S}_0(x)$ which is (a) slowly varying (on the scale of the cutoff) and (b) solution of the equations of motion i.e. it extremizes the euclidean action $S = \int d^dx\, \mathcal{L}$

Since $\vec{S}^2 = 1$ and $\vec{S}_0^2 = 1$ any local variation must be essentially orthogonal to $\vec{S}_0(x)$. Let us define a local coordinate system by

$$\{ \vec{S}_0(x), \{\vec{e}_i(x)\} \} \qquad i = 1, \dots, N-1$$



Consider configurations $\vec{S}(x)$ which vary <u>rapidly</u> compared to $\vec{S}_0(x)$ (Polyakov)

$$\vec{S}(x) = F(x)\ \vec{S}_0(x) + \sum_{i=1}^{N-1} \phi_i(x)\ \vec{e}_i(x)$$

$$\vec{S}^{\,2} = 1 = F^2(x) + \sum_{i=1}^{N-1} \phi_i^{\,2}(x)$$

$$\vec{S}_0 \cdot \vec{e}_i = 0 \qquad \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

$$\Rightarrow \quad F(x) = \sqrt{1 - \vec{\phi}^{\,2}(x)}$$

$$\vec{S}(x) = \sqrt{1-\vec{\phi}^{\,2}(x)}\ \ \vec{S}_0(x) + \sum_{i=1}^{N-1} \phi_i(x)\ \vec{e}_i(x)$$

(Berezinskii)

Local changes in $\vec{S}_0$ and $\vec{e}_i$ can also be parametrized

① $\vec{S}_0 \cdot \nabla_\mu \vec{S}_0(x) = 0 \Rightarrow \qquad \nabla_\mu S_0^{\,a}(x) = \sum_{i=1}^{N-1} B_\mu^{\,i}\ e_i^{\,a}(x)$

$$\Rightarrow \boxed{B_\mu^{\,i}(x) = \cancel{\vec{e}_l(x)\nabla_\mu \vec{e}_l}\ \ \vec{e}_i(x) \cdot \nabla_\mu \vec{S}_0(x)}$$

$$\cancel{\nabla_\mu e_i^a = - B_\mu e_i}$$

$$\nabla_\mu e_i^{\,a} = \sum_j A_\mu^{\,ij}\ e_j^{\,a}(x) - B_\mu^{\,i}\ S_0^{\,a}(x)$$

$$\boxed{A_\mu^{\,ij}(x) = \vec{e}_i(x) \cdot \nabla_\mu \vec{e}_j(x)}$$

These gauge fields reflect the ambiguities in the choice of a local frame.

Let us substitute and find the action for $\phi_i(x)$

(unrestricted!)

$$\nabla_\mu S^a = \nabla_\mu\left(\sqrt{1-\vec{\phi}^2}\right) s_0{}^a + \sqrt{1-\vec{\phi}^2}\, \nabla_\mu s_0{}^a +$$

$$+ \sum_{i=1}^{N-1} (\nabla_\mu \phi_i)\, \vec{e}_i{}^a + \sum_{i=1}^{N-1} \phi_i\, \nabla_\mu e_i{}^a$$

$$= \left[\nabla_\mu \left(\sqrt{1-\vec{\phi}^2}\right)\right] s_0{}^a + \sqrt{1-\vec{\phi}^2} \sum_{i=1}^{N-1} B_\mu{}^i\, e_i{}^a$$

$$+ \sum_{i=1}^{N-1} (\nabla_\mu \phi_i)\, e_i{}^a + \sum_{i=1}^{N-1} \phi_i \sum_{j=1}^{N-1} A_\mu^{ij}\, e_j{}^a -$$

$$- \left(\sum_{i=1}^{N-1} \phi_i\, B_\mu{}^i\right) s_0{}^a =$$

$$= \left[\nabla_\mu\left(\sqrt{1-\vec{\phi}^2}\right) - \sum_{i=1}^{N-1} \phi_i\, B_\mu{}^i\right] s_0{}^a +$$

$$+ \sum_{i=1}^{N-1} e_i{}^a \left[\nabla_\mu \phi_i + \sqrt{1-\vec{\phi}^2}\, B_\mu{}^i + A_\mu^{ij}\, \phi_j\right]$$

