Thus at a fixed point we expect the system to be **scale invariant** \(\Rightarrow\) All correlation functions should depend on distances in such a way that the functions should retain their form under a rescaling of all coordinates: 

\[
\vec{r}_i \rightarrow 2\vec{r}_i
\]

Hence, the correlation functions must be homogeneous functions of the coordinates. \(\ast\)

E.g. an operator \(A(\vec{r})\) should have a corr. function

\[
\langle A(\vec{r}) \, A(\vec{r}') \rangle^* = \text{const} \frac{X_A}{|\vec{r} - \vec{r}'|^{2\Delta_A}}
\]

\(X_A\) is an exponent.

By dimensional analysis we see that the units or dimension

\[
\Delta_A \text{ of the op. } A(\vec{r}) \text{ at } \mu^* \text{ are}
\]

\[
[A] = \mu^{2\Delta_A}
\]

With \(\Delta_A = \frac{X_A}{2}\) \(\Leftrightarrow X_A = 2\Delta_A\)

Notice that for a free field \(\ast\) (or \(\Delta_A\)) are

simply \(\ast\) fractions. In general they are not so simple!

\(\ast\) \(f(x)\) is homogeneous if \(f(\lambda x) = \lambda^k f(x)\)

\(k\): "degree" \(\Rightarrow\) i.e. \(f(x)\) transforms simply ("irreducibly") under dilations
Simple examples of fixed points: Free massless fields

1. Consider a free massless scalar field in \( D \) dimensions

\[
\mathcal{L} = \frac{1}{2} (\partial \mu \phi)^2
\]

\[ [\mathcal{L}] = L^{-D} \Rightarrow [\phi] = L^{-\left( D-2 \right)/2} \]

\[ \Rightarrow \Delta \phi = \frac{D-2}{2} \quad \text{(scaling dimension)} \]

\[ \Rightarrow \langle \phi(x) \phi(y) \rangle \sim \frac{1}{|x-y|^{2\Delta \phi}} \]

\[ \Rightarrow 2\Delta \phi = D-2 \]

Other operators:

1. \( \phi^n \), \( \Delta_n = \Delta [\phi^n] = L^{-n\Delta \phi} \)

\[ \Rightarrow \Delta_n = n \Delta \phi \]

\[ \Rightarrow \langle \phi^n(x) \phi^n(y) \rangle \sim \frac{1}{|x-y|^{2\Delta_n}} \]

\[ 2\Delta_n = n(D-2) \]

(Here \( A = A - \langle A \rangle \))

which is correct since for a free field

\[ \langle \phi^n(x) \phi^n(y) \rangle = \# \langle \phi(x) \phi(y) \rangle^n \]
(c) Gauge Fields:

\[ L = \frac{1}{4} F_{\mu \nu}^2 \]

\[ [L] = L^{-D} \quad \text{and} \quad F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + g [A, A] \]

\[ \Rightarrow [A] = L^{-\frac{(D-2)}{2}} \]

\[ A_A = \frac{D-2}{2} \]

\[ \text{and} \quad S = \int d^D x \ L \quad \text{(formally)} \]

We will now see that to assess the stability of these fixed points under perturbative fluctuations, they play a crucial role and that in many cases we will find that these simple fixed points are not stable.
Hence under a scale transformation

\[ x \to \lambda x \]

\[ \phi(\lambda x) \to \lambda^{-\Delta} \phi(x) \]

\[ S(\phi(\lambda x)) = \int dx^D \lambda^D \lambda^{-2} \lambda^{-2\Delta} \phi^2 \left( \frac{\partial \phi}{\partial x} \right)^2 \]

but \( \lambda^{(D-2)-2\Delta} = 1 \)

\[ \Rightarrow S \text{ is invariant.} \]

\( \square \) Free massless Dirac fermion.

\[ S = \int dx^D \bar{\psi} \gamma^\mu \partial_\mu \psi \]

Note: the \( \gamma \)-matrices change with \( D \).

\[ \Rightarrow \{ \psi \} = \lambda^{- (D-1)/2} \]

\[ \Rightarrow \Delta \psi = \frac{D-1}{2} \]

and \( \langle \bar{\psi}(x) \psi(y) \rangle \sim \frac{1}{|x-y|^{2\Delta}} \)

\[ 2\Delta = \frac{\chi(D-1)}{2} \]

Likewise \( \Delta(\bar{\psi}\psi)^n = 2n \Delta \psi = 2n \frac{D-1}{2} \)

\[ \Rightarrow \text{under } x \to \lambda x \text{, } \psi(\lambda x) \to \lambda^\Delta \psi(x) \]

