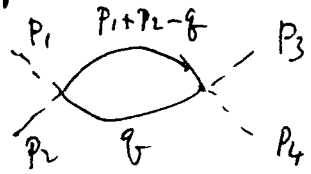
<u>Regularization</u>: We have to face now the problem that our ~~Before we~~ Feynman diagrams diverge. In order to perform any calculati[on] we have to render them finite first. There are many ways of doing that: these are the regularization schemes.

The simplest, and more intuitive, method is a <u>cutoff</u>, either in momentum space (Λ) or position space (a).

As a prototype let's think of the diagram (in ϕ^4 theory)



$= I$

(a) <u>Momentum space cutoff</u>: (sharp cutoff)

$$I = -\frac{\lambda^2}{2} \int_{|q|\leq\Lambda} \frac{1}{(q^2+m_0^2)((P-q)^2+m_0^2)} + 2 \text{ permut}$$

$$P \equiv p_1 + p_2$$

(b) <u>Lattice cutoff a</u>: By placing the theory on a lattice of spacing a we replace the propagators by

$$q^2 + m_0^2 \rightarrow \frac{2}{a^2}\sum_{\mu=1}^{d}\left[1 - \cos q_\mu a\right] + m_0^2$$

and the integral by $\int_{|q|\leq\frac{\pi}{a}}$

Both procedures are intuitively appealing but suffer from some major technical problems: (1) the integrals become very hard very quickly, (2) they violate space symmetries explicitly.

For these reasons other methods have been invented.

(C) Pauli-Villars regularization.

This method, applied first to the renormalization of QED in the 30's, consists of performing a number of subtractions to make the integrals converge. Firstly one invents some fake, unphysical, unobservable very heavy fields with masses M_i. The propagators are then modified

$$\Delta(m) \rightarrow \mathrm{reg}_M \, \Delta(m_0) = \Delta(m_0) + \sum_i C_i \, \Delta(M_i)$$

and the C_i's are determined by the condition that the regularized function be smooth. The simplest case has just one mass M

$$\frac{1}{p^2+m_0^2} \longrightarrow \mathrm{reg}_M \frac{1}{p^2+m_0^2} \equiv \frac{1}{p^2+m_0^2} - \frac{1}{p^2+M^2}$$

This procedure makes the propagator fall off like $\sim \frac{1}{p^4}$ for $p \gg M$

If this procedure does not suffice to make a given diagram finite then additional subtractions will be necessary. In general the C_i's are determined in such a procedure.

(d) Dimensional Regularization. (t'Hooft and Veltman, Bollini - Giambiagi)

This procedure is the least intuitive but is mathematically most powerful (except in some important cases (Anomalies)).

If we look at the integral in arbitrary dimension d, we can consider it as an ~~analytic~~ analytic function of the complex variable d. Then if the dimension is not too high the integral will converge in the ultraviolet (no infrared divergences will arise if $m_0 \neq 0$ otherwise the momentum transfer cannot be set to zero). Thus one is lead to study the analyticity property of I as a function of d. By defining the integral first in the region where it converges we can later define its analytic continuation beyond that region. Such methods will work only near the leading singularity and provided everything in the theory can be continued in d smoothly. This excludes anomalies.

I will now compute I within this 4 procedures.

(A) Momentum Cutoff:

Instead of the sharp momentum cutoff discussed before I will use a smooth cutoff Λ. It's done as follows.

$$\int_{|q|\le\Lambda}\left[\frac{1}{(q^2+m_0^2)\left((P-q)^2+m_0^2\right)}\right] \to \int_{\text{all } q}\left[\qquad\right]\left[\frac{\Lambda^2}{q^2+\Lambda^2}\right]^p$$

where the power p is determined by the condition that the integral be finite. The integral now scales (for $q^2 \gg \Lambda^2$) like

$$\underbrace{\Lambda^{d-4-2p}}_{\text{from integral}} \times \underbrace{\Lambda^{2p}}_{\text{factorizes}}$$

and the integral will converge in the u.v. even at $d=4$ (up to $d \le 4+2$

~~let open~~

$$f_\Lambda(p)=\frac{\Lambda^2}{p^2+\Lambda^2}\sim\begin{cases}1 & p^2\ll\Lambda^2\\ \left(\Lambda/p\right)^2 & p^2\gg\Lambda^2\end{cases}$$

so that the IR behavior is not affected.