$\Rightarrow$

$$\mathcal{L} = \frac{1}{2g} (\nabla_\mu S^a)^2 =$$

$$= \frac{1}{2g} \Big\{ \Big(\nabla_\mu \sqrt{1-\vec{\phi}^2} - \sum_{i=1}^{N-1} \vec{\phi} \cdot \vec{B}_\mu \Big)^2 +$$

$$+ \sum_{i=1}^{N-1} \Big[\nabla_\mu \phi_i + A_\mu^{ij} \phi_j + \sqrt{1-\vec{\phi}^2} \, B_\mu^i \Big]^2 \Big\}$$

Expand in powers of $\vec{\phi}$

$$\mathcal{L} = \frac{1}{2g} \Big\{ \big[\vec{\phi} \cdot (\nabla_\mu \vec{\phi} + \sqrt{1-\vec{\phi}^2} \, \vec{B}_\mu) \big]^2 \frac{1}{1-\vec{\phi}^2} +$$

$$+ \sum_{i=1}^{N-1} \big(\nabla_\mu \cancel{\vec{\phi}}\,\phi^i + A_\mu^{ij} \phi^j + \sqrt{1-\vec{\phi}^2} \, B_\mu^i \big)^2 \Big\}$$

$$\cancel{\cong \frac{1}{2g} \Big\{ (1-\vec{\phi}^2)(\vec{\phi}\cdot\vec{B}_\mu)^2 + 2(\vec{\phi}) \cdot \nabla_\mu \vec{\phi}}$$

$$\cong \frac{1}{2g} \Big\{ \cancel{1+\vec{\phi}}\, (\vec{\phi} \cdot \vec{B}_\mu)^2 + (1-\vec{\phi}^2)\, \vec{B}_\mu^2 +$$

$$+ (\nabla_\mu \phi^i + A^{ij} \phi^j)^2 +$$

$$+ 2 (\nabla_\mu \phi^i + A_\mu^{ij} \phi^j) \big\{ B_\mu^i \big(1 - \cancel{\tfrac{\vec{\phi}^2}{2}} \big) \Big\}$$

$$= \frac{1}{2g} \Big\{ \phi_i \phi_j \big[B_\mu^i B_\mu^j - \delta^{ij} \vec{B}_\mu^2 \big] + \vec{B}_\mu^2 + (\nabla_\mu \phi^i + A^{ij} \phi^j)^2$$

$$+ 2 \phi^i (-\nabla_\mu B_\mu^i + A_\mu^{ij} B_\mu^j) \Big\}$$

The last term vanishes if $\vec{S}_0$ is a sol. of the equation of motion

$$D_\mu^{ij} B_\mu^j \equiv (\nabla_\mu \delta^{ij} - A_\mu^{ij}) B_\mu^j = 0$$

The quadratic piece is

$$\mathcal{L}^{(2)} \simeq \frac{1}{2g} \left\{ \sum_{i,j=1}^{N-1} \phi_i \phi_j (B_\mu^i B_\mu^j - \delta^{ij} \vec{B}_\mu^2) + (\nabla_\mu \phi^i + A_\mu^{ij} \phi_j)^2 \right\}$$

$$+ \frac{1}{2g} \vec{B}_\mu^2 \qquad \frac{1}{\sqrt{g}} \phi_i = \phi_i'$$

or

$$\mathcal{L}^{(2)} = \frac{1}{2\cancel{g}} \left\{ B_\mu^i B_\mu^j (\phi'^i \phi'^j - \vec{\phi}'^2 \delta^{ij}) + (\nabla_\mu \phi'^i + A_\mu^{ij} \phi_j')^2 \right\}$$

$$+ \frac{1}{2g} \vec{B}_\mu^2$$

Let's compute the contribution to the effective action coming from fluctuations inside a shell $b\Lambda \leq |P| \leq \Lambda$

~~(b→1)~~
$(b \uparrow 1)$

(Recall that $\vec{B}_\mu^2 = (\nabla_\mu \vec{S}_0)^2$)

