Li2

R.G. approach to ϕ^4 theory: momentum shell Integration

I return now to ~~...~~ ϕ^4 theory. Let's try a momentum-shell approach.

Thus we first split the field $\phi(x)$ in slow and fast components

$$\phi(x) = \phi_<(x) + \phi_>(x) \qquad (b<1)$$

$$\phi_<(x) = \int_{|p|<b\Lambda} \frac{d p^d}{(2\pi)^d} \, e^{ip\cdot x} \, \phi(p)$$

$$\phi_>(x) = \int_{\text{shell}} \frac{d p^d}{(2\pi)^d} \, e^{ip\cdot x} \, \phi(p)$$

$m_0^2 = t\Lambda^2$
$\lambda = u\Lambda^\epsilon$
t, u dimensionless
$\epsilon = 4-d$

The free energy:

$$S = \int dx^d \left[\frac{1}{2}(\nabla\phi)^2 + \frac{t\Lambda^2}{2}\phi^2 + \frac{u\Lambda^\epsilon}{4!}\phi^4 \right]$$

$$= \int dx^d \left[\frac{1}{2}(\nabla\phi_<)^2 + \frac{t\Lambda^2}{2}\phi_<^2 + \frac{u\Lambda^\epsilon}{4!}\phi_<^4 \right] + \qquad \leftarrow S_<$$

$$+ \int dx^d \left[\frac{1}{2}(\nabla\phi_>)^2 + \frac{t\Lambda^2}{2}\phi_>^2 + \frac{u\Lambda^\epsilon}{4!}\phi_>^4 \right] + \qquad \leftarrow S_>$$

$$+ \int dx^d \, \frac{u\Lambda^\epsilon}{4!} \left[4\phi_>^3\phi_< + 6\phi_>^2\phi_<^2 + 4\phi_>\phi_<^3 \right] \qquad \leftarrow S_{int}$$

Note: there are no quadratic mixed terms

$$\int \frac{1}{2}\left[(\nabla\phi)^2 + t\Lambda^2\phi^2\right] dx^d = \int \frac{dp^d}{(2\pi)^d} \frac{1}{2}(p^2 + t\Lambda^2)\phi(p)\phi(-p) \quad \text{which splits.}$$

I'll now integrate out $\phi_>(x)$ in p.t.

$$Z = \int \mathcal{D}\phi\, e^{-S^\Lambda(\phi)} = \int \mathcal{D}\phi_>\, \mathcal{D}\phi_<\, e^{-S^\Lambda(\phi_>,\phi_<)} =$$

$$= \int \mathcal{D}\phi_<\, e^{-S_<^\Lambda(\phi_<)} \int \mathcal{D}\phi_>\, e^{-S_>(\phi_>) - S_{int}(\phi_>,\phi_<)}$$

$$= \int \mathcal{D}\phi_<\, e^{-S_<^{b\Lambda}(\phi_<)}$$

$$e^{-S_<^{b\Lambda}(\phi_<)} = \cancel{\int} e^{-S_<^\Lambda(\phi_<)} \int \mathcal{D}\phi_>\, e^{-S_>^\Lambda(\phi_>) - S_{int}^\Lambda(\phi_>,\phi_<)}$$

$$\int \mathcal{D}\phi_>\, e^{-S_>^\Lambda(\phi_>) - S_{int}^\Lambda(\phi_>,\phi_<)} = \int \mathcal{D}\phi_>\, e^{-S_0^\Lambda(\phi_>)}\, e^{-S_{int}^\Lambda(\phi_>) - S_{int}^\Lambda(\phi_>,\phi_<)}$$

$$= \cancel{\sum_{n=0}^{\infty}} \qquad = \int \mathcal{D}\phi_>\, e^{-S_0^\Lambda(\phi_>)} \left\langle e^{-S_{int}^\Lambda(\phi_>) - S_{int}^\Lambda(\phi_>,\phi_<)} \right\rangle_{0,>}$$

$$= \left[\int \mathcal{D}\phi_>\, e^{-S_0^\Lambda(\phi_>)}\right] \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \left[S_{int}^\Lambda(\phi_>) + S_{int}^\Lambda(\phi_>,\phi_<)\right]^n \right\rangle_{0,>}$$

$$S_{int} = S_{int}(\phi_>) + S_{int}(\phi_>,\phi_<)$$

$$= \int d^dx \left\{ \frac{u\Lambda^\epsilon}{4!} \left[\phi_>^4 + 4\phi_>^3\phi_< + 6\phi_>^2\phi_<^2 + 4\phi_>\phi_<^3 \right] \right.$$

