Lecture 18 (3/7)

In the past few lectures we have used p.t. to evaluate the N-point functions of ϕ^4 theory. Immediately we encountered trouble in the form of divergent contributions. This is, at first sight, unusual. In Q.M. every order in p.t. is finite while the sum of the p.t. series may or may not converge. Here we have infinities at every order. To handle the infinities we discussed a procedure: define effective (or renormalized) quantities. This procedure looks like sweeping the issue under the rug and in fact it is so.

There is a physical origin of the divergencies. In a classical field theory (i.e. defined by a P.D.E.) every dimensionfull quantity is determined by the dimensionfull parameters of the P.D.E. In Q.F.T. (or in S.M.) we cannot isolate <u>a single scale</u> responsible for the physical behavior of macroscopic quantities. So, when we study the corrections to the vertex function $\Gamma^{(4)}$ we found a 1st correction proportional to

$$\int_p^{\Lambda} \frac{1}{(p^2+m_0^2)^2}$$

where the integral is cutoff at $|p| \gtrsim \Lambda$. In this integral we find that the effective coupling constant has contributions not from a single length scale p^{-1} but from a whole range

$$\xi^{-1} = m_0 \leq |p| \leq \Lambda$$

$$a = \Lambda^{-1} \leq |x| \leq \xi$$

and we must sum over all such contributions. The contributions for $|p| \approx 0$ are important since the denominator is small but the contribution of $|p| \sim \Lambda$ are even bigger since the phase space factor can be big $\sim \Lambda^d$

Thus we should expect divergencies to occur since there is nothing to stop fluctuations on scales between a and ξ from happening.

Brief history of RG: Gell-Mann - Low thy. Bogoliubov-Shirkov || attempts of defining a continuum limit $\Lambda \to \infty$ by means of renormalization.

relate to Riemann's def. of an integral.

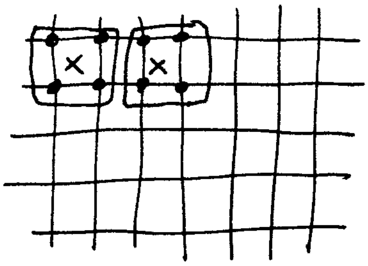
In S.M. Widom - Fisher (etc.) defining scaling

Kadanoff picture: The physics at long distances (i.e. $|x| \gg a$) should not depend on the physics at short distances ($|x| \sim a$). Thus while there are many configuration of $\{\sigma\}$ in

$$Z = \Sigma_\sigma \, e^{\beta \sigma \sigma} \qquad (\text{e.g. } \sigma = \pm 1)$$

that contribute at scales $|x| \sim a$ they give a small contribution at $|x| \gg a$ (since they may average out to zero on such scales)

Kadanoff proposed an iterative picture.



$$\vec{\mu} = \sum_{i\in A} \vec{s}_i \Big/ \Big\| \sum_{i\in A} \vec{s}_i \Big\| \qquad \vec{s}^{\,2} = 1 \Rightarrow \vec{\mu}^{\,2} = 1$$

He argued that at long distance only smoothed out config. should be important. He then proposed to coarse grain the system by defining block spin-variables

Let H be the Hamiltonian def. on a scale a (lattice const.) and ~~will~~ we'll try to change the lattice spacing (i.e. the cutoff) by integrating out the variables $\vec{s}$.

Def. a block spin transf.

$$T[\vec{\mu}|\vec{s}]$$

for example $\quad T[\vec{\mu}|\vec{s}] = \delta\left(\vec{\mu} - \sum_i \vec{s}_i / \| \sum_i \vec{s}_i \|\right)$

s.t. $\quad \sum_{\{\vec{\mu}\}} T[\vec{\mu}|\vec{s}] = 1$

Then

$$Z = \sum_{\{\vec{s}\}} e^{-H[\vec{s}]} = \sum_{\{\vec{s}\}} \sum_{\{\vec{\mu}\}} T\{\vec{\mu}|\vec{s}\}\, e^{-H\{\vec{s}\}} =$$

$$Z = \sum_{\{\vec{\mu}\}} \sum_{\{\vec{s}\}} T\{\vec{\mu}|\vec{s}\}\, e^{-H\{\vec{s}\}}$$

Def. $H_{eff}\{\vec{\mu}\}$ s.t.

