Renormalization and Regularization

* Dimensional ~~Analysis~~ Analysis

I quickly survey the dimensional analysis of different theories

(a) Scalar theories $(N=1)$

$$\mathcal{H} = \frac{1}{2}(\nabla_\mu \phi)^2 + \frac{m_0^2}{2}\phi^2 + \sum_{r=3}^{\infty} \frac{\lambda_r}{r!}\phi^r$$

$\int d^dx\, \mathcal{H}[\phi(x)]$ is dimensionless $\Rightarrow$

$[\mathcal{H}] = L^{-d}$

$\Rightarrow$ (i) $L^{-d} = L^{-2}[\phi]^2 \Rightarrow [\phi] = L^{-(d-2)/2} \equiv \Lambda^{(d-2)/2}$

(ii) $L^{-d} = [m_0^2][\phi]^2 \Rightarrow [m_0] = L^{-1} = \Lambda$

(iii) $[\lambda_r][\phi]^r = L^{-d} \Rightarrow [\lambda_r] = L^{-d}[\phi]^{-r} = L^{-d} L^{(d-2)\frac{r}{2}}$

$$[\lambda_r] = L^{(d-2)\frac{r}{2} - d} = \Lambda^{d - \frac{r}{2}(d-2)} = \Lambda^{\delta_r}$$

$$\delta_r = d + r - \frac{rd}{2}$$

We now ask the question: when is λ_r dimensionless

$$\Rightarrow \delta_r = 0 \Rightarrow d_{uc}(r) = \frac{r}{-1+\frac{r}{2}}$$

$d_{uc}(3) = 6$ (ϕ^3)

$d_{uc}(4) = 4$ (ϕ^4)

$d_{uc}(6) = 3$ (ϕ^6)

$d_{uc}(\infty) = 2$ $\Rightarrow$ at $d=2$ all ϕ^r are dimensionless (since ϕ itself is dimensionless!)

(b) Spin ~~zero~~-one half theories (Fermions)

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m_0)\psi + g(\bar{\psi}\psi)^2$$

$$[\mathcal{L}] = L^{-d} = \Lambda^{d}$$

(i) $L^{-d} = [\psi]^2 L^{-1} \Rightarrow [\psi] = L^{-(d-1)/2} = \Lambda^{(d-1)/2}$

(ii) $\Lambda^{+d} = [m_0][\bar{\psi}\psi] = [m_0]\ \Lambda^{(d-1)} \Rightarrow [m_0] = \Lambda$

(iii) $\Lambda^{d} = [g]\ \Lambda^{2(d-1)}$

$[g] = \Lambda^{-d+2} = \Lambda^{-(d-2)} = L^{d-2}$ $\Rightarrow [g] = 1$ $\boxed{d=2}$

Note that g scales like the coupling constant of the NLσ-model

$$\mathcal{H} = \frac{1}{2g}(\nabla\vec{S})^2 \qquad [\delta(S^2-1)]$$

$$L^{-d} = [g]^{-1} L^{-2} \Rightarrow [g] = L^{d-2} \Rightarrow [g] = \Lambda^{-(d-2)}$$

(c) Spin-one theories (Gauge-Theories)

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \qquad \text{(Non-Abelian case)}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

f^{abc} are the structure constants of the gauge group.

$$[F^{\mu\nu}_a] = L^{-1}[A]$$

$$[\mathcal{L}] = L^{-d} = \Lambda^{d} = [F^{\mu\nu}]^2$$

$$\Lambda^{d} = \Lambda^2 [A]^2 \Rightarrow [A] = \Lambda^{(d-2)/2} = L^{-(d-2)/2}$$

$$[F] = \Lambda\ \Lambda^{(d-2)/2} = \Lambda^{d/2} = L^{-d/2} = [g]\ [A]^2$$

$$\Rightarrow [g] = \Lambda^{\frac{d}{2}}\Lambda^{-d+2} \Rightarrow [g] = \Lambda^{2-\frac{d}{2}} = L^{(d-4)/2}$$

and the coupling constant is dimensionless at $\boxed{d=4}$

Gauge Coupling: $\quad \not{D} = \not{\partial} - ig \not{A} \quad$ (Abelian)

$$[\not{D}] = \Lambda = [g][\not{A}]$$

$$[g] = \Lambda \, \Lambda^{-(d-2)/2} = \Lambda^{2-d/2} = L^{(d-4)/2}$$

$\Rightarrow$ the coupling to a gauge field is also dimensionless at $d=4$.