$$e^{-S_{eff}} = e^{-S_<(\phi_<)} \left[\int \mathcal{D}\phi_> e^{-S_0^\Lambda(\phi_>)}\right] I$$

$$I = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{u\Lambda^\epsilon}{4!}\right)^n \int_{\{x_j\}} \Big\langle \prod_{j=1}^{n} \Big[\phi_>^4(x_j) + 4\phi_>^3(x_j)\phi_<(x_j) + 6\phi_>^2(x_j)\phi_<^2(x_j) +$$

$$+ 4\phi_>(x_j)\phi_<^3(x_j)\Big]\Big\rangle_0$$

$$= \sum_{n=0}^{\infty} \frac{I_n}{n!} \left(-\frac{u\Lambda^\epsilon}{4!}\right)^n$$

$\boxed{n=1}$

$$I_1 = \int dx_1 \left\langle \phi_>^4(x_1) + 4\phi_>^3(x_1)\phi_<(x_1) + 6\phi_>^2(x_1)\phi_<^2(x_1) + 4\phi_>(x_1)\phi_<^3(x_1)\right\rangle$$

$$\langle \phi_>^{2s+1}(x)\rangle = 0$$

$$\Rightarrow I_1 = \int dx_1 \left[\langle \phi_>^4(x_1)\rangle + 6\langle\phi_>^2(x_1)\rangle\, \phi_<^2(x_1)\right]$$

$\boxed{n=2}$

$$I_2 = \int dx_1 dx_2 \Big[\langle \phi_>^4(x_1)\phi_>^4(x_2)\rangle + 4^2\langle\phi_>^3(x_1)\phi_<(x_1)\phi_>^3(x_2)\phi_<(x_2)\rangle +$$

$$+ 6^2\langle\phi_>^2(x_1)\phi_<^2(x_1)\phi_>^2(x_2)\phi_<^2(x_2)\rangle + 4^2\langle\phi_>(x_1)\phi_<^3(x_1)\phi_>(x_2)\phi_<^3(x_2)\rangle$$

$$+ 2\times4\langle\phi_>^4(x_1)\phi_>^3(x_2)\phi_<(x_2)\rangle + 2\times6\langle\phi_>^4(x_1)\phi_>^2(x_2)\phi_<^2(x_2)\rangle$$

$$+ 2\times4\langle\phi_>^4(x_1)\phi_<(x_2)\phi_>^3(x_2)\rangle + 2\times4\times6\langle\phi_>^3(x_1)\phi_<(x_1)\phi_>^2(x_2)\phi_<^2(x_2)\rangle$$

$$+ 2\times4\times4\langle\phi_>^3(x_1)\phi_<(x_1)\phi_>(x_2)\phi_<^3(x_2)\rangle + 2\times6\times4\langle\phi_>^2(x_1)\phi_<^2(x_1)\phi_>(x_2)\phi_<^3(x_2)\rangle$$

Dropping terms with odd averages (they vanish) I get

$$I_2 = \int dx_1 dx_2 \Big[\langle \phi_>^4(x_1)\, \phi_>^4(x_2) \rangle + 4^2 \langle \phi_>^3(x_1)\, \phi_>^3(x_2) \rangle\, \phi_<(x_1) \phi_<(x_2)$$

$$+ 6^2 \langle \phi_>^2(x_1)\, \phi_>^2(x_2) \rangle\, \phi_<^2(x_1) \phi_<^2(x_2) + 4^2 \langle \phi_>(x_1)\, \phi_>(x_2) \rangle\, \phi_<^3(x_1) \phi_<^3(x_2)$$

$$+ 2\times 6 \langle \phi_>^4(x_1)\, \phi_>^2(x_2) \rangle\, \phi_<^2(x_2) + 2\times 4\times 4 \langle \phi_>^3(x_1)\, \phi_>(x_2) \rangle\, \phi_<(x_1) \phi_<^3(x_2) \Big]$$

$$F = \ln I(x) \qquad I(0) = 1 \Rightarrow F(0) = 0$$

$$F'(x) = \frac{I'(x)}{I(x)} \Rightarrow \qquad F'(0) = I'(0) = I_1$$

$$F''(x) = \frac{I''(x)}{I(x)} - \left(\frac{I'}{I}\right)^2 \Rightarrow \quad F''(0) = I_2 - (I_1)^2$$