$$e^{-H_{eff}\{\vec{\mu}\}} = \sum_{\{\vec{s}\}} T\{\vec{\mu}|\vec{s}\}\, e^{-H\{\vec{s}\}} \Rightarrow Z = \sum_{\{\vec{\mu}\}} e^{-H_{eff}\{\vec{\mu}\}}.$$

Lecture 19 (3/8)

Obviously $H'\{\vec{\mu}\} \neq H\{\vec{\mu}\}$

The block transf. has reduced the # of degrees of freedom by a factor b per linear dimension ($b=2$ in the example)

Also

$$\langle \vec{\mu}(\vec{R}) \cdot \vec{\mu}(\vec{R}') \rangle_{H_{eff}} = \left\langle \frac{\sum_i \vec{S}_i(\vec{R})}{\| \sum \vec{S}_i \|} \cdot \frac{\sum_j \vec{S}_j(\vec{R}')}{\| \sum \vec{S}_j \|} \right\rangle_H$$

$ba \longleftarrow a$

If the Hamiltonian H has a correlation length $\xi(\vec{S}) \Rightarrow$ the new Hamiltonian will have $\xi(\vec{\mu}) = \dfrac{\xi(\vec{S})}{b}$ simply because we have b times fewer linear sites.

Suppose now that there is a complete set of local operators $O_\alpha\{\vec{S}\}$ (conveniently normalized) $\Rightarrow$

$$\boxed{L.9} \quad H\{\vec{S}\} = \sum_\alpha h_\alpha \, O_\alpha\{\vec{S}\}$$

Example: $\vec{S}_i$; $\vec{S}_i \cdot \vec{S}_j$; $(\vec{S}_i \cdot \vec{S}_j)^2$ etc.

but also Heff will be expandable

$$H_{eff}\{\vec{\mu}\} = \sum_\alpha h_\alpha^{eff}(b)\, O_\alpha\{\vec{\mu}\}$$

where the $O_\alpha\{\vec{\mu}\}$ are def. in terms of block variables.

Suppose that we've been clever enough to pick the $O_\alpha\{\vec{S}\}$ such that under a block transformation (which involves a rescaling)

$$h_\alpha^{eff}(b) = b^{y_\alpha}\, h_\alpha \qquad \text{where the } y_\alpha\text{'s are some numbers}$$

then if $y_\alpha > 0$ the contribution of O_α is bigger in Heff than in H itself. Conversely if $y_\alpha < 0$ we find the opposite.

The first type of operator (with $y_\alpha > 0$) is said to be a <u>relevant</u> operator while the second class ($y_\alpha < 0$) is <u>irrelevant</u>. The limiting case ($y_\alpha = 0$) is known as a marginal operator.

It is clear now that if a block spin transf. is repeated very many times a limit will be reached in which all irrelevant op. will have dropped out. At that point the Hamiltonian will, up to a rescaling factor, begin to reproduce itself: ~~a fixed p~~ <u>an invariant Hamiltonian or fixed point</u> will have been reached (Wilson).

In general we will have

$$H = \underbrace{H^*}_{\text{fixed point}} + \int d^dx \underbrace{\sum_\alpha h_\alpha \, O_\alpha\{\vec{s}(x)\}}_{\text{will only include relevant and marginal perturbations.}}$$

At H^* we cannot have any scale left over. If there was a scale then ~~the~~ H could not be invariant.

In order to achieve such a F.P. we have to rescale the lattice spacing in Heff back to ~~at~~ what it was in H ~~i.e.~~ i.e. we've got to divide all lengths by b

Hence an RG transf. involves 2 steps

(1) a block spin transf. that eliminates a fraction of the degrees of freedom

(2) a rescaling of lengths.

In this form an H^* can be reached. We see that a new form of invariance has appeared: scale invariance (also known as conformal invariance) i.e. the invariance under arbitrary (global!) changes of scale. At H^* scale invariance is reached and ξ must either be zero or ∞.

Types of F.P.: there are great many classes.