Clearly the leading correction is

$$\langle \mathcal{L}^{(2)} \rangle = \frac{1}{2g} \vec{B}_\mu^2 + \frac{1}{2} B_\mu^i B_\mu^j \langle \phi_i' \phi_j' - \vec{\phi}'^2 \delta_{ij} \rangle_{shell}$$

$$\langle \phi_i' \phi_j' - \vec{\phi}'^2 \delta_{ij} \rangle = \delta_{ij} [1 - (N-1)] \int_{b\Lambda < |\vec{P}| < \Lambda} \frac{1}{\vec{P}^2}$$

$$\int_{b\Lambda < |\vec{P}| < \Lambda} \frac{1}{\vec{P}^2} = \frac{S_d}{(2\pi)^d} \int_{b\Lambda}^{\Lambda} dp \, \frac{p^{d-1}}{p^2} =$$

$$= \frac{S_d}{(2\pi)^d} \Lambda^{d-2} (1-b)$$

$$b = e^{-\delta l} \simeq 1 - \delta l$$

$$1 - b = \delta l$$

$$\equiv (\nabla_\mu \vec{S}_0)^2$$

$$\langle \mathcal{L}^{(2)} \rangle = \frac{1}{2} \left[\frac{1}{g} + (2-N) \int_{\text{shell}} \frac{1}{\vec{p}^2} \right] \vec{B}_\mu^2 + \cdots$$

$$g = T a^{d-2} = T \Lambda^{2-d}$$

$$\frac{1}{g'} = \frac{1}{g} + (2-N) \frac{S_d}{(2\pi)^d} \Lambda^{d-2} \delta l$$

rel. to the new cutoff

$$\frac{1}{T' b^{2-d} \Lambda^{2-d}} = \frac{1}{T \Lambda^{2-d}} + (2-N) \frac{S_d}{(2\pi)^d} \Lambda^{d-2} \delta l$$

$$\frac{1}{T'} = \frac{b^{2-d}}{T} + (2-N) \frac{S_d}{(2\pi)^d} b^{2-d} \delta l$$

$$\frac{1}{T'} \cong \frac{1}{T} \left[1 + (d-2)\delta l \right] + (2-N) \frac{S_d}{(2\pi)^d} \cancel{\Lambda^{d-2}} \delta l$$

$$\frac{1}{T'} - \frac{1}{T} = \delta l \left[\frac{d-2}{T} - (N-2) \frac{S_d}{(2\pi)^d} \right]$$

$$T' = \frac{1}{\frac{1}{T} - \delta l\left[-\frac{(d-2)}{T} + (N-2)\frac{S_d}{(2\pi)^d}\right]} = \frac{T}{1 - \delta l\left[-(d-2) + (N-2)\frac{S_d}{(2\pi)^d} T\right]}$$

$$T' \simeq T\left[1 + \delta l\left(-(d-2) + (N-2)\frac{S_d}{(2\pi)^d} T\right)\right]$$

$$\frac{T'-T}{\delta l} \simeq -(d-2)T + (N-2)\frac{S_d}{(2\pi)^d} T^2 + \cdots$$

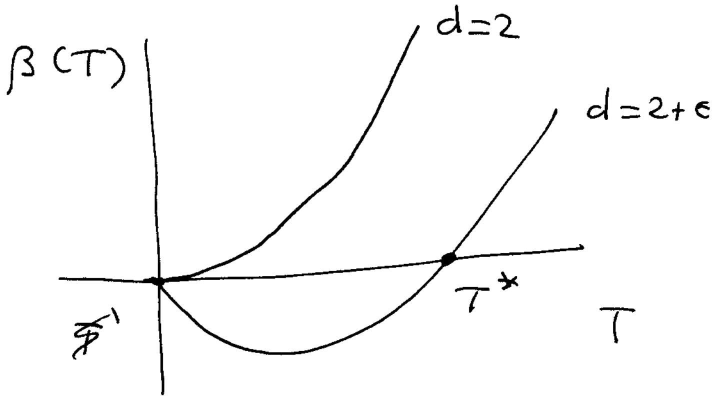
$$\beta(T) = \frac{dT}{dl} = -(d-2)T + (N-2)\frac{S_d}{(2\pi)^d} T^2$$

