Thus at a fixed point we expect the system to be

Scale invariant $\Rightarrow$ All correlation functions should depend on distances in such a way that the functions should retain their form under a rescaling of all coordinates

$$\vec{r}_i \to \lambda \vec{r}_i$$

Hence, the correlation ~~function~~ functions must be homogeneous functions of the coordinates. (*)

e.g. an operator $A(\vec{r})$ should have a corr. function

$$\langle A(\vec{r})\, A(\vec{r}\,')\rangle^* = \frac{\text{const}}{|\vec{r}-\vec{r}\,'|^{x_A}}$$

x_A is an exponent.

By dimensional analysis we see that the units or dimensions Δ_A of the op. $A(\vec{r})$ at H^* are

$$[A] = L^{-\Delta_A}$$

with $\Delta_A = \frac{x_A}{2} \iff x_A = 2\Delta_A$

Notice that for a free field x_A (or Δ_A) are simple ~~...~~ fractions. In general they are not so simple!

(*) $f(x)$ is homogeneous if $f(\lambda x) = \lambda^k f(x)$

k: "degree" ; i.e. $f(x)$ transforms simply ("irreducibly") under dilations

Simple examples of fixed points: Free massless fields

(A) Consider a free massless scalar field in D dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2$$

$$[\mathcal{L}] = L^{-D} \Rightarrow \qquad [\phi] = L^{-(D-2)/2}$$

$$\Rightarrow \Delta_\phi = \frac{D-2}{2} \qquad \text{(scaling dimension)}$$

$$\Rightarrow \langle \phi(x)\, \phi(y) \rangle \sim \frac{1}{|x-y|^{2\Delta_\phi}}$$

$$\Rightarrow 2\Delta_\phi = D-2 \quad \checkmark$$

Other operators:

(1) ϕ^n, $\quad \Delta_n = \Delta[\phi^n] = L^{-n\Delta_\phi}$

$$\Rightarrow \Delta_n = n\Delta_\phi$$

$$\Rightarrow \langle :\phi^n(x):\ :\phi^n(y): \rangle \sim \frac{1}{|x-y|^{2\Delta_n}}$$

$$2\Delta_n = n(D-2)$$

(here $:A: = A - \langle A \rangle$)

which is correct since for a free field

$$\langle :\phi^n(x):\ :\phi^n(y): \rangle = \# \langle \phi(x)\, \phi(y) \rangle^n$$

(c) Gauge Fields :

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2$$

$$[\mathcal{L}] = L^{-D} \qquad \text{and} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A,A]$$

$$\Rightarrow \quad [A] = L^{-(D-2)/2}$$

$$\Delta_A = \frac{D-2}{2}$$

and $S = \int d^Dx \, \mathcal{L}$ is (formally) scale invariant

We will now see that to assess the stability of these fixed points under perturbations fluctuations play a crucial ~~role~~ role and that in many cases we will find that these simple fixed points are not stable.

Hence under a scale transformation

$x \to \lambda x$

$\phi(\lambda x) \to \lambda^{-\Delta\phi}\, \phi(x)$

$\Rightarrow \quad S[\phi(\lambda x)] = \int d^D x \; \lambda^D \; \lambda^{-2} \; \lambda^{-2\Delta\phi} \; \frac{1}{2}(\partial_\mu \phi)^2$

but $\quad \lambda^{(D-2)-2\Delta\phi} = 1$

$\Rightarrow$ S is invariant.

(B) Free massless Dirac fermions.

$S = \int d^D x \; \bar{\psi}\, i \gamma^\mu \partial_\mu \psi$

note: the γ-matrices change with D.

$\Rightarrow \quad [\psi] = L^{-(D-1)/2}$

$\Rightarrow \quad \Delta_\psi = \frac{D-1}{2}$

and $\langle \bar{\psi}(x) \; \psi(y) \rangle \sim \frac{1}{|x-y|^{2\Delta_\psi}}$

$2\Delta_\psi = \cancel{2}\left(\frac{D-1}{\cancel{2}}\right) \quad \checkmark$

Likewise $\quad \Delta((\bar{\psi}\psi)^n) = 2n\,\Delta_\psi = \cancel{2}n \frac{(D-1)}{\cancel{2}}$

$\Rightarrow$ under $\quad x \to \lambda x \quad , \psi(\lambda x) \to \lambda^{-\Delta_\psi}\, \psi(x)$

$S(\psi)$ is <u>invariant</u>

At a critical point the system exhibits <u>scaling</u>. This is so because the correlation length ξ <u>diverges</u> (i.e $L \gtrsim \xi \gg a_0$)

The two point functions satisfy

$$\langle A(0)\, A(\vec{r})\rangle = \frac{1}{|\vec{r}|^{2\Delta_A}}$$ where Δ_A is the scaling dimension of A which is normalized such that the coefficient is one.