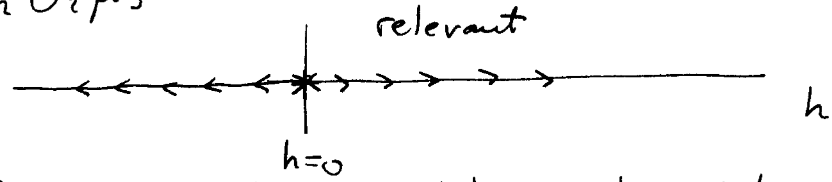
(a) Problem with one relevant perturbation.

$$H = H^* + \int dx \; h \; O\{\vec{S}\}$$

$\Rightarrow$ block-spin + rescaling yields

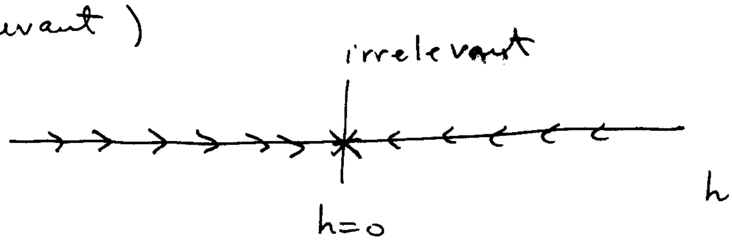
$$H' = H^* + \int dx \; b^y h \; O\{\vec{\mu}\}$$

$(y > 0)$



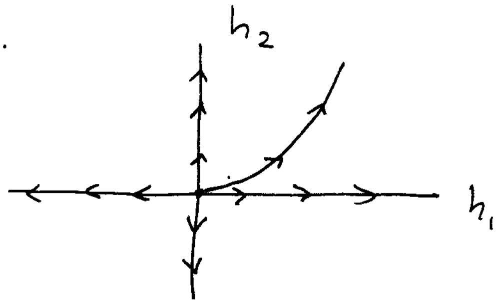
unstable F.P. (Infrared since we iterate with $b>1 \Rightarrow$ long distances)

$(y < 0)$ (irrelevant)



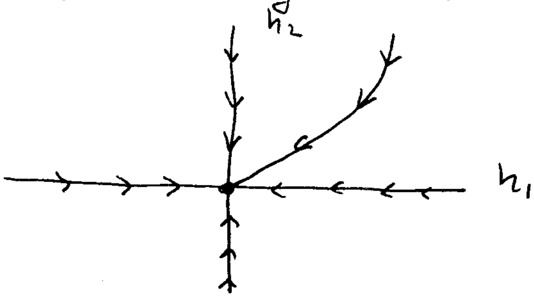
More general case: two operators

(a) two relevant op.

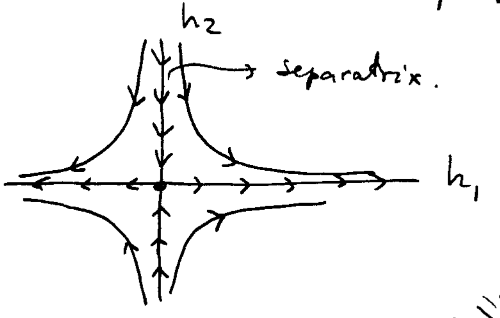


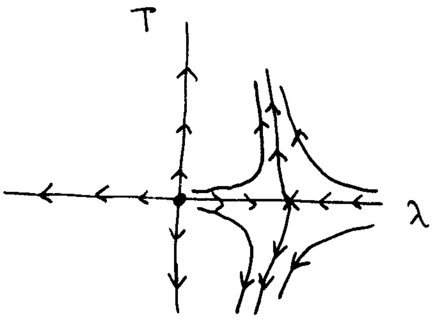
(totally unstable)

(b) two irrelevant op. (totally stable)

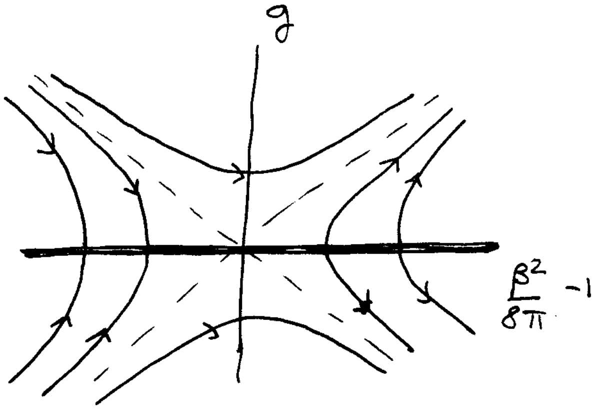


(c) ~~one stable~~ a relevant and an irrelevant op.