$$\frac{dT}{dl} = a\frac{dT}{da}$$

Fixed points $\quad T^* = 0, \quad \frac{(2\pi)^d}{S_d}\frac{(d-2)}{(N-2)}$

for T^* to be close to zero we need $d = 2 + \epsilon$

$$T^* = 0, \quad \frac{2\pi\epsilon}{N-2}$$



$$\boxed{\beta(T) = -\epsilon T + \frac{N-2}{2\pi} T^2}$$

$$\beta'(T^*) = -\epsilon + \frac{N-2}{\pi} T^*$$

$$\beta'(0) = -\epsilon$$

$$\beta'\left(\frac{2\pi\epsilon}{N-2}\right) = -\epsilon + \frac{\cancel{N-2}}{\cancel{\pi}}\,\frac{2\cancel{\pi}\epsilon}{\cancel{N-2}} = 2\epsilon - \epsilon = \epsilon$$

Let us rescale s_3 : $\quad s_3 = \sqrt{g}\,\varphi$

$$\Rightarrow \mathcal{H} = \cancel{\tfrac{1}{2g}}\ \frac{1}{2}(\nabla_\mu \varphi)^2 + \frac{1}{2g}\left(1-g\varphi^2\right)(\nabla_\mu \theta)^2 + \frac{1}{2}\,\frac{g}{1-g\varphi^2}\,(\varphi\nabla_\mu \varphi)^2$$

Let's assume we are interested in the behavior for g small $\Rightarrow$ let's expand in powers of g

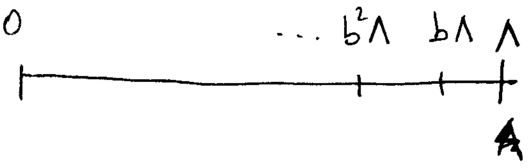
$$\mathcal{H} \simeq \frac{1}{2}(\nabla_\mu \varphi)^2 + \frac{1}{2g}\left(1-g\varphi^2\right)(\nabla_\mu \theta)^2 + \frac{g}{2}(\varphi\nabla_\mu \varphi)^2 + \frac{g^2}{2}\varphi^2(\varphi\nabla_\mu \varphi)^2 + O(g^2)$$

We see that the fluctuations of φ do affect those of θ but the converse is not true (at least not strongly so)

This Hamiltonian is supplemented by the cutoff condition $|p| < \Lambda$

Obviously the fluctuations for $|p| \sim \Lambda$ should not affect very much the behavior for $\vec{p} \sim 0$ (long wavelengths) if a continuum limit is to exist. The natural thing to do is to integrate out such fluctuations.

However in the absence of an obvious scale separation one is forced to slice up the momentum range and evaluate the contrib. of each slice, just as much as we did by blocking up the degrees of freedom in position space.



$b < 1$

Let us calculate the effective Hamiltonian by integrating out the fluctuations with momenta near the cutoff.

affect very strongly phenomena taking place ~~near~~ for $|\vec{p}| \sim 0$

Thus, it is natural to integrate out such <u>fast</u> modes.

Let b be a real number close to one, $b < 1$. Consider

the fluctuations in the momentum shell $b\Lambda < |\vec{p}| < \Lambda$.

We can write

$$e^{-S_{eff}[\vec{m}]} = e^{-\frac{1}{2u a_0^{D-2}} \int_{b\Lambda} dx^D (\nabla_i \theta)^2} \times$$

$$\times \int_{b\Lambda < |\vec{p}| < \Lambda} \mathcal{D}\varphi \quad e^{-S[\varphi, \theta]}$$

$$e^{-H_{eff}} = e^{-\frac{1}{2g(b)}\int_{b\Lambda} d^dx\,(\nabla_\mu\theta)} \int_{b\Lambda<|p|<\Lambda} \mathcal{D}\varphi \; e^{-H[\varphi,\theta]} \times \int_{0\le|p|<b\Lambda}$$

$$g = T a^{d-2} = T\Lambda^{2-d} = \cancel{T(b\Lambda)^{2-d}} \; b^{d-2}$$

$\cancel{g(b) = g\, b^{2-d}}$

small

$$\int_{b\Lambda<|p|<\Lambda} \mathcal{D}\varphi \; e^{-H[\varphi,\theta]} = \int_{b\Lambda<|p|<\Lambda} \mathcal{D}\varphi(p)\; e^{-\frac{1}{2g}\int d^dx\left[\frac{1}{2}(\nabla\varphi)^2 + \varphi^2(\nabla\theta)^2 + g\,\varphi^2(\nabla\varphi)^2\right]}$$