Let A, B, C be three operators of dimensions Δ_A, Δ_B and Δ_C. We will show later that scale invariance (plus rotational and translational invariance) implies the scaling form for the three point function.

$$\langle A(r_1)\, B(r_2)\, C(r_3)\rangle = \frac{K_{ABC}}{|\vec{r}_1-\vec{r}_2|^{\Delta_{12}}\, |\vec{r}_1-\vec{r}_3|^{\Delta_{13}}\, |\vec{r}_2-\vec{r}_3|^{\Delta_{23}}}$$

where K_{ABC} is a universal constant and

$$\Delta_{12} = \Delta_A + \Delta_B - \Delta_C$$

$$\Delta_{13} = \Delta_A + \Delta_C - \Delta_B$$

$$\Delta_{23} = \Delta_B + \Delta_C - \Delta_A$$

Operator Product Expansion (Wilson - Kadanoff - Polyakov)
~1969

$\{ O_m(x) \}$ is a "complete" set of operators

$$\Rightarrow \lim_{x'\to x} O_m(x)\, O_{m'}(x') = \lim_{x'\to x} \sum_k \frac{C_{mm'k}}{|x-x'|^{\Delta_m+\Delta_n-\Delta_k}} O_k(\tfrac{1}{2}(x+x'))$$

dimensional analysis.

This is an identity in the weak sense, i.e. when inserted in an arbitrary exp. value. (explain)

what are the coeff. C_{mnk} ?

$$\lim_{x'\to x} \langle O_m(x)\, O_n(x')\, O_p(z) \rangle = \lim_{x'\to x} \sum_k \frac{C_{mnk}}{|x-x'|^{\Delta_m+\Delta_n-\Delta_k}} \langle O_k(\tfrac{1}{2}(x+x'))\, O_p(z)\rangle$$

The operators $\{ O_m(x) \}$ are "independent" (we'll see what we mean below)

$$\langle O_p(z)\, O_n(x) \rangle = \frac{\delta_{pn}}{|x-z|^{2\Delta_n}}$$

(we will prove this statement later)

$$\Rightarrow \lim_{y\to x} \langle O_m(x)\, O_n(y)\, O_p(z) \rangle = C_{mnp} \lim_{y\to x} \frac{\langle O_p(x)\, O_p(z)\rangle}{|y-x|^{\Delta_m+\Delta_n-\Delta_p}}$$

$$C_{mnp} = \lim_{y\to x} \left[\langle O_m(x)\, O_n(y)\, O_p(z)\rangle\, |y-x|^{\Delta_m+\Delta_n-\Delta_p}\, |x-z|^{2\Delta_p} \right]$$

$$\langle O_m(x)\, O_n(y)\, O_p(z) \rangle = \frac{K_{mnp}}{|x-y|^{\Delta_m+\Delta_n-\Delta_p}\, |y-z|^{\Delta_n+\Delta_p-\Delta_m}\, |z-x|^{\Delta_p+\Delta_m-\Delta_n}}$$

$\Rightarrow \quad C_{mnp} = \cancel{\quad} K_{mnp}$

Perturbing about a Fixed Point

Let S^* be a fixed point (i.e. a ~~scale~~ scale invariant action) and δS be a set (complete) of perturbations $\Rightarrow$

$$S = S^* + \delta S \equiv S^* + \sum_i \int \frac{d^d x}{a^d}\, g_i\, a^{\Delta_i}\, \phi_i(x)$$

$\{\phi_i(\vec{x})\}$ are a complete set of operators (in the sense that the OPE closes)

$$Z = \operatorname{tr} e^{-S[\phi]}$$
$$= \operatorname{tr} e^{-S^*[\phi]}\, e^{-\sum_i \int d^d x\, g_i\, a^{\Delta_i - d}\, \phi_i(x)}$$

$$= \operatorname{tr} e^{-S^*[\phi]} \Big\{ 1 - \sum_i \int d^d x\, g_i\, a^{\Delta_i - d}\, \phi_i(x) +$$
$$+ \frac{1}{2!} \sum_{ij} \int d^d x_1\, d^d x_2\, g_i g_j\, a^{\Delta_i + \Delta_j - 2d}\, \phi_i(x_1)\, \phi_j(x_2)$$
$$- \frac{1}{3!} \sum_{ijk} \int d^d x_1\, d^d x_2\, d^d x_3\, g_i g_j g_k\, a^{\Delta_i + \Delta_j + \Delta_k - 3d}\, \phi_i(x_1)\, \phi_j(x_2)\, \phi_k(x_3)$$
$$+ \cdots \Big\}$$

$$Z^* = \mathrm{tr}\, e^{-S^*}$$

$$\Rightarrow Z = Z^* \Big\{ 1 - \int \frac{d^d x}{a^{d-\Delta_i}} g_i \langle \phi_i(x) \rangle_{S^*}$$