\( S(\psi) \) is invariant.
At a critical point the system exhibits scaling. This is so because the correlation length is divergent (i.e. 
\[ L \gg 1, L \gg 100 \] )

The two point functions satisfy

\[ \langle A(0) B(\vec{r}) \rangle = \frac{1}{|\vec{r}|^{2 \Delta_A}} \text{ where } \Delta_A \text{ is the scaling dimension of } A \text{ which is normalized such that the coefficient is one.} \]

Let \( A, B, C \) be three operators of dimensions \( \Delta_A, \Delta_B, \) and \( \Delta_C \). We will show later that scale invariance (plus rotational and translational invariance) implies the scaling form for the three point function.

\[ \langle A(\vec{r}_1) B(\vec{r}_2) C(\vec{r}_3) \rangle = \frac{K_{ABC}}{|\vec{r}_1 - \vec{r}_2|^{\Delta_{12}} |\vec{r}_2 - \vec{r}_3|^{\Delta_{13}} |\vec{r}_1 - \vec{r}_3|^{\Delta_{23}}} \]

where \( K_{ABC} \) is a universal constant and

\[ \begin{align*}
\Delta_{12} &= \Delta_A + \Delta_B - \Delta_C \\
\Delta_{13} &= \Delta_A + \Delta_C - \Delta_B \\
\Delta_{23} &= \Delta_B + \Delta_C - \Delta_A
\end{align*} \]
Operator Product Expansion (Wilson - Kadanoff - Polyakov) ~1969

\[ \{ O_m(x) \} \text{ is a "complete" set of operators} \]

\[ \lim_{x' \to x} O_m(x) O_{m'}(x') = \lim_{x' \to x} \sum_{k} \frac{C_{m n k}}{|x-x'| \Delta_m \Delta_n \Delta_k} O_{k}(\frac{1}{2}(x+x')) \]

\( \text{dimensional analysis.} \)

This is an identity in the weak sense, i.e. when inserted in an arbitrary exp. value. (explain)

What are the coeff. \( C_{m n k} \)?

\[ \lim_{x' \to x} \langle O_m(x') O_n(x') O_p(z) \rangle = \lim_{x' \to x} \sum_{k} \frac{C_{m n k}}{|x-x'| \Delta_m \Delta_n \Delta_k} \langle O_k(\frac{1}{2}(x+x')) O_p(z) \rangle \]

The operators \( \{ O_m(x) \} \) are \( \Gamma \) "independent" \( \Gamma \) (we'll see what we mean below)

\[ \langle O_p(z) O_n(x) \rangle = \frac{\delta_{p n}}{|x-z|^{2\Delta_n}} \]

(we will prove this statement later)

\[ \lim_{y \to x} \langle O_m(x) O_n(y) O_p(z) \rangle = C_{m n p} \lim_{y \to x} \frac{\langle O_p(z) O_n(y) \rangle}{|y-x|^{2\Delta_n - 2\Delta_p}} \]

\[ C_{m n p} = \lim_{y \to x} \left[ \frac{\langle O_m(x) O_n(y) O_p(z) \rangle}{|y-x|^{2\Delta_p}} \right] |y-x|^{\Delta_m + \Delta_n - \Delta_p} \]

\[ \langle O_m(x) O_n(y) O_p(z) \rangle = \frac{K_{m n p}}{|y-z|^{2\Delta_p + \Delta_n} |x-z|^{\Delta_m + \Delta_n - \Delta_p} |z-x|^{\Delta + \Delta - \Delta}} \]
Perturbing about a Fixed Point

Let $S^*$ be a fixed point (i.e., a scale invariant action) and $\delta S$ be a set (complete) of perturbations $\Rightarrow$

$$S = S^* + \delta S = S^* + \sum_i \int \frac{d^d x}{a^d} g_i \cdot a^{\phi_i}(x)$$

$\phi_i(x)$ are a complete set of operators (in the sense that the OPE closes).

$$Z = \text{tr} \ e^{-S}[\phi]$$

$$= \text{tr} \ e^{-S^*[\phi]} \ e^{-\sum_i \int \frac{d^d x}{a^d} g_i \cdot a^{\phi_i}(x)}$$

$$= \text{tr} \ e^{-S^*[\phi]} \left\{ 1 - \sum_i \int \frac{d^d x}{a^d} g_i \cdot a^{\phi_i}(x) + \right.$$  

$$+ \frac{1}{2!} \sum_{ij} \int \frac{d^d x}{a^d} \frac{d^d y}{a^d} \ g_i \cdot g_j \cdot a^{\phi_i(y)} \phi_j(x) +$$  