Since we are interested in θ's which are smoothly varying we can assume $(\nabla\theta)^2$ to be nearly constant (or momentum indep.) for $b\Lambda<|p|<\Lambda$ provided $b\to 1$

$$\int_{b\Lambda<|p|<\Lambda} \mathcal{D}\varphi(p)\; e^{-\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}p^2|\varphi(p)|^2} \; e^{+\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}|\varphi(p)|^2(\nabla\theta)^2} =$$

$$= \# \; \exp\left\{\frac{1}{2}\int d^dx \; \langle\varphi^2\rangle_\varphi \; (\nabla\theta)^2\right\}$$

$$= \prod_{b\Lambda<|p|<\Lambda}\left[\frac{2\pi}{p^2-(\nabla\theta)^2}\right]^{1/2} = \exp\left\{\frac{1}{2}\int_{b\Lambda<|p|<\Lambda}\frac{d^dp}{(2\pi)^d}\left[\ln 2\pi - \ln\left(p^2-(\nabla\theta)^2\right)\right]\right\}$$

$$\cong \exp\left\{\frac{1}{2}\int_{b\Lambda<|p|<\Lambda}\frac{d^dp}{(2\pi)^d}\ln\frac{2\pi}{p^2} + \frac{(\nabla\theta)^2}{2}\int_{b\Lambda\le|p|<\Lambda}\frac{d^dp}{(2\pi)^d}\frac{1}{p^2} + \cdots\right\}$$

Lecture 23 (3/16)

$$= \# \exp \left\{ \frac{1}{2} (\nabla\theta)^2 \int_{b\Lambda<|p|<\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} \right\}$$

The effective hamiltonian is

$$H_{eff}[\theta] = \left[\frac{1}{g} - \int_{b\Lambda<|p|<\Lambda} \frac{1}{p^2} \right] \frac{1}{2} \int d^dx \, (\nabla\theta)^2$$

$$d \simeq 2 \qquad \int = -\frac{\Lambda^\epsilon}{2\pi} \ln b > 0$$

$d = 2+\epsilon$ $\qquad$ ϵ small

$$b^\epsilon = e^{\epsilon \ln b} \cong 1 + \epsilon \ln b$$

$$H_{eff} = \frac{1}{2g'} (\nabla\theta)^2$$

$$g' = T' a'^\epsilon = T' \Lambda'^{-\epsilon} = T' \Lambda^{-\epsilon} b^{-\epsilon}$$

$$\frac{\Lambda^\epsilon}{T'} = b^{-\epsilon} \left[\frac{\Lambda^\epsilon}{T} + \frac{\Lambda^\epsilon}{2\pi} \ln b \right]$$

$$\frac{1}{T'} = b^{-\epsilon} \left[\frac{1}{T} + \frac{1}{2\pi} \ln b \right]$$

$$\frac{1}{T'} - \frac{1}{T} = \left(-\frac{\epsilon}{T} + \frac{1}{2\pi} \right) \ln b$$

$$\frac{a'}{a} = 1 + \frac{da}{a}$$

Let $\ln b = -dl = \ln \frac{\Lambda'}{\Lambda} = -\ln \frac{a'}{a} = -\frac{da}{a}$

$$\& \quad d \frac{1}{T} = -\left(-\frac{\epsilon}{T} + \frac{1}{2\pi} \right) \frac{da}{a}$$

$$+ \frac{dT}{T^2} = \left(-\frac{\epsilon}{T} + \frac{1}{2\pi} \right) \frac{da}{a}$$

$$\cancel{a\frac{dg}{da} = -\epsilon g + \frac{g^2}{2\pi}} \qquad \boxed{a\frac{dT}{da} = -\epsilon T + \frac{T^2}{2\pi}}$$

Thus by integrating out the fluctuations at scales of the order of the cutoff we discover that the effective coupling constant ~~g~~ T becomes scale dependent. In this case, it satisfies a differential equation. Of course I've only proved that the term $\frac{1}{2g}(\nabla\theta)^2$ is reproduced up to a renormalization of g. I still have to show that the other terms are also reproduced with ~~with~~ the correct powers of g. This is

L.11 necessary to keep the O(3) invariance of the original system.