$$F(x) = I_1 x + \frac{1}{2!} \left(I_2 - (I_1)^2\right) x^2 + O(x^3)$$

Simple calculation:

$$I_2 - I_1^2 = \int dx_1 dx_2 \Big\{ \Big[\langle \phi_>^4(x_1) \phi_>^4(x_2) \rangle - \langle \phi_>^4(x_1) \rangle \langle \phi_>^4(x_2) \rangle \Big] +$$

$$+ 6^2 \Big[\langle \phi_>^2(x_1)\, \phi_>^2(x_2) \rangle - \langle \phi_>^2(x_1) \rangle \langle \phi_>^2(x_2) \rangle \Big] \phi_<^2(x_1) \phi_<^2(x_2)$$

$$+ 2\times 6 \Big[\langle \phi_>^4(x_1)\, \phi_>^2(x_2) \rangle - \langle \phi_>^4(x_1) \rangle \langle \phi_>^2(x_2) \rangle \Big] \phi_<^2(x_2)$$

$$+ 4^2 \langle \phi_>^3(x_1)\, \phi_>^3(x_2) \rangle\, \phi_<(x_1) \phi_<(x_2) + 4^2 \langle \phi_>(x_1) \phi_>(x_2) \rangle\, \phi_<^3(x_1)\, \phi_<^3(x_2)$$

$$+ 2\times 4\times 4 \langle \phi_>^3(x_1)\, \phi_>(x_2) \rangle \quad \phi_<(x_1)\, \phi_<^3(x_2) \Big\}$$

The effective action is $\qquad \left(\int \mathcal{D}\phi_> \, e^{-S_0(\phi_>)} = e^{-E_0} \right)$

$$S_{eff} = S^{b\Lambda}(\phi_<) = \int dx^d \left[\frac{1}{2}(\nabla\phi_<)^2 + \frac{t\Lambda^2}{2}\phi_<^2 + \frac{u\Lambda^\epsilon}{4!}\phi_<^4(x) \right] + E_0$$

$$- \left(-\frac{u\Lambda^\epsilon}{4!} \right) I_1 - \frac{1}{2!}\left(\frac{u\Lambda^\epsilon}{4!} \right)^2 I_2 + O\left(\left(\frac{u\Lambda^\epsilon}{4!} \right)^3 \right)$$

$$:A: = A - \langle A \rangle$$

$$\langle\langle AB \rangle\rangle = \langle AB \rangle - \langle A \rangle\langle B \rangle = \langle :A::B: \rangle$$

$$S_{b\Lambda}(\phi_<) = \int dx^d \left[\frac{1}{2}(\nabla\phi_<)^2 + \frac{t\Lambda^2}{2}\phi_<^2 + \frac{u\Lambda^\epsilon}{4!}\phi_<^4(x) \right] +$$

$$+ \Big[E_0 + \frac{u\Lambda^\epsilon}{4!} \int dx_1 \langle \phi_>^4(x_1) \rangle - \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!} \right)^2 \int dx_1 dx_2 \langle\langle \phi_>^4(x_1)\phi_>^4(x_2) \rangle\rangle$$

$$+ O(u^3) \Big]$$

$$+ \Big[+ \int dx_1 \; 6 \langle \phi_>^2(x_1) \rangle \phi_<^2(x_1) \left(\frac{u\Lambda^\epsilon}{4!} \right) -$$

$$- \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!} \right)^2 \int dx_1 dx_2 \Big\{ 6^2 \langle\langle \phi_>^2(x_1)\phi_>^2(x_2) \rangle\rangle \phi_<^2(x_1)\phi_<^2(x_2) +$$

$$+ 2\times 6 \langle\langle \phi_>^4(x_1)\phi_>^2(x_2) \rangle\rangle \phi_<^2(x_2)$$

$$+ 4^2 \langle \phi_>(x_1)\phi_>(x_2) \rangle \phi_<^3(x_1)\phi_<^3(x_2) + 4^2 \langle \phi_>^3(x_1)\phi_>^3(x_2) \rangle \phi_<(x_1)\phi_<(x_2) +$$

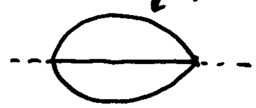
$$+ 2\times 4\times 4 \langle \phi_>^3(x_1)\phi_>(x_2) \rangle \phi_<(x_1)\phi_<^3(x_2) \Big\} \Big]$$

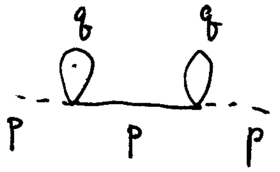
We'd like to rewrite this expression in terms of local operators such that $\phi^{n}(x)$, $\nabla_i \nabla_j \phi^n(x)$ etc.