(ϕ^4 $d<4$)

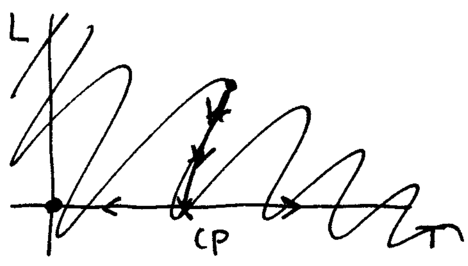


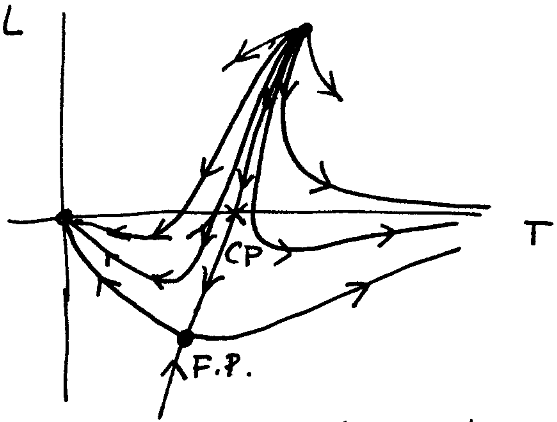
Sine-gordon.
(i.e. Kosterlitz-Thouless)

October 21 (3PM)

L.10

In the previous lectures I introduced the ideas of the renormalization group at a "plausibility" level and presented the concept of a fixed point hamiltonian. We have also discussed, in a simple illustration, how these "block-spin" ideas can be realized and found a simple realization of such a fixed point. In practice, however, there are severe limitations to the applicability of such simple schemes. Realistic schemes do not guarantee the invariance of a nearest neighbor hamiltonian. That is to say there is no reason for the F.P. Hamiltonian to have only nearest neighbor interactions. In practice it will have contributions from a number of local operators (like next-nearest neighbors, 4 spin, etc) but the F.P. properties will still be the same.

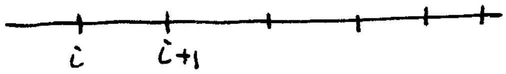




Thus the main effect of an irrelevant operator will be to change the position of the CP (i.e. the actual transition) but it will not change the critical properties which are determined by the F.P.

Lecture 20 (3/9)

A trivial example: 1d Ising Model

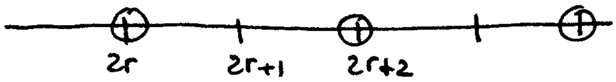


$$Z = \sum_{\{\sigma_i\}} \exp\left\{ \sum_{i=1}^{N} \beta \, \sigma_i \sigma_{i+1} \right\}$$

$\beta = \frac{1}{T}$ P.B.C.

Integrate out every other spin (decimation) $i = 2r$ $\sigma_{2r} \equiv \mu_r$

$$\sum_{\sigma_{2r+1}} \exp\{\beta \sigma_{2r+1} (\mu_r + \mu_{r+1})\} = 2\cosh\beta(\mu_r + \mu_{r+1})$$



$$= \exp\{\alpha + \beta' \mu_r \mu_{r+1}\}$$

$$\Rightarrow \begin{cases} e^{\alpha+\beta'} = 2\cosh 2\beta \\ e^{\alpha-\beta'} = 2 \end{cases} \Rightarrow \begin{array}{l} e^{2\alpha} = 4\cosh 2\beta \\ e^{2\beta'} = \cosh 2\beta \end{array}$$

$$\boxed{\beta' = \frac{1}{2} \ln\cosh 2\beta}$$

$$\Rightarrow H' = \text{const.} + \frac{N}{4} \ln(4\cosh 2\beta) + \beta' \sum_{r=1}^{N/2} \mu_r \mu_{r+1}$$

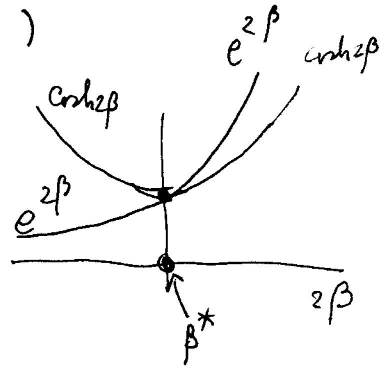
Up to a shift, the Hamiltonian has the same form