* Lecture 24

At $d=2$ something very special happens. First of all g becomes dimensionless in $d=2$. Remember that g is the only dimensional quantity in this Hamiltonian. We may then suspect that if there are no dimensional quantities then there is no privileged scale and the correlation length is either zero or infinite. If we look at the recursion formulas for $d=2$ ($\epsilon=0$)

$$a\frac{dg}{da} = \frac{g^2}{2\pi}$$

we discover that $g^*=0$ is a fixed point and a marginally unstable one. Let's assume there is a finite correlation length $\xi(g)$. Dimensionally it must be true that

$$\xi(T) = a \; \Phi(T(a)) \qquad \text{where } \Phi \text{ is dimensionless}$$

$$a \frac{\partial \xi}{\partial a} = 0 \qquad (\text{since } \xi \text{ is a physical scale})$$

$$0 = \Phi + \Phi' a \frac{dg}{da}$$

$$0 = \Phi[T] + \frac{d\Phi}{dT} \; \beta(T)$$

$$\frac{d\Phi}{\Phi} = - \frac{dT}{\beta(T)}$$

$$d \ln \Phi = - \frac{dT}{\beta(T)}$$

$$\Rightarrow \ln \Phi(T_1) - \ln \Phi(T_0) = - \int_{T_0}^{T_1} \frac{dt}{\beta(t)}$$

$$\xi(T_1) = a_1 \; \Phi(T(a_1)) \qquad T_1 = T(a_1)$$

$$\xi(T_0) = a_0 \; \Phi(T(a_0))$$

$$\frac{\xi(T_1)}{\xi(T_0)} = \frac{a_1}{a_0} \frac{\Phi(T(a_1))}{\Phi(T(a_0))} = \frac{a_1}{a_0} \exp\left\{ - \int_{T(a_0)}^{T(a_1)} \frac{dt}{\beta(t)} \right\}$$

If T_0 is very small $(T_0 \gtrsim T^*)$ $\Rightarrow$ $\Phi_1 \approx 1$ and $\xi(T_1(a_1)) \equiv \xi(T_1) \approx a_1$

$$\Rightarrow \xi(T_0) = a_0 \left[\frac{\xi(T(a_1))}{a_1} \right] \exp\left\{ \int_{T_0}^{T_1} \frac{dt}{\beta(t)} \right\}$$

$\boxed{d=2}$ $$\int_{T_0}^{T_1} \frac{dt}{\beta(t)} = 2\pi \int_{T_0}^{T_1} \frac{dt}{t^2} = -\frac{2\pi}{T_1} + \frac{2\pi}{T_0}$$

$$\xi(T_0) = a_0 \left[\frac{\xi(T_1)}{a_1} e^{-2\pi/T_1} \right] \exp\left\{ \frac{2\pi}{T_0} \right\}$$

$$f(T_1)$$

$f(T_1)$ is a smooth function of T_0

$\Rightarrow \xi(T) = a N \exp\left\{\frac{2\pi}{T}\right\}$ and the correlation length diverges with an essential singularity as $T \to 0$ (T^*)

$\boxed{d>2}$

$T_1 >> T_0 \gtrsim T^* = 2\pi\epsilon$

$\int_{T_0}^{T_1} \frac{dt}{\beta(t)} =$ it is dominated by the pole at the lower end

smooth

$$\int_{T_0}^{T_1} \frac{dt}{\beta(t)} \simeq F(T_1) - \int_{T_0} \frac{dt}{\beta'(t^*)(t-t^*)} = F(T_1) - \frac{\ln(T_0 - T^*)}{\beta'(T^*)}$$

$$\Rightarrow \xi(T_0) = a_0 N \exp\left\{ -\frac{1}{\beta'(T^*)} \ln(T_0 - T^*) \right\}$$

$$\xi(T) = a N \, |T - T^*|^{-\nu}$$

$$\nu = \frac{1}{\beta'(T^*)} = \frac{1}{y_T} = \frac{1}{\epsilon}$$

Of course we do not know yet if ξ is finite or not. All we've learned is that the RG is compatible with that provided ξ can be shown to be of the order of the lattice spacing for g sufficiently large (i.e $T \gg 1$) But we know this is true from the high T expansion.