We can do that provided that the kernels are sufficiently local.

Example $\int dx_1\, dx_2 \; \phi_<(x_1)\, \phi_<(x_2) \; \langle \phi_>^3(x_1)\, \phi_>^3(x_2) \rangle_0 \; 4^2$

p's on the momentum shell

$=$ 

$\Big[$ 

$\equiv 0 \Big]$

$$= 4^2 \times 2 \times 3 \int dx_1 \int dx_2 \; \phi_<(x_1)\, \phi_<(x_2) \; \left[G_>(x_1 - x_2) \right]^3$$

with $G_>(x_1 - x_2) = \int_{\underline{\underline{\text{shell}}}} \frac{dp^d}{(2\pi)^d} \frac{e^{i p \cdot x}}{p^2 + t\Lambda^2}$, Assume that $G_>(x)$ decays rapidly

$$\phi_<(x_2) = \phi_<(x_2 - x_1 + x_1) \cong$$
$$= \phi_<(x_1 + \ell) \cong \phi_<(x) + l_i \nabla_i \phi_<(x_1) +$$
$$+ \frac{1}{2} l_i l_j \nabla_i \nabla_j \phi_<(x_1) + \cdots$$

$$\text{Example} = 4^2 \times 2 \times 3 \int dx \; \phi_<(x) \Big[\phi_<(x) \int (G_>(\ell))^3 d\ell \; +$$
$$+ \int d\ell \; l_i \left[G_>(\ell) \right]^3 \; \nabla_i \phi_<(x) \; +$$
$$+ \frac{1}{2} \left(\int d\ell \; l_i l_j \left[G_>(\ell) \right]^3 \right) \nabla_i \nabla_j \phi_<(x) + \cdots \Big]$$

$$= 4^2 \times 2 \times 3 \int dx \left[\phi_<^2(x) \int_\ell [G_>(\ell)]^3 + \frac{1}{2}\left[\int_\ell \ell_i \ell_j (G_>(\ell))^3\right] \phi_<(x) \nabla_i \nabla_j \phi_<(x) + \cdots \right]$$

$$\equiv (4^2 \times 2 \times 3) \int dx \left\{ \left[\int_\ell (G_>(\ell))^3\right] \phi_<^2(x) - \left[\frac{1}{d} \int_\ell \ell^2 (G_>(\ell))^3\right] \frac{1}{2} (\nabla \phi_<(x))^2 + \cdots \right\}$$

Thus I get

$$S_{b\Lambda}(\phi_<) = \int d^dx \left[\frac{A}{2} (\nabla\phi_<)^2 + B\,\phi_<^2 + C\,\phi_<^4 \right] + \text{const.} + \text{higher operators.}$$

$$A = \left\{ 1 - \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!}\right)^2 (-1) \times (4^2 \times 2 \times 3) \left[\frac{1}{d} \int_\ell (G_>(\ell))^3 \ell^2\right] \right\} + O(u^3)$$

$$B = \left\{ \frac{t\Lambda^2}{2} + 6\,\frac{u\Lambda^\epsilon}{4!}\, G_>(0) - \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!}\right)^2 2 \times 6 \times 4 \times 3\; G_>(0) \int_\ell (G_>(\ell))^2 \right.$$

$$\left. - \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!}\right)^2 (4^2 \times 2 \times 3) \int_\ell (G_>(\ell))^3 + O(u^3) \right\}$$

$$C = \left\{ \frac{u\Lambda^\epsilon}{4!} - \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!}\right)^2 \left[6^2 \times 2 \times \int d\ell\, (G_>(\ell))^2 + 2 \times 4 \times 4 \times 3\; G_>(0) \int_\ell G_>(\ell) \right] \right.$$

$$G_>(\ell) = \int_{b\Lambda < |\vec{p}| < \Lambda} \frac{d^dp}{(2\pi)^d} \frac{e^{i\vec{p}\cdot\vec{\ell}}}{p^2 + t\Lambda^2}$$