If $\beta = \beta' = \beta^* \Rightarrow$ we have a fixed point (after rescaling the lattice spacing)

$$2\beta^* = \ln\cosh 2\beta^* \Rightarrow \beta^* = 0, \infty$$

$$T^* = \infty, 0$$

$(d=1)$



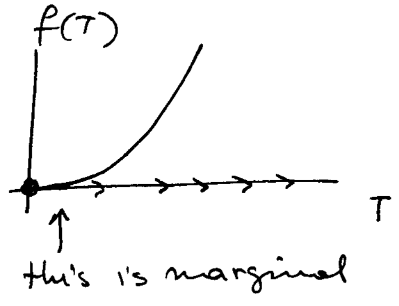
$$\beta' = \frac{1}{2}\ln\cosh 2\beta$$

$$T' = \frac{2}{\ln\cosh\frac{2}{T}} \qquad \text{near } T \to 0$$

$$T' \cong \frac{2}{\ln\frac{e^{2/T}}{2}(1+e^{-4/T})} = \frac{2}{\frac{2}{T} - \ln 2 + \ln(1+e^{-4/T})}$$

$$T' \simeq \frac{2}{\frac{2}{T} - \ln 2} = \frac{T}{1-\frac{T}{2}\ln 2} \approx T\left(1 + \frac{T^2}{2}\ln 2\right)$$

$$T' - T = f(T) = \frac{T^2}{2}\ln 2$$

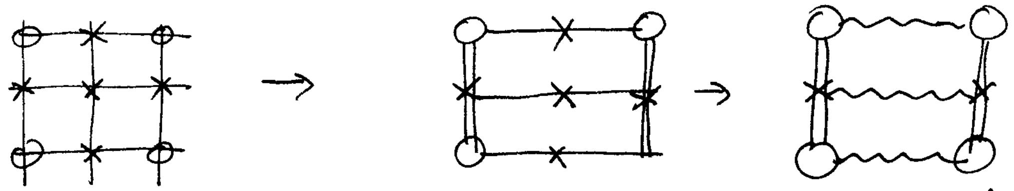


Let's do something a little less trivial.

Migdal proposed an RG which is qualitatively right and exact as $d \to 1$

It is a combination of the decimation process I just described + some "bond moving"

$d=2$



b (scale factor b)

$\sim\sim = \beta_1' = \frac{1}{2}\ln\cosh 2\beta_1$ $\Rightarrow$ symmetrize $\beta_1'' = 2\beta_1'$

$= \; : \; \beta_2' = 2\beta_2$ $\beta_2'' = \frac{1}{2}\ln\tanh 2\beta_2'$

b

In d dimension $\beta_1^{(d)} = \cancel{2\beta}\; 2^{d-1}\,\frac{1}{2}\ln\cosh 2\beta_1$

and similar expressions for the other β's. (they are $\neq$!)

$$\beta' = 2^{d-2}\ln\cosh 2\beta$$

$$T' = \frac{2^{2-d}}{\ln\cosh\frac{2}{T}}$$ analytic function of d.

$d\to 1 \Rightarrow T^*\to 0$ $\qquad d = 1+\epsilon \qquad 2^{2-d} = 2^{1-\epsilon} \cong 2\left[1-\epsilon\ln 2\right]$

$$T' \approx \frac{2(1\mp\epsilon\ln 2)}{\ln\frac{e^{2/T}}{2}} = \frac{2(1-\epsilon\ln 2)}{\frac{2}{T}-\ln 2}$$

$T \lesssim \epsilon$

$$T' = T\,\frac{(1-\epsilon\ln 2)}{1-\frac{T}{2}\ln 2} \cong T\,(1-\epsilon\ln 2)\left(1+\frac{T}{2}\ln 2\right)$$