Let $p=2$

$$\Rightarrow I_{reg}(P)=\int\frac{d^dq}{(2\pi)^d}\left(\frac{\Lambda^2}{q^2+\Lambda^2}\right)\left[\frac{1}{(q^2+m_0^2)}\,\frac{1}{(P-q)^2+m_0^2}\right]\left(\frac{\Lambda^2}{(P-q)^2+\Lambda^2}\right)^2$$

Partial fractions:

$$\frac{1}{q^2+\Lambda^2}\times\frac{1}{q^2+m_0^2}=\left[\frac{1}{q^2+m_0^2}-\frac{1}{q^2+\Lambda^2}\right]\left[\frac{1}{\Lambda^2-m_0^2}\right]$$

$$I_{reg}(P)=\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{q^2+m_0^2}-\frac{1}{q^2+\Lambda^2}\right]\left[\frac{1}{(P-q)^2+m_0^2}-\frac{1}{q^2+\Lambda^2}\right]\left[\frac{\Lambda^2}{\Lambda^2-m_0^2}\right]^2$$

Thus this procedure is equivalent to some form of Pauli-Villars regularization

Note that to regularize every propagator is certainly sufficient but by no means necessary since we could have defined

$$I'(P) = \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2+m_0^2} \frac{1}{(P-q)^2+m_0^2} \frac{\Lambda^2}{q^2+\Lambda^2}$$

which will converge provided $d < 6$ and P^2 small

For $P^2 \approx \Lambda^2 \gg m_0^2$ $\quad I'(P)$ could diverge even for $d=4$

$$\left[\frac{\Lambda^2}{\Lambda^2-m_0^2}\right]^2 \approx 1 \qquad (\Lambda \to \infty)$$

I will do the integral I explicitly to show how does this wo

$$I_d(P) = \int \frac{d^dq}{(2\pi)^d} \left(\frac{1}{q^2+m_0^2} - \frac{1}{q^2+\Lambda^2}\right)\left(\frac{1}{(P-q)^2+m_0^2} - \frac{1}{(P-q)^2+\Lambda^2}\right)$$

(Schwinger-Feynman)

I use

$$\frac{1}{A} = \frac{1}{2}\int_0^\infty dx\, e^{-x\frac{A}{2}}$$

to raise all denom. to an exponent.

$$I_d(p) = \frac{1}{4}\int_0^\infty dx \int_0^\infty dy \int \frac{d^dq}{(2\pi)^d} \left(e^{-\frac{x}{2}(q^2+m_0^2)} - e^{-\frac{x}{2}(q^2+\Lambda^2)}\right)\left(e^{-\frac{y}{2}((P-q)^2+m_0^2)} - m_0 \leftrightarrow \Lambda\right)$$

Use the Formula (without cutoff):

$$I_d(p) = \frac{1}{4}\int_0^\infty dx \int_0^\infty dy \int_q e^{-\frac{x}{2}(q^2+m_0^2) - \frac{y}{2}((p-q)^2+m_0^2)}$$

$$\int \frac{d^d q}{(2\pi)^d}\; e^{-\frac{A}{2}\vec{q}^{\,2} - B\vec{P}\cdot\vec{q}} = \frac{e^{+B^2P^2/2A}}{(2\pi A)^{d/2}}$$

$$I_d(P) = \frac{1}{4}\int_0^\infty dx \int_0^\infty dy \left\{ \frac{e^{-(x+y)\frac{m_0^2}{2}}\, e^{-y\frac{P^2}{2}}\, e^{+y^2P^2/2(x+y)}}{(2\pi(x+y))^{d/2}} + \right.$$

$$+ \frac{e^{-(x+y)\frac{\Lambda^2}{2} - y\frac{P^2}{2} + \frac{y^2P^2}{2(x+y)}}}{(2\pi(x+y))^{d/2}} - \frac{e^{-\frac{x m_0^2}{2} - y\frac{\Lambda^2}{2}}\, e^{-\frac{yP^2}{2} + \frac{y^2P^2}{2(x+y)}}}{(2\pi(x+y))^{d/2}}$$

$$\left. - \frac{e^{-\frac{x\Lambda^2}{2} - y\frac{m_0^2}{2} - y\frac{P^2}{2} + \frac{y^2P^2}{2(x+y)}}}{(2\pi(x+y))^{d/2}} \right\}$$