$\boxed{L.13}$ I will only compute to leading order

$$\Rightarrow \quad A = 1 + O(u^2)$$

$$B = \frac{t\Lambda^2}{2} + 6\,\frac{u\Lambda^\epsilon}{4!}\,G_>(0) + O(u^2)$$

$$C = \frac{u\Lambda^\epsilon}{4!} - \frac{1}{2}\left(\frac{u\,\Lambda^\epsilon}{4!}\right)^2 36\times 2 \int d\ell\,[G_>(\ell)]^2 + O(u^3)$$

note $\int G_>(\ell)\, d^d\ell = \bullet = \int_{shell} \frac{d^dp}{(2\pi)^d}\,\frac{(2\pi)^d\,\delta(p)}{p^2+t\Lambda^2} = 0$

$$G_>(0) = \int_{shell} \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+t\Lambda^2} \simeq \frac{\Lambda^{d-2}}{1+t}\,\frac{S_d}{(2\pi)^d}\,(1-b) = \frac{\Lambda^{d-2}}{1+t}\,\frac{S_d}{(2\pi)^d}\,\delta\ell$$

$$\int d^d\ell\,[G_>(\ell)]^2 = \int_{shell} \frac{d^dp}{(2\pi)^d}\,\frac{1}{(p^2+t\Lambda^2)^2} \approx \frac{\Lambda^{d-4}}{(1+t)^2}\,\frac{S_d}{(2\pi)^d}\,(1-b) \simeq \frac{\Lambda^{d-4}}{(1+t)^2}\,\frac{S_d}{(2\pi)^d}\,\delta\ell$$

$$b = e^{-\delta\ell} \simeq 1 - \delta\ell$$

$$1 - b = \delta\ell$$

$$A = 1 + O(u^2)$$

$$B = \frac{\Lambda^2}{2}\left[t + \frac{u\cancel{\Lambda^\epsilon}}{2}\,\frac{\cancel{\Lambda^{-\epsilon}}}{1+t}\,\frac{S_d}{(2\pi)^d}\,\delta\ell + O(u^2)\right]$$

$$C = \frac{\Lambda^\epsilon}{4!}\left[u - \frac{3u^2}{2}\,\cancel{\Lambda^\epsilon}\,\frac{\cancel{\Lambda^{-\epsilon}}}{(1+t)^2}\,\frac{S_d}{(2\pi)^d}\,\delta\ell + O(u^3)\right]$$

$$A = 1 + O(u^2)$$

$$B = \frac{\Lambda^2}{2}\left[t + \frac{u}{2(1+t)}\frac{S_d}{(2\pi)^d}\,\delta l + O(u^2)\right] \simeq \frac{\Lambda^2}{2}\left[t + \frac{(u - ut)}{2}\frac{S_d}{(2\pi)^d}\,\delta l + \dots\right]$$

$$C = \frac{\Lambda^\epsilon}{4!}\left[u - \frac{3}{2}\frac{u^2}{(1+t)^2}\frac{S_d}{(2\pi)^d}\,\delta l + O(u^3)\right] \simeq \frac{\Lambda^\epsilon}{4!}\left[u - \frac{3}{2}u^2\frac{S_d}{(2\pi)^d}\,\delta l + \dots\right]$$

expand in t and keep terms $O(t)$, $O(u_*)$, $O(ut)$, $O(u^2)$

we'd like to write $S_{b\Lambda}(\phi_<) = S'(\phi')$

$$\sqrt{Z_\phi}\,\phi' = \phi_<$$

$$\vec{x}' = \vec{x}\, b$$

$$S'(\phi') = \int d^dx'\; b^{-d}\left[\frac{A}{2} Z_\phi b^2 (\nabla'\phi')^2 + B Z_\phi \phi'^2 + C Z_\phi^2 (\phi')^4 + \dots\right]$$

require:

$$b^{2-d} A Z_\phi = 1$$

$$b^{-d} B Z_\phi = \frac{t'\Lambda^2}{2}$$

$$b^{-d} C Z_\phi^2 = \frac{u'\Lambda^\epsilon}{4!}$$

$$\Rightarrow Z_\phi = b^{d-2}(1 + O(u^2)) \qquad (A=1)$$

$$t' = \left(\frac{2}{\Lambda^2}\right) b^{-2} B =$$

$$t' = b^{-2}\left[t + \frac{u}{2}\frac{S_d}{(2\pi)^d}\,\delta l - \frac{ut}{2}\frac{S_d}{(2\pi)^d}\,\delta l + \dots\right]$$

$$u' = b^{-\epsilon}\left[u - \frac{3}{2}u^2\frac{S_d}{(2\pi)^d}\,\delta l + \dots\right]$$

$$t' = b^{-2}\left(t + \frac{(u - ut)}{2}\frac{S_d}{(2\pi)^d}\,\delta l\right)$$

$$u' = b^{-\epsilon}\left(u - \frac{3}{2}u^2\frac{S_d}{(2\pi)^d} + \cdots\right)$$