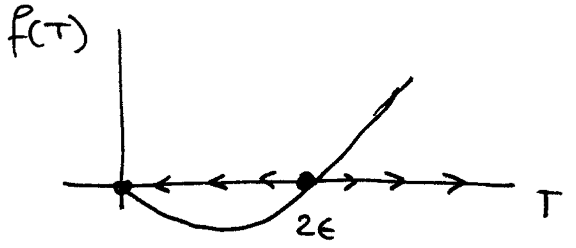
$$T' \cong T\left(1-\epsilon\ln 2+\frac{T}{2}\ln 2\right)$$

$$f(T) = T'-T \approx -T\epsilon\ln 2+\frac{T^2}{2}\ln 2$$

$$f(T) = \frac{\ln 2}{2}\left(T^2-2T\epsilon\right)$$

$f(T)=0 \Rightarrow \qquad T^2-2T\epsilon=0 \Rightarrow T=0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T=2\epsilon$



$f'(0) = -\epsilon \ln 2$

$f''(2\epsilon) = +\epsilon \ln 2$

$$H = \sum_i \beta^* \sigma_i \sigma_{i+1} + \sum_i \delta\beta\, \sigma_i\sigma_{i+1}$$

$$H' = \sum_i \beta^* \sigma_i \sigma_{i+1} + \sum_i \delta\beta'\, \sigma_i\sigma_{i+1}$$

$$\delta\beta' = b^y\, \delta\beta \qquad\qquad \delta\beta = \delta \frac{1}{T} = -\frac{\delta T}{T^2}$$

$\delta\beta' = -\frac{\delta T'}{T'^2}$

$\delta\beta' = b^y\, \delta\beta$

$\frac{\delta T'}{\cancel{T^{*2}}} = b^y \frac{\delta T}{\cancel{T^{*2}}}$

* Lecture 21 (3/11)

$$T' = T + \frac{\ln 2}{2}\,(T^2 - 2T\epsilon)$$

$$T' - 2\epsilon = T - 2\epsilon + \frac{\ln 2}{2}\, T(T-2\epsilon)$$

$$\delta T' = \delta T\,[1+\epsilon \ln 2] \approx \delta T\; 2^{\epsilon}$$

$\Rightarrow \boxed{y_T = \epsilon}$ the temperature T is relevant at $T^* = 2\epsilon$ (the transition

However at $T^* = 0$

$\delta T' = \delta T\,[1 - \epsilon \ln 2] \qquad \Rightarrow y_T = -\epsilon \qquad$ at $T^* = 0$

(it is irrelevant $\Rightarrow T^* = 0$ is stable)

At every iteration we've got a reduction of ξ

$\xi' = \frac{\xi}{2} \quad \Rightarrow$ after n iterations $\quad \xi_n = \frac{\xi}{2^n}$

If T is large $\Rightarrow \quad \xi \simeq 1$ (lattice constant)

Also $\quad \delta T' = 2^y \delta T$

$\Rightarrow \quad \delta T^{(n)} = 2^{ny} \delta T$ (assuming we are still in the linear regime)

$$2^n = \left[\frac{\delta T^{(n)}}{\delta T}\right]^{1/y} = \frac{\xi}{\xi_n}$$

$\Rightarrow$ for large n $\quad \delta T^{(n)} \approx T^*$ and $\xi_n \gtrsim 1$

$$\xi \approx \xi^* \left(\frac{T^*}{\delta T}\right)^{1/y}$$

as $\delta T \to 0 \Rightarrow \xi \to \infty \quad$ like $\xi \sim (\delta T)^{-\nu} \qquad \nu = \frac{1}{y}$

$\Rightarrow \nu = \frac{1}{y} = \frac{1}{\epsilon}$ (in this case)

and the correlation length diverges (in lattice units) as $\delta T \to 0$

for n large $\quad \delta T^{(n)} \sim T^*$ and $\xi_n \sim a$ (finite). I parametrize my results in terms of ξ_n and $\delta T^{(n)}$ ("renormalized parameters"). Thus I'll keep them fixed while varying the "bare" parameters and the lattice spacing

* Landau theory: $\quad \nu = \frac{1}{2} \qquad \leftarrow$ Valid when $d=4$
* Here: $\quad \nu = \frac{1}{\epsilon} = \frac{1}{d-1} + O(\epsilon) \qquad \leftarrow$ valid when $d \downarrow 1$

$d=2$ (exact solution) $\quad \nu = 1$ (this is an accident)

$\Rightarrow$ the exponents vary with $\underline{d}$. They also vary with the symmetries!