~~without cutoff ($\Lambda \to \infty$) $I_d(P) =$~~

$$I_d(P^2) = \frac{1}{4}\int_0^\infty dx \int_0^\infty dy\; \frac{e^{-\frac{xyP^2}{2(x+y)}}}{(2\pi(x+y))^{d/2}} \left\{ e^{-(x+y)\frac{m_0^2}{2}} + e^{-(x+y)\frac{\Lambda^2}{2}} - e^{-\frac{x\Lambda^2}{2} - y\frac{m_0^2}{2}} - e^{-\frac{x m_0^2}{2} - \frac{\Lambda^2 y}{2}} \right\}$$

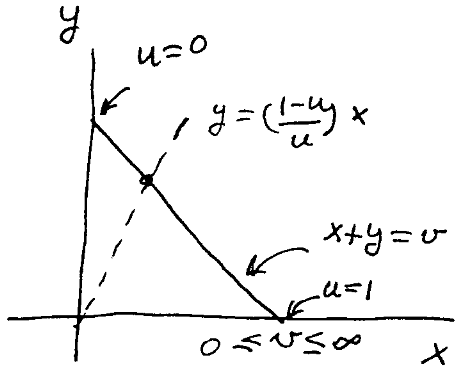
$$\frac{y^2}{x+y} - y = \frac{y^2 - y(x+y)}{x+y} = -\frac{yx}{x+y}$$

$$I_d(P^2) = \frac{1}{4}\int_0^\infty dx \int_0^\infty dy\; \frac{e^{-\frac{xyP^2}{2(x+y)}}}{(2\pi(x+y))^{d/2}} \left(e^{-\frac{x m_0^2}{2}} - e^{-\frac{x\Lambda^2}{2}}\right)\left(e^{-\frac{y m_0^2}{2}} - e^{-\frac{y\Lambda^2}{2}}\right)$$

Change variables to:

$\Lambda \to \infty$ $\quad I_d(P^2) \to \frac{1}{4}\int_0^\infty dx \int_0^\infty dy\; \frac{e^{-\frac{xyP^2}{2(x+y)} - \frac{x m_0^2}{2} - \cdots}}{(2\pi(x+y))^{d/2}}$

$$\left.\begin{aligned} x &= uv \\ y &= (1-u)v \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} x+y &= v \\ \frac{x}{x+y} &= u \end{aligned}$$



$$\frac{\partial\, xy}{\partial\, uv} = \begin{vmatrix} v & u \\ -v & (1-u) \end{vmatrix} = v$$

where $\quad 0 \leq v \leq \infty \qquad 0 \leq u \leq 1$

$$\frac{xy}{x+y} = \frac{uv^{\cancel{2}}(1-u)}{\cancel{v}} = uv(1-u)$$

Jacobian

$$I_d(P^2) = \frac{1}{4}\int_0^1 du \int_0^\infty dv\, v\, \frac{e^{-\frac{uv(1-u)P^2}{2}}}{(2\pi v)^{d/2}} \left(e^{-uv\frac{m_0^2}{2}} - e^{-uv\frac{\Lambda^2}{2}}\right) \cdot \left(e^{-(1-u)v\frac{m_0^2}{2}} - e^{-(1-u)\frac{v\Lambda^2}{2}}\right)$$

$$I_d(P^2) = \frac{1}{4(2\pi)^{d/2}} \int_0^1 du \int_0^\infty dv\; v^{1-\frac{d}{2}}\, e^{-\frac{uv}{2}(1-u)P^2} \times$$