Def. $v = u\dfrac{S_d}{(2\pi)^d}$

$$\begin{cases} t' = b^{-2}\left(t + \frac{1}{2}(v - vt)\,\delta l + \cdots\right) \\ v' = b^{-\epsilon}\left(v - \frac{3}{2}v^2\delta l + \cdots\right) \end{cases}$$

$$b^{-\nu} = e^{\nu\delta l} \simeq 1 + \nu\delta l$$

$$\frac{t' - t}{\delta l} = \frac{dt}{dl} = \beta_t = 2t + \frac{1}{2}(v - vt)$$

$$\frac{v' - v}{\delta l} = \frac{dv}{dl} = \beta_v = \epsilon v - \frac{3}{2}v^2$$

$$\Rightarrow \begin{cases} \beta_t = 2t + \frac{1}{2}(v - vt) \\ \beta_v = \epsilon v - \frac{3}{2}v^2 \end{cases}$$

F.P. = $v^* = 0$ (Gaussian)

$v^* = \frac{2}{3}\epsilon > 0$ ($d<4$)

$\delta l = -\ln b$

$\dfrac{ds}{dl} = -\dfrac{ds}{\ln b}$

$b = \dfrac{\Lambda'}{\Lambda} = \dfrac{a}{a'}$

$a' = a - \delta a$

$\dfrac{a'}{a} = 1 - \dfrac{\delta a}{a}$

$\ln\dfrac{a'}{a} = -\dfrac{\delta a}{a}$

$$\boxed{\frac{ds}{dl} = +a\frac{ds}{da}}$$

Linearize around the F.P.

$$v = v^* + \bar{v}$$

$$\begin{cases} \beta_t = \left(2 - \frac{v^*}{2}\right) t + \frac{v^*}{2} + \frac{\bar{v}}{2} - \frac{\bar{v} t}{2} \\ \beta_{\bar{v}} = -\epsilon \bar{v} \end{cases}$$

$$t = \bar{t} + c$$

$$c = -\frac{\epsilon}{6}$$

$$\begin{cases} \beta_{\bar{t}} = \left(2 - \frac{\epsilon}{3}\right) \bar{t} + \frac{\bar{v}}{2}\left(1 + \frac{\epsilon}{6}\right) - \cancel{\frac{\bar{v}\bar{t}}{2}} \\ \beta_{\bar{v}} = -\epsilon \bar{v} + \dots \end{cases}$$

$$y = \bar{v}$$

$$x = a\bar{t} + b\bar{v}$$

$$\frac{dx}{dl} = a\left(2 - \frac{\epsilon}{3}\right)\bar{t} + a\frac{\bar{v}}{2}\left(1 + \frac{\epsilon}{6}\right) - b\epsilon\bar{v}$$

$$\cancel{\frac{dx}{dl} = \left[\frac{a}{2}\left(1 + \frac{\epsilon}{2}\right) - b\epsilon\right] y}$$

$$\frac{dx}{dl} = \left(2 - \frac{\epsilon}{3}\right)(x - by) + \left(\frac{a}{2}\left(1 + \frac{\epsilon}{6}\right) - b\epsilon\right) y$$

$$\frac{dx}{dl} = \left(2 - \frac{\epsilon}{3}\right) x + y\left[\frac{a}{2}\left(1 + \frac{\epsilon}{6}\right) - b\epsilon - b\left(2 - \frac{\epsilon}{3}\right)\right].$$

require $\frac{a}{2}\left(1 + \frac{\epsilon}{6}\right) = b\left(2 + \frac{\epsilon}{3}\right)$