$$+ \frac{1}{2} \sum_{ij}^{*} \int \frac{d^d x_1 \, d^d x_2}{a^{2d-\Delta_i-\Delta_j}} g_i g_j \langle \phi_i(x_1) \phi_j(x_2) \rangle_{S^*}$$

$$- \frac{1}{3!} \sum_{ijk} \int \frac{d^d x_1 \, d^d x_2 \, d^d x_3}{a^{3d-\Delta_i-\Delta_j-\Delta_k}} g_i g_j g_k \langle \phi_i(x_1) \phi_j(x_2) \phi_k(x_3) \rangle_{S^*}$$

$$+ \cdots \Big\}$$

Since S^* is scale invariant $\Rightarrow$ the integrals over the correlators will have IR divergences $\Rightarrow$ we need a finite size L to cut off the IR behavior.
But, precisely because S^* is scale invariant it will ~~be~~ turn out that we will need essentially the short distance singular behavior, which is given by the OPE.

This expansion looks like the low density expansion of a gas in the ~~Grand~~ Grand Canonical Ensemble with the particles labelled by the coordinates $\{\vec{r}_k\}$ and the index $\underline{\underline{i}}$ representing different "species" of particles.

Each species has a "fugacity" g_i. As a short distance cutoff we will stop the integrals at some distance $\underline{a}$ between the particles in real space $\Rightarrow \cancel{L} \geq |\vec{r}_i - \vec{r}_j| \geq a$

Renormalization procedure: we will attempt to change the short distance cutoff by some amount, and then compute the change of the action that compensates that change by requiring that Z is fixed.

① we rescale the cutoff $\quad a \to b\,a \quad , b > 1$

$b = 1 + \delta l$ (we increase the short distance cutoff)

How do we change the couplings $\{g_i\}$ / Z is invariant? The cutoff $\underline{a}$ appears in 3 places

ⓐ in divisors: $a^{d-\Delta_i}$, here Δ_i: dimension of the operator

ⓑ as a cutoff of the integrals

ⓒ through their dependence on L, L/a.

If $\delta l \ll 1$ $\Rightarrow$ we need only linear changes as

$$a \to ba = (1+\delta l)\, a$$

$$\Rightarrow \quad \frac{g_i}{a^{d-\Delta_i}} \longrightarrow \frac{g_i}{a^{d-\Delta_i}\, b^{d-\Delta_i}}$$

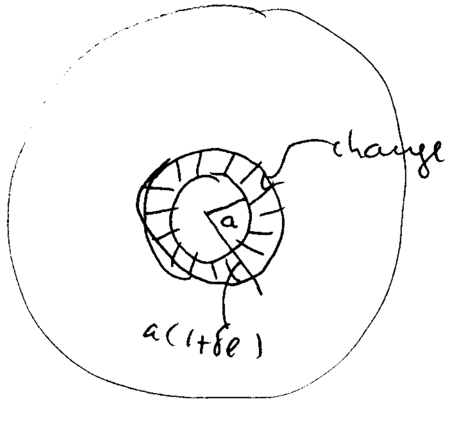
$$\Rightarrow \quad g_i \to g_i\, b^{d-\Delta_i} = g_i (1+\delta l)^{d-\Delta_i}$$

$$\approx g_i \left(1 + (d-\Delta_i)\, \delta l\right)$$

(1) $\Rightarrow g_i \to g_i + (d-\Delta_i)\, g_i\, \delta l + \cdots$

(2) Changing the short distance cutoff in the integrals:

$$\int_{|\vec{r}_1-\vec{r}_2| > a(1+\delta l)} = \int_{|r_1-r_2|>a} - \int_{a(1+\delta l) > |r_1-r_2| > a}$$



- First Term: we get back the original contribution to Z
- Second term: we can compute this contribution using the OPE, i.e. (all x's $\leftrightarrow$ Δ's)

$$\frac{1}{2}\sum_{ij}\sum_{k} c_{ijk}\, a^{x_k - x_i - x_j} \int_{a(1+\delta l) > |r_1 - r_2| > a} \left\langle \phi_k\left(\frac{r_1+r_2}{2}\right)\right\rangle_{S^*} \frac{dr_1^d\, dr_2^d}{a^{2d-x_i-x_j}}$$

$$\approx \frac{1}{2}\sum_{ij}\sum_{k} c_{ijk}\, a^{x_k - x_i - x_j}\int d\vec{r}^{\,d} \,\langle \phi_k(\vec{r})\rangle \frac{S_d\, a^d\, \delta l}{a^{2d-x_i-x_j}}$$

$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ = area of hypersphere in d dimensions

$S_d\, a^d\, \delta l$: volume of the shell

$$= \frac{1}{2}\sum_{ij}\sum_k c_{ijk} \int \frac{d\vec{r}^{\,d}}{a^{d-x_k}} \langle \phi_k(\vec{r})\rangle\, S_d\, \delta l$$

$$\Rightarrow g_k \to g_k - \frac{1}{2} S_d \sum_{ij} g_i g_j c_{ijk}\, \delta l$$

In fact because of the structure of the OPE similar changes happen to all orders in the expansion.