$$\times \left\{ e^{-v\frac{m_0^2}{2}} + e^{-v\Lambda^2/2} - e^{-uv\frac{m_0^2}{2} - (1-u)\frac{v\Lambda^2}{2}} - e^{-uv\frac{\Lambda^2}{2} - (1-u)v\frac{m_0^2}{2}} \right\}$$

$$\int_0^\infty dv\; v^{1-d/2}\, e^{-v\gamma} = \frac{1}{\gamma^{2-d/2}} \int_0^\infty dt\; t^{1-d/2} e^{-t}$$

$$= \frac{1}{\gamma^{2-\frac{d}{2}}}\, \Gamma\!\left(2-\frac{d}{2}\right)$$

$$\int_0^\infty dt\; t^{z-1} e^{-t} = \Gamma(z) \qquad \text{Euler's } \Gamma\text{-function.}$$

$$I = \int_0^1 du \; \frac{\Gamma(2-d/2)}{4(2\pi)^{d/2}} \left\{ \left[\frac{1}{\frac{u(1-u)P^2}{2} + \frac{m_0^2}{2}} \right]^{2-\frac{d}{2}} + \left[\frac{1}{\frac{u(1-u)P^2}{2} + \frac{\Lambda^2}{2}} \right]^{2-\frac{d}{2}} - \right.$$

$$\left. - \left[\frac{1}{\frac{u(1-u)P^2}{2} + \frac{u m_0^2}{2} + \frac{(1-u)\Lambda^2}{2}} \right]^{2-\frac{d}{2}} - \left[\frac{1}{\frac{u(1-u)P^2}{2} + \frac{u\Lambda^2}{2} + \frac{(1-u)m_0^2}{2}} \right]^{2-\frac{d}{2}} \right\}$$

$$d = 4-\epsilon \Rightarrow 2 - \frac{d}{2} = \frac{\epsilon}{2}$$

$$\frac{1}{\gamma^{\epsilon/2}} \simeq 1 - \frac{\epsilon}{2} \ln\gamma + O(\epsilon^2)$$

$\Lambda \to \infty$

$$I_d = \int_0^1 du \; \frac{\Gamma(2-\frac{d}{2})}{4(2\pi)^{d/2}} \left(\frac{u(1-u)P^2 + m_0}{2} \right.$$

$$\frac{1}{\gamma_1^{\epsilon/2}} + \frac{1}{\gamma_2^{\epsilon/2}} - \frac{1}{\gamma_3^{\epsilon/2}} - \frac{1}{\gamma_4^{\epsilon/2}} = -\frac{\epsilon}{2} \ln\left(\frac{\gamma_1 \gamma_2}{\gamma_3 \gamma_4} \right)$$

$d \to 4$

$$I \simeq -\frac{2^{\epsilon/2}}{4(2\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \int_0^1 du \frac{\epsilon}{2} \ln\left[\frac{(u(1-u)P^2 + m_0^2)(u(1-u)P^2 + \Lambda^2)}{(u(1-u)P^2 + u m_0^2 + (1-u)\Lambda^2)\;(m_0^2 \leftrightarrow \Lambda^2)} \right]$$

$$x\Gamma(x) = \Gamma(x+1)$$

$$\Gamma\left(2-\frac{d}{2}\right) = \frac{\Gamma(3-\frac{d}{2})}{2-\frac{d}{2}} \xrightarrow[\epsilon\to 0]{} \frac{2}{\epsilon}\Gamma(1) = \frac{2}{\epsilon}$$

$$I_4 \simeq -\frac{1}{16\pi^2} \; \cancel{\frac{2}{\epsilon}} \int_0^1 du \, \cancel{\frac{\epsilon}{2}} \ln \left[\quad \right]$$

$\Lambda \to \infty$

$\boxed{L15}$ $$I_4(P^2) \cong -\frac{1}{16\pi^2}\int_0^1 du\left[\ln\left(\frac{u(1-u)P^2+m_0^2}{\mu^2}\right)+\ln\frac{\Lambda^2}{\mu^2}-\ln\frac{(1-u)\Lambda^2}{\mu^2}-\ln\frac{u\Lambda^2}{\mu^2}\right]$$