$$- \frac{1}{3!} \sum_{ijk} \int \frac{d^d x}{a^d} \frac{d^d y}{a^d} \frac{d^d z}{a^d} \ g_i \cdot g_j \cdot g_k \cdot a^{\phi_i(y_1)} \phi_j(y_2) \phi_k(y_3) +$$  

$$+ \cdots \right\}$$
\[ Z^* = \tau e^{-S^*} \]

\[ \Rightarrow Z = Z^* \left[ 1 - \int \frac{d^d x}{a^{d-\delta}} \langle \phi_c(\mathbf{r}) \rangle_{S^*} \right. \]

\[ + \frac{1}{2} \sum_{i,j} \int \frac{d^d x_i \cdot d^d x_j}{a^{2d-\delta_i-\delta_j}} \langle \phi_i(\mathbf{x_i}) \phi_j(\mathbf{x_j}) \rangle_{S^*} \]

\[ - \frac{1}{3!} \sum_{i,j,k} \int \frac{d^d x_i \cdot d^d x_j \cdot d^d x_k}{a^{3d-\delta_i-\delta_j-\delta_k}} \langle \phi_i(\mathbf{x_i}) \phi_j(\mathbf{x_j}) \phi_k(\mathbf{x_k}) \rangle_{S^*} \]

\[ + \ldots \}

Since \( S^* \) is scale invariant \( \Rightarrow \) the integrals over the correlators will have IR divergences \( \Rightarrow \) we need a finite size \( L \) to cut off the IR behaviour.

But, precisely because \( S^* \) is scale invariant \( \Rightarrow \) it will turn out that we will need essentially the short distance singular behaviour, which is given by the OPE.

This expansion looks like the slow density expansion of a gas in the grand canonical ensemble with the particles labelled by the coordinates \( \{ \mathbf{x}_i \} \) and the index \( i \) representing different "species" of particles.
Each space has a "fugacity" $g_i$. As a short distance cutoff, we will stop the integrals at some distance $a$ between the particles in real space $\Rightarrow |\vec{r}_i - \vec{r}_j| > a$.

Renormalization procedure: We will attempt to change the short distance cutoff by some amount, and compute the change of the action that compensates the change by requiring that $\mathcal{Z}$ is fixed.

1. We rescale the cutoff $a \rightarrow b \cdot a$, $b > 1$
   $b = 1 + \epsilon \ell$ (we increase the short distance cutoff)

How do we change the couplings $g_i g_j / \mathcal{Z}$ is invariant? The cutoff $a$ appears in three places:

- In divisors: $a^{\Delta_i}$, where $\Delta_i$ is dimension of the operator
- As a cutoff of the integrals
- Through their dependence on $L$, $1/a$. 


If $\delta l \ll 1 \Rightarrow$ we need only linear changes:

$$a \rightarrow b = (1 + \delta l) a$$

$$\Rightarrow \quad \frac{g_i}{a^{d-\delta l}} \rightarrow \frac{g_i}{a^{d-\delta l} b^{d-\delta l}}$$

$$\Rightarrow \quad g_i \rightarrow g_i b^{d-\delta l} = g_i (1 + \delta l)^{d-\delta l}$$

$$\Rightarrow \quad g_i \rightarrow g_i (1 + (d-\delta l) \delta l)$$

(1) \quad \Rightarrow \quad g_i \rightarrow g_i + (d-\delta l) g_i \delta l + \ldots$$

(2) Changing the shift distance cutoff in the integrals:

$$\int_{|\vec{r}_1 - \vec{r}_2| > a(1 + \delta l)} \rightarrow \int_{|\vec{r}_1 - \vec{r}_2| > a(1 + \delta l)} \cap \int_{|\vec{r}_1 - \vec{r}_2| > a}$$

$\delta l \ll 1$, change
\[ \frac{1}{2} \sum_{ij} \sum_{k} c_{ijk} a^{x_k - x_i - x_j} \int d^d r \langle \phi_k(r) \sigma_{l1} \sigma_{l2} \rangle \frac{d^d r_1 d^d r_2}{\alpha^{d-2|x_i - x_j|}} \]

\[ \approx \frac{1}{2} \sum_{ij} \sum_{k} c_{ijk} a^{x_k - x_i - x_j} \int d^d r \langle \phi_k(r) \rangle \frac{S_d a^d d^2 \delta l}{\alpha^{d-2|x_i - x_j|}} \]

\[ S_d = \frac{2 \pi^{d/2}}{\Gamma(d/2)} \text{ area of hypersphere in } d \text{ dimensions} \]

\[ S_d a^d \delta l: \text{ volume of the shell} \]

\[ = \frac{1}{2} \sum_{ij} \sum_{k} c_{ijk} \frac{2 \pi}{\alpha^{d-2|x_k|}} \int d^d r \langle \phi_k(r) \rangle \ S_d \delta l \]

\[ \Rightarrow g_k \rightarrow g_k - \frac{1}{2} S_d \sum_{ij} g_i g_j c_{ijk} \delta l \]

In fact because of the structure of the OPE, similar changes happen to all orders in the expansion.