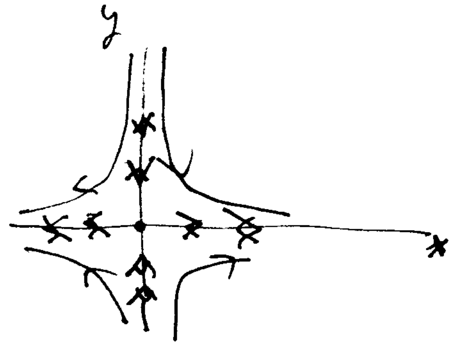
$$\Rightarrow \begin{cases} \dfrac{dx}{dl} = \left(2 - \dfrac{\epsilon}{3}\right) x \\ \dfrac{dy}{dl} = -\epsilon y \end{cases}$$

$$\Rightarrow \quad a = 4b\left(1 + \frac{\epsilon}{6}\right)$$

$$\Rightarrow a = 4 \quad b = 1 - \frac{\epsilon}{6} + \dots$$

$$\begin{cases} x = 4\bar{t} + \left(1 - \frac{\epsilon}{6}\right)\bar{v} \\ y = \bar{v} \end{cases} \qquad \text{(scaling fields)}$$

Thus x is relevant and y is irrelevant



relevant ↓ irrelevant ↓

Note the change in eigenvalues $\quad y_x = 2 - \frac{\epsilon}{3}$, $y_y = -\epsilon$

$$\Rightarrow \nu = \frac{1}{y_x} = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)$$

Gaussian: $v^* = 0$

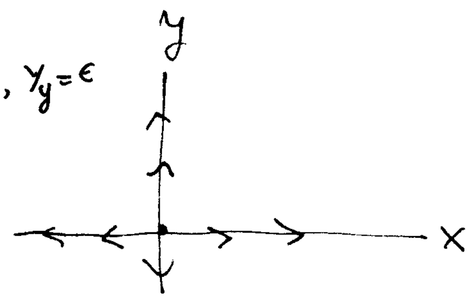
$$\frac{dt}{dl} = 2t + \frac{v}{2} \qquad \frac{dv}{dl} = \epsilon v$$

$$x = at + bv \qquad a = 2(2+\epsilon), \; b = 1$$

$$y = v$$

$$\frac{dx}{dl} = 2x \qquad y_x = 2, \; y_y = \epsilon$$

$$\frac{dy}{dl} = \epsilon y$$



unstable

Let's consider the behavior of the correlation length ξ.

$$\xi = \frac{1}{\Lambda} f(t, u) \qquad \text{(by dim. analysis)}$$

ξ is a physical scale $\Rightarrow$ is invariant

$$\frac{\partial \xi}{\partial \Lambda} = 0 = -\frac{1}{\Lambda^2} f + \frac{1}{\Lambda} \frac{\partial f}{\partial \Lambda}$$

$$\frac{\partial f}{\partial \Lambda} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial \Lambda} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial \Lambda}$$

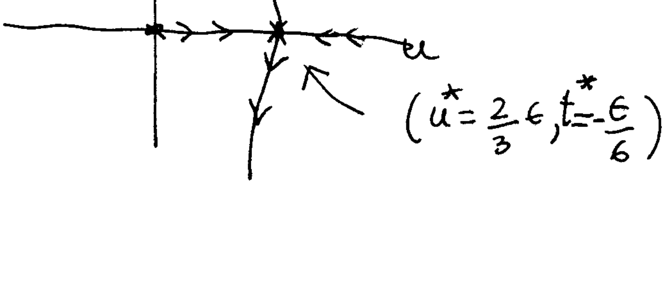
$$\beta_u = \frac{\partial u}{\partial l} = -\Lambda \frac{\partial u}{\partial \Lambda} = a_0 \frac{\partial u}{\partial a_0}$$

$$\beta_t = \frac{\partial t}{\partial l} = -\Lambda \frac{\partial t}{\partial \Lambda}$$

$$\Rightarrow \boxed{0 = f + \frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial t} \beta_t}$$

f obeys a P. diff. eqn (Callan-Symanzik)

Let's solve it for the trajectory which is connected to the non-trivial f.p.



Close to the F.P. we can still linearize the flow. and work with the scaling fields x, y

$$x = 4\bar{t} + \left(1 - \frac{\epsilon}{6}\right)\bar{v} \qquad \bar{t} = t + \frac{\epsilon}{6}$$

$$y = \bar{v} \qquad \bar{v} = v - \underset{\text{F.P.}}{\underset{\uparrow}{v^*}}$$

$$\frac{S_d}{(2\pi)^d} u = v$$

$$0 = f + \frac{\partial f}{\partial x}\beta_x + \frac{\partial f}{\partial y}\beta_y$$