μ^2 arbitrary scale.

$$I_4(P^2) = +\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{\mu^2} - \frac{1}{16\pi^2}\int_0^1 du \ln\left[\frac{u(1-u)P^2+m_0^2}{\mu^2}\right]$$

$$+\frac{1}{16\pi^2}\int_0^1 du \ln u(1-u)$$

$$\int_0^1 du \ln u(1-u) = -2$$

Also $(m_0 \neq 0)$

$$I_4(P^2) = \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m_0^2} - \underbrace{\frac{1}{16\pi^2}\int_0^1 du \ln\left[\frac{1+u(1-u)\frac{P^2}{m_0^2}}{u(1-u)}\right]}_{\text{finite part.}}$$

$$I_4 = \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{m_0^2} - \frac{1}{8\pi^2} - \frac{1}{16\pi^2}\int_0^1 du \ln\left[1+u(1-u)\frac{P^2}{m_0^2}\right]$$

And we have pulled out the divergent part from a finite part.

Massless case : $m_0 \to 0$

$$I_4(P^2) \cong +\frac{1}{16\pi^2}\ln\frac{\Lambda^2}{\mu^2} - \frac{1}{8\pi^2} - \frac{1}{16\pi^2}\int_0^1 du \left[\frac{u(1-u)P^2}{\mu^2}\right]$$

$$I_4(P^2) \simeq \frac{1}{16\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{1}{8\pi^2} + \frac{2}{16\pi^2} - \frac{1}{16\pi^2}\ln\frac{P^2}{\mu^2}$$

$$\Rightarrow \quad I_4(P^2) \simeq \frac{1}{16\pi^2}\ln\left(\frac{\Lambda^2}{P^2}\right) \equiv \frac{1}{16\pi^2}\left[\ln\frac{\Lambda^2}{\mu^2} - \ln\frac{P^2}{\mu^2}\right]$$

and obviously we cannot set $P \to 0$ (IR divergence (log))

Dimensional regularization:

In the previous calculation we showed that the cutoff Λ (or regulator) enters in a rather essential way in the result.

Let's do the same computation within the dimensional regularization scheme.

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$I_d(P) = \frac{1}{4}\int_0^\infty dx \int_0^\infty dy \; \frac{e^{-xyP^2/2(x+y)}\, e^{-(x+y)m_0^2/2}}{(2\pi(x+y))^{d/2}}$$

and no "artifitial" regulator is imposed. However $I_d(P)$ will generally diverge. We can define though an analytic function of d in the region where $I_d(P)$ converges and then define $I_d(P)$ (for all d) as an analytic continuation.

$$I_d(P) = \frac{1}{4} \int_0^1 \frac{du}{(2\pi)^{d/2}} \int_0^\infty dv \; v^{1-\frac{d}{2}} \; e^{-v\left[\frac{u(1-u)P^2+m_0^2}{2}\right]}$$

Let $\Gamma(z) = \int_0^\infty dt \; t^{z-1} \, e^{-t}$

$$I_d(P) = \frac{1}{4} \int_0^1 \frac{du}{(2\pi)^{d/2}} \left[\frac{2\mu^2}{u(1-u)P^2+m_0^2}\right]^{2-\frac{d}{2}} \mu^{d-4} \; \Gamma\left(2-\frac{d}{2}\right)$$

$I_d(P)$ is analytic if $\quad 2-\frac{d}{2} > 0 \quad \Rightarrow \quad d < 4$

At $d=4$ $\quad \Gamma(2-\frac{d}{2})$ has a pole

$\epsilon = 4-d$

$$\Gamma\left(2-\frac{d}{2}\right) = \frac{\Gamma(3-\frac{d}{2})}{2-\frac{d}{2}} \quad \text{etc.}$$

$$\Gamma(\varepsilon) \simeq \frac{1}{\varepsilon} - C + O(\epsilon)$$

where C is Euler's constant $\quad C = 0.5772\ldots$

$$I_d(P) = \frac{\mu^{-\epsilon}}{4(2\pi)^{d/2}} \, 2^{\epsilon/2} \int_0^1 du \left[\frac{\mu^2}{u(1-u)P^2+m_0^2}\right]^{\varepsilon/2} \Gamma\left(\frac{\varepsilon}{2}\right)$$

If we keep $m_0 \neq 0$ then we ~~only~~ should only worry about u.v. divergencies. Where are they gone? They are reflected in the singularities of $\Gamma(\frac{\varepsilon}{2})$.