3) A dependence due to L dependence:

But we should not change L since otherwise we would have not changed anything.

\[ \Rightarrow \text{this dependence is trivial.} \]
\[ \frac{dg_k}{dl} = (a - x_k) g_k - \frac{1}{2} S_{ij} \sum c_{ijk} g_i g_j + \cdots \]

\[ \Rightarrow \text{RG eigenvalues at the trivial } (S^k) \text{ fixed point } g_k = a - x_k \]

Redefine the coupling constants:

\[ g_k \to \frac{2}{S_{ij}} g_k \]

\[ \frac{dg_k}{dl} = \sum g_k \sum_{ij} c_{ijk} g_i g_j + \cdots \]

is the RG equation.

To get the h.o.t. we must use renormalized pert. theory (such as dimensional reg.)
Examples of OPE's

\[ \phi^n \text{ theory} \]

Define \( \phi_n = : \phi^n : \)

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 + t a^{-2} \phi^2 + h a^{-\frac{d}{2} - 1} \phi \]

\[ + \ u a^{d-4} \phi^4 + \ldots \]

\[ : \phi^2 : = \phi^2 - <\phi^2> \]

\[ : \phi^4 : = \phi^4 - 3 <\phi^2> \phi^2 \]

\[ + \ldots \]

\( \Rightarrow \) \( \phi_1 \cdot \phi_1 = : \phi_1 \phi_1 : = \frac{1}{r_1^{d-2}} + : \phi^2 (r_1 r_2) : + \ldots \)

\[ \Rightarrow \]

\[
\begin{cases}
\phi_1 \cdot \phi_1 = 1 + \phi_2 \\
\phi_1 \cdot \phi_2 = 2 \phi_1 + \phi_3
\end{cases}
\]

\[ : \phi^2 (r_1) : : : \phi^2 (r_2) : = \frac{2}{r_1^{2d-4}} + \frac{4}{r_2^{d-2}} \phi^2 + : \phi^4 : \]

\( \Rightarrow \)

\[
\begin{cases}
\phi_2 \cdot \phi_2 = 2 + 4 \phi_2 + \phi_4 \\
\phi_1 \cdot \phi_4 = 4 \phi_3 + \ldots \\
\phi_2 \cdot \phi_4 = 12 \phi_2 + 8 \phi_4 \\
\phi_4 \cdot \phi_4 = 24 + 96 \phi_2 + 72 \phi_4 + \ldots
\end{cases}
\]
\[ g_1 = h, \quad g_2 = t, \quad g_3 = 0, \quad g_4 = u \]

\[
\begin{align*}
\frac{dh}{dl} &= \left(\frac{d^2}{2} + 1\right) h - 2.2ht + \cdots \\
\frac{dt}{dl} &= 2t - h^2 - 4t^2 - 2.12tu - 96u^2 + \cdots \\
\frac{du}{dl} &= t u - t^2 - 2.8tu - 72u^2 + \cdots
\end{align*}
\]

\[ \varepsilon = 4 - d \]

\[ h = 0 \Rightarrow \begin{cases} 
\frac{dt}{dl} &= 2t - 4t^2 - 24tu - 96u^2 \\
\frac{du}{dl} &= t u - t^2 - 16tu - 72u^2 
\end{cases} \]

Fixed Points:

1. **Trivial:** \( u^* = t^* = 0 \)
2. **Willem-Friske** (\( \varepsilon > 0 \))

\[ h^* = 0, \quad u^* = \frac{\varepsilon}{72} + O(\varepsilon^2) \]

\[ t^* = O(\varepsilon^2) \]

\[ \Rightarrow \quad \left. \frac{dt}{dl} \right|_{WF} = 2t - 24u^*t = \left(2 - \frac{41}{72} \varepsilon\right)t + \cdots \]

\[ \Rightarrow \quad \left. \frac{du}{dl} \right|_{WF} = 2 - \frac{\varepsilon}{3} + O(\varepsilon^2) \]

\[ \Rightarrow \quad u = \frac{1}{6t} = \frac{1}{2 - \varepsilon/3} \approx \frac{1}{2} \left(1 + \frac{\varepsilon}{6} + \cdots\right) = \frac{1}{2} + \frac{\varepsilon}{12} + \cdots \]