(3) a dependence due to L dependence:

But we should <u>not</u> change L since otherwise we ~~~~ would have not changed anything.

$\Rightarrow$ this dependence is trivial.

$$\Rightarrow \quad \frac{dg_k}{dl} = (d - x_k)\, g_k - \frac{1}{2} S_d \sum_{ij} c_{ijk}\, g_i g_j + \dots$$

$\Rightarrow$ RG eigenvalues at the trivial (S^*) fixed point $y_k = d - x_k$

redefine the coupling constants:

$$g_k \rightarrow \frac{2}{S_d}\, g_k$$

$$\Rightarrow \boxed{\frac{dg_k}{dl} = y_k\, g_k - \sum_{ij} c_{ijk}\, g_i g_j + \dots}$$

is the RG equation.

To get the h.o.t. we must use renormalized pert. theory (such as dimensional reg.)

Examples of OPE's

ϕ^4 theory:

$$J = h\, a^{-\frac{d}{2}-1}$$
$$\lambda = u\, a^{d-4}$$
$$, m^2 = t\, a^{-2}$$

define $\quad \phi_n = :\phi^n:$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + t a^{-2}\phi^2 + h a^{-\frac{d}{2}-1}\phi + u a^{d-4}\phi^4 + \cdots$$

$$:\phi^2: \equiv \phi^2 - \langle\phi^2\rangle$$

$$:\phi^4: \equiv \phi^4 - 3\langle\phi^2\rangle\phi^2$$

etc.

$$\Rightarrow \phi_1 \cdot \phi_1 = :\phi(r_1)::\phi(r_2): = \frac{1}{r_{12}^{d-2}} + :\phi^2\left(\tfrac{r_1+r_2}{2}\right): + \cdots$$

$$\Rightarrow \boxed{\phi_1 \cdot \phi_1 = 1 + \phi_2}$$

$$\boxed{\phi_1 \cdot \phi_2 = 2\phi_1 + \phi_3}$$

$$:\phi^2(r_1)::\phi^2(r_2): = \frac{2}{r_{12}^{2d-4}} + \frac{4}{r_{12}^{d-2}}:\phi^2: + :\phi^4:$$

$$\Rightarrow \boxed{\phi_2 \cdot \phi_2 = 2 + 4\phi_2 + \phi_4}$$

$$\phi_1 \cdot \phi_4 = 4\phi_3 + \cdots$$
$$\phi_2 \cdot \phi_4 = 12\phi_2 + 8\phi_4$$
$$\phi_4 \cdot \phi_4 = 24 + 96\phi_2 + 72\phi_4 + \cdots$$

$$g_1 = h \quad , \quad g_2 = t , \quad g_3 = 0 , \quad g_4 = u$$

$$\begin{cases} \dfrac{dh}{dl} = \left(\dfrac{d}{2}+1\right) h - 2 \cdot 2 h t + \cdots \\[2ex] \dfrac{dt}{dl} = 2t - h^2 - 4t^2 - 2 \cdot 12\, t u - 96\, u^2 + \cdots \\[2ex] \dfrac{du}{dl} = \epsilon u - t^2 - 2 \cdot 8\, t u - 72\, u^2 + \cdots \end{cases}$$

$$\epsilon = 4 - d$$

$$h = 0 \Rightarrow \begin{cases} \dfrac{dt}{dl} = 2t - 4t^2 - 24 t u - 96 u^2 \\[2ex] \dfrac{du}{dl} = \epsilon u - t^2 - 16 t u - 72 u^2 \end{cases}$$

Fixed Points: (a) Trivial: $u^* = t^* = 0$

(b) Wilson-Fisher ($\epsilon > 0$)

$$h^* = 0 , \quad u^* = \frac{\epsilon}{72} + O(\epsilon^2)$$

$$t^* = O(\epsilon^2)$$

$$\Rightarrow \left.\frac{dt}{dl}\right|_{WF} = 2t - 24 u^* t = \left(2 - \frac{\cancel{24}\,1}{\cancel{72}\,3} \epsilon\right) t + \cdots$$

$$\Rightarrow y_t\big|_{WF} = 2 - \frac{\epsilon}{3} + O(\epsilon^2)$$

$$\Rightarrow \nu = \frac{1}{y_t} = \frac{1}{2 - \epsilon/3} \approx \frac{1}{2}\left(1 + \frac{\epsilon}{6} + \cdots\right) = \frac{1}{2} + \frac{\epsilon}{12} + \cdots$$