$\Gamma(z)$ is a function which is analytic for $\mathrm{Re}\, z > 0$ and has simple poles at $z=-n$ $(n \in \mathbb{N})$. To see that one writes down the so-called Weierstrass representation.

$$\Gamma(z)=\int_0^\infty dt\, t^{z-1} e^{-t} = \int_0^1 dt\, e^{-t} t^{z-1} + \int_1^\infty dt\, t^{z-1} e^{-t}$$

The second integral is clearly finite since the lower end of it does not get down to $t=0$

$$\Gamma(z)= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 dt\, t^{z+n-1} + \Gamma_{reg}(z)$$

$$\Gamma(z)= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z+n)} + \int_1^\infty dt\, t^{z-1} e^{-t}$$

$\Gamma(z)$ has then ~~single~~ simple poles at $z=-n$ $(n \in \mathbb{N})$

If ε is small (i.e. $d \to 4^-$) we can write

$$I(P) = \frac{2^{\epsilon/2}\mu^{-\epsilon}}{4\,(2\pi)^{2-\frac{\epsilon}{2}}} \left[\frac{2}{\epsilon} - C + O(\epsilon)\right] \int_0^1 du \left[\frac{\mu^2}{u(1-u)P^2+m_0^2}\right]^{\epsilon/2}$$

$$M^\epsilon = 1 + \epsilon \ln M + O(\epsilon^2)$$

$$I(P) \simeq \mu^{-\epsilon} \frac{(2)^{\epsilon/2}}{4\,(2\pi)^{2-\epsilon/2}} \left[\frac{2}{\epsilon} - C + O(\epsilon)\right]\left[1 + \frac{\epsilon}{2}\int_0^1 du \ln\left[\frac{\mu^2}{u(1-u)P^2+m_0^2}\right] + O\right.$$

$$I(P) = \mu^{-\epsilon}\left[\frac{1}{8\pi^2\epsilon} + \frac{\ln 4\pi}{16\pi^2} - \frac{C}{16\pi^2} + \int_0^1 \frac{du}{16\pi^2} \ln\left[\frac{\mu^2}{u(1-u)P^2+m_0^2}\right] + O(\epsilon)\right]$$

$$I(p) = \mu^{-\epsilon}\left[\frac{1}{8\pi^2\epsilon} + \frac{1}{16\pi^2}\ln 4\pi - \frac{C}{16\pi^2} + \frac{1}{16\pi^2}\int_0^1 du \ln\left[\frac{\mu^2}{u(1-u)P^2 + m_0^2}\right] + \right.$$

$$I(p) = \frac{\mu^{-\varepsilon}}{16\pi^2}\left[\frac{2}{\varepsilon} + \ln 4\pi - C + \int_0^1 du \ln\left(\frac{\mu^2}{u(1-u)P^2 + m_0^2}\right) + \cdots\right]$$

$m_0 \to 0$

$$I(p) = \frac{\mu^{-\varepsilon}}{16\pi^2}\left[\frac{2}{\varepsilon} + \ln 4\pi - C + \ln\left(\frac{\mu^2}{p^2}\right) + 2 + \cdots\right] \qquad (D.R)$$

Compare this result with

$$I(p) = \frac{1}{16\pi^2}\left[\ln\frac{\Lambda^2}{\mu^2} + \ln\frac{\mu^2}{p^2}\right] \qquad (P\text{-}V)$$

Thus the log divergence $\left(\ln\frac{\Lambda^2}{\mu^2}\right)$ is traded off by the pole $\frac{2}{\varepsilon}$

Note that the finite parts <u>are different</u>.