We then arrive to a very interesting situation. In $d=2$ we have a Hamiltonian which has no scale in it.

$$H = \frac{1}{2g} \int d^2x \, (\nabla_\mu \vec{S})^2 \qquad [\, \delta(S^2-1) \,]$$

If I change all lengths by a factor $\lambda \Rightarrow H$ is invariant since both g and $\vec{S}$ are dimensionless $\Rightarrow$ H is scale invariant (i.e. conformally invariant). This is a classical statement. In the classical theory no cutoffs are necessary. As soon as we want to discuss a quantum (or fluctuating) theory we have to impose a cutoff or else U.V. divergences will become overwhelming. But a cutoff theory does have a scale (the cutoff scale). Only at the F.P. (e.g. $g^* = 0$) we can remove the cutoff. Anywhere else we have to live with it and the result is a massive theory (i.e. a theory with finite correlation length). Close enough to the F.P. the cutoff may enter in a rather uninteresting way (though essential!) e.g.

$$\xi = \text{number} \times a \times e^{+2\pi/g}$$

In QFT one speaks about masses, etc. Dimensionally a mass scales like a momentum cutoff $\Lambda \sim \frac{1}{a}$ $\Rightarrow$ all masses should scale ~~like~~ like ξ^{-1}

$$M \sim \frac{1}{a} \# \; e^{-2\pi/g}$$

A massive theory can be defined as a limit $a \to 0$ (i.e. continuum) provided $g \to 0$ by keeping M constant. Then M is said to be the RG invariant mass. This limit ($a \to 0$, $g \to g^*$ M fixed) is known as the scaling limit. Every F.P. defines one such scaling limit and therefore a possible Field Theory.

How should g vary with a so that M is fixed. This is found by solving the R.G. equation.

$$a \frac{dg}{da} = \beta(g)$$

$$\frac{dg}{\beta(g)} = d\ln a$$

$$\ln \frac{a_1}{a_0} = \int_{g_0}^{g_1} \frac{dg}{\beta(g)}$$

or $\int^{g} \frac{dg'}{\beta(g')} - \ln a = \text{constant.}$ (R.G. invariant)

$$2\pi \int^{g} \frac{dg'}{g'^2} - \ln a = C$$

$$-\frac{2\pi}{g} - \ln a = C$$

$$\frac{2\pi}{g(\Lambda)} - \ln \Lambda = c' \Rightarrow \frac{2\pi}{g(\Lambda)} - \frac{2\pi}{g(M)} = \ln\left(\frac{\Lambda}{M}\right)$$

$$\mathfrak{g}(\Lambda) = \frac{\mathfrak{g}(M)}{1 + \frac{\mathfrak{g}(M)}{2\pi} \ln \frac{\Lambda}{M}} \simeq \mathfrak{g}(M) \left[1 - \frac{\mathfrak{g}(M)}{2\pi} \ln \frac{\Lambda}{M} + \left(\frac{\mathfrak{g}(M)}{2\pi} \right)^2 \ln^2 \frac{\Lambda}{M} - \dots \right]$$

Thus the effective coupling constant at ~~short~~ a scale Λ becomes a series in powers of $\ln(\frac{\Lambda}{M})$. This is something we've already seen before! We see that the RG is summing up the leading logarithms in the coupling constant renormalization. We'll see that this is always the case. Please note that if $\mathfrak{g}(M)$ is fixed then $\mathfrak{g}(\Lambda)$ becomes very small if $\Lambda \to \infty$ $(\Lambda \gg M)$

$$\mathfrak{g}(\Lambda) \simeq \frac{2\pi}{\ln \frac{\Lambda}{M}} \qquad (\Lambda \gg M)$$

with log precission. This statement (that the effective coupling constant is small at short distances) has been ~~[illegible]~~ named: Asymptotic Freedom. It means that ~~the~~ at distances short compared with typical ~~mass~~ length scales (inverse mass or correlation length) (but still much bigger than the cutoff a) perturbation theory should be valid.

Cases of AF QFT: QCD ($N_f < 16$), GN model ($d = 1+1$). This is why renormalized p.t. can be used in QCD to predict cross sections etc. for processes with large momentum transfer p_T (large compared with $(1\,fm)^{-1}$)