$$\beta_x \approx \left(2 - \frac{\epsilon}{3}\right)x$$

$$\beta_y \approx -\epsilon y$$

at $y^* = 0 \Rightarrow \beta_y = 0$

$$0 = f + \frac{\partial f}{\partial x}\beta_x$$

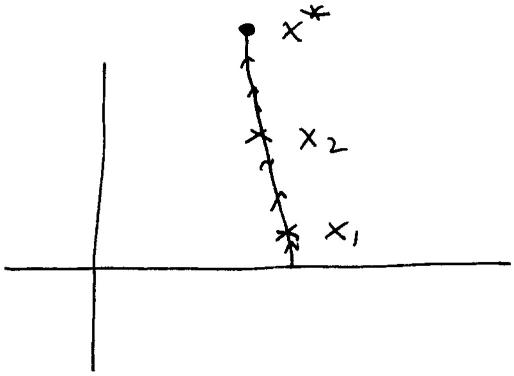
$$0 = 1 + \frac{\partial \ln f}{\partial x}\beta_x$$

$$\frac{\partial \ln f}{\partial x} = -\frac{1}{\beta_x}$$

$$\ln f = \text{const} - \int \frac{dx}{\beta_x}$$

$$\ln \frac{f(x)}{f(x_0)} = \cancel{\text{const}} - \int_{x_0}^{x} \frac{dx'}{\beta(x')}$$

$$f(x) = f(x_0)\, e^{-\int_{x_0}^{x} \frac{dx'}{\beta(x')}}$$



Consider two different starting points at t_1 and t_2 (or x_1 and x_2) and iterate all the way to same t^* (or x^*) well into the high temperature phase (with the same initial cutoff Λ_0)

$$\xi(x_1) = \frac{1}{\Lambda_0} f(x_1)$$

$$\xi(x_2) = \frac{1}{\Lambda_0} f(x_2)$$

$$f(x_1) = f(x^*)\, e^{+\int_{x_1}^{x^*} \frac{dx}{\beta(x)}}$$

$$f(x_2) = f(x^*)\, e^{+\int_{x_2}^{x^*} \frac{dx}{\beta(x)}}$$

$$\frac{\xi(x_1)}{\xi(x_2)} = \frac{\frac{1}{\cancel{\Lambda_0}} f(x_1)}{\frac{1}{\cancel{\Lambda_0}} f(x_2)} = \frac{\cancel{f(x^*)}\, e^{+\int_{x_1}^{x^*} \frac{dx}{\beta(x)}}}{\cancel{f(x^*)}\, e^{+\int_{x_2}^{x^*} \frac{dx}{\beta(x)}}}$$

$$= e^{+\int_{x_1}^{x^*} + \int_{x^*}^{x_2}} = e^{+\int_{x_1}^{x_2} \frac{dx}{\beta(x)}}$$

$$\xi(x_1) = \xi(x_2)\, e^{+\int_{x_1}^{x_2} \frac{dx}{\beta(x)}}$$

Suppose that x_2 is well in the high $\underbrace{\text{temp.}}_{}$ (symmetric) phase

(i.e. $\xi(x_2) \sim a_0$) and x_2 is close to the F.P.

$$\xi(x_1) \sim a_0 \, e^{+\int_{x_1}^{x_2} \frac{dx}{\beta(x)}} = a_0 \, e^{+\int_{x_1}^{x_2} \frac{dx}{(2-\frac{\epsilon}{3})x}}$$

$$\xi(x_1) \sim a_0 \, e^{+\frac{1}{(2-\frac{\epsilon}{3})} \ln \frac{x_2}{x_1}}$$

$$\xi(x_1) \sim a_0 \, e^{-\frac{1}{2-\frac{\epsilon}{3}} \ln \frac{x_1}{x_2}}$$

$$\xi(x_1) \sim a_0 \left| \frac{x_1}{x_2} \right|^{-\nu}$$

$$\nu = \frac{1}{2-\frac{\epsilon}{3}}$$ and $2 - \frac{\epsilon}{3} = \beta'(x)$ at the F.P

$\uparrow$
eigenvalue!

$$y = \bar{v} = 0$$

$$x = 4\bar{t} = 4(t - t^*) \qquad t^* = -\frac{\epsilon}{6}$$

$$\xi(t) \sim \frac{a_0}{|t-t^*|^{\nu}}$$

$\nu = \frac{1}{2}$ is the classical value ($m^2 = T - T_0 \Rightarrow \xi \sim \frac{1}{\sqrt{T-T_0}}$)

$$\nu = \frac{1}{2-\frac{\epsilon}{3}} = \frac{1}{2(1-\frac{\epsilon}{6})} = \frac{1}{2}\left[1 + \frac{\epsilon}{6} + \cdots\right]$$