Green's functions:

L14

$$G^{(N)}(x_1, \dots, x_N) = \langle \phi(x_1) \cdots \phi(x_N) \rangle$$

$$[G^{(N)}] = [\phi]^N = \Lambda^{(d-2)\frac{N}{2}}$$

$$[G^{(N)}(\{k_i\})] = [G^N(x_i)] \underbrace{\Lambda^{-Nd}}_{\text{phase space}}$$

$$[G^{(N)}(\{k_i\})] = \Lambda^{(d-2)\frac{N}{2} - Nd} = \Lambda^{-N(\frac{d}{2}+1)}$$

$$G = \bar{G}\, \delta^{(d)}(k_i) \Rightarrow [\bar{G}] = \Lambda^{d - N(\frac{d}{2}+1)}$$

Vertex functions: $\quad [\Gamma^{(N)}(x_i)] = [G^{(N)}(x_i)] \; V^{-N} [G^{(2)}(x_i)]^{-N}$

$$= \Lambda^{(d-2)\frac{N}{2} + Nd - N(d-2)} = \Lambda^{N(\frac{d}{2}+1)}$$

$$[\Gamma^N(k_i)] = \Lambda^{-N(\frac{d}{2}-1)}$$

$$[\bar{\Gamma}^N(k_i)] = \Lambda^{N + d - \frac{Nd}{2}}$$

All of these is just classical. It simply means that if all length scales are scaled by $\frac{1}{\alpha}$ (i.e. moments by α) $\Rightarrow$

$[G] = \Lambda^g \Rightarrow \quad G \to \alpha^g G \quad$ is the transformation law under a dilatation of scale.

We are going to see in what follows that ~~these~~ naive or canonical dimensions are modified. The interacting theory does not transform under scale transformation as the free theory does.

Divergencies:

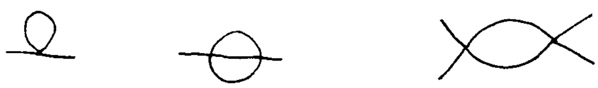
In our previous attempts at evaluating Feynman graphs we found that a good number of them are divergent (if we try to remove the cutoff naively)

If we look at these diagrams more closely we find out that at an arbitrary number of loops $N \geq 2$ some of the divergence come from insertions from ~~lower~~ orders

e.g.

etc.

The renormalization program ~~as~~ we discussed before takes care of these divergencies. There are divergencies that do not arise from low order insertions.



ϕ^4 at $d=4$

these divergencies are called <u>primitive divergencies</u>

* Superficial degree of divergence of a graph:

In any theory we'll have to discuss primarily primitively divergent graphs. These graphs depend only on the nature of the interaction, the dimensionality and the vertex. (and the order of P.T

Consider ϕ^r theory. A graph of n-th order, for a vertex function with N external legs will behave like

$$\Lambda^{\delta(r,d,N,n)} = \Lambda^{Ld-2I} \qquad \text{(sup. deg. of div.)}$$

where $L = \#$ of loops (or indep. integrals) and I the # of internal lines (or propagators)

$$L = I - \underbrace{(n-1)}_{\delta\text{-functions.}}$$

$$2I = nr - N = \#\text{ of lines.}$$

$$\Rightarrow \quad \delta = Ld - 2I = d(I-(n-1)) - 2I = (d-2)I - (n-1)d$$

$$\delta = \frac{(d-2)}{2}(nr-N) - (n-1)d$$

$$\delta_r = r + d - \frac{rd}{2}$$

$$\boxed{\delta = -n\delta_r + d + N - \frac{Nd}{2}}$$

Thus the primitive degree of divergence of a graph in a given order n of P.T. is determined by the canonical dimension of the interaction. Moreover if $\delta_r = 0 \Rightarrow \delta$ is independent of n (the order in P.T.). If $d = d_{0c} \Rightarrow$ the coupling constant is dimensionless and the degree of divergence (superficial) depends only on the vertex in question and does not change with the order in p.t.

If $d > d_c \Rightarrow$ the ~~theory~~ theory is said to be non-renormalizable ($\delta_r < 0$). This means that $\delta \sim n|\delta_r|$ and grows with n

On the other hand if $d < d_c \Rightarrow \delta_r > 0 \Rightarrow$ the degree of primitive divergence decreases with n: the theory is superrenormalizable.

At $d = d_c$ the theory is said to be renormalizable. This means that perturbation theory may be made finite with a <u>finite</u> number of renormalization constants.

At d_c what are the vertex functions which diverge?

$$\delta \geq 0 \quad \Rightarrow \delta_c = N + d_c - \frac{N d_c}{2} \geq 0$$

$$\Rightarrow \quad N \leq \frac{d_c}{\frac{d_c}{2} - 1} = r$$

(a) $\phi^4 \Rightarrow N \leq 4$ are primitively divergent at $d = d_c = 4$

$\Rightarrow$ we have to study $\Gamma^{(2)}$ (quadratic) and $\Gamma^{(4)}$ (logarithmic)

(b) $\phi^6 \Rightarrow N \leq 6 \Rightarrow \Gamma^{(2)}, \Gamma^{(4)}, \Gamma^{(6)}$ etc.

<u>Composite Operators</u>