$\boxed{L16}$ Renormalization Conditions:

In lecture #17 I showed to you that ϕ^4 theory (in 4d) could be made finite to two loops provided the mass, the coupling constant and the field were renormalized.

In particular we discussed that one could find ~~these functions~~ m_0^2, λ, Z_ϕ as functions of m_R^2, g_R and the cutoff scale Λ

$$m_R^2 = Z_\phi \, m^2(m_0^2, \lambda, \Lambda)$$

$$g_R = Z_\phi^2 \, g(m_0^2, \lambda, \Lambda)$$

~~where $Z_\phi \equiv Z_\phi$~~

alternatively

$$m_0^2 = m_0^2(m_R^2, g_R, \Lambda)$$

$$\lambda = \lambda(m_R^2, g_R, \Lambda)$$

$$Z_\phi = Z_\phi(m_R^2, g_R, \Lambda)$$

Then the renormalized 1PI vertex functions

$$\Gamma_R^{(N)}(\{k_i\}, m_R^2, g_R, \Lambda) = Z_\phi^{N/2} \, \Gamma^{(N)}(\{k_i\}; m_0^2, \lambda, \Lambda)$$

and $\Gamma_R^{(N)}$ are finite as $\Lambda \to \infty$ (at $d=4$) at every order in an expansion in powers of g_R.

How does one ~~do~~ this in practice? One first computes the bare

1PI vertex function $\Gamma^{(N)}(\{k_i\}; m_0^2, \lambda, \Lambda)$. Then ~~one~~ writes down m_0^2, λ, Z_ϕ as power series in g_R in which the coeff. are functions of m_R^2 and Λ. Then one finds a way to choose the coeff. so that $\cancel{\Gamma_R^{(N)}}$ ~~effects~~ $Z_\phi^{N/2}\,\Gamma^{(N)}$ is finite at every order in g_R (as $\Lambda \to \infty$).

$\boxed{\epsilon = 4-d}$

Dimensionally we have

$$m_0^2\,(m_R^2, g_R, \Lambda) = \Lambda^2\, \bar{m}\left(\frac{m_R}{\Lambda},\ m_R^{-\epsilon} g_R\right)$$

where $\bar{m}$ is a dimensionless and thus it can only depend on dimensionless ratios $\frac{m_R}{\Lambda}$ and $m_R^{-\epsilon} g_R$ where I have used the fact that $[g] = \Lambda^\epsilon$

Likewise

$$\lambda\,(m_R^2, g_R, \Lambda) = \Lambda^\epsilon\, \bar{\lambda}\left(\frac{m_R}{\Lambda},\ m_R^{-\epsilon} g_R\right)$$

$$Z_\phi(m_R^2, g_R, \Lambda) = \bar{Z}_\phi\left(\frac{m_R}{\Lambda},\ m_R^{-\epsilon} g_R\right) \qquad \text{etc.}$$

At the tree level this is trivial:

$$m_0^2 = m_R^2 \Rightarrow \bar{m}^2 = \left(\frac{m_R}{\Lambda}\right)^2 + O(g_R)$$

$$\lambda_0 = g \Rightarrow \bar{\lambda} = \left(\frac{m_R}{\Lambda}\right)^\epsilon (m_R^{-\epsilon} g_R) + O(g_R^2)$$

$$Z_\phi = \bar{Z}_\phi = 1 + O(g_R)$$

There is a great deal of freedom to choose ~~the~~ a way relate renormal. and bare quantities. The most intuitive way is to choose renormalization conditions.

Example: Renormalization of a massive theory at zero momentum.

$$\Gamma_R^{(2)}(0, m_R^2, g_R) = m_R^2 \qquad \leftarrow \text{fixes the renorm. mass}$$

$$\frac{\partial}{\partial k^2} \Gamma_R^{(2)} (0, m_R^2, g_R) = 1 \qquad \leftarrow \text{fixes } Z_\phi$$

$$\Gamma_R^{(4)} (\{k_i = 0\}; m_R^2, g_R) = g_R \qquad \leftarrow \text{fixes } g$$

This is what we did before.

$$\Rightarrow \begin{cases} m_R^2 = Z_\phi \, \Gamma^{(2)}(0, m_0^2, \lambda, \Lambda) \\ 1 = Z_\phi \dfrac{\partial}{\partial k^2} \Gamma^{(2)} (0, m_0^2, \lambda, \Lambda) \\ g_R = Z_\phi^2 \, \Gamma^{(4)} (k_i = 0; m_0^2, \lambda, \Lambda) \end{cases}$$

This is OK except for the fact that it does not work for the massless (i.e. critical) theory. The reason is that infrared divergencies will arise and we will not be able to set $m_R \to 0$.

The solution is to renormalize the theory at an arbitrary momentum scale K (the so-called renormalization point)

We can impose the conditions

$$\Gamma_R^{(2)}(0, g_R) = 0 \qquad \text{(i.e. } m_R = 0\text{)}$$

$$\frac{\partial}{\partial k^2} \Gamma_R^{(2)}(k; g_R)\Big|_{k^2 = K^2} = 1$$

$$\Gamma_R^{(4)}(k_i; g_R)\Big|_{S.P.} = g_R$$

where the symmetric point S.P. is defined as

$$\vec{k}_i \cdot \vec{k}_j = \frac{K^2}{4}(4\delta_{ij} - 1)$$

i.e. $\vec{k}_i^{\,2} = \frac{3}{4}K^2 \qquad \vec{k}_i \cdot \vec{k}_j = -\frac{K^2}{4}$

This choice makes the calculation more symmetric.

$\Rightarrow \quad P^2 = (\vec{k}_i + \vec{k}_j)^2 = K^2 \quad$ indep. of $\vec{k}_i, \vec{k}_j$

(we could also have defined $\Gamma_R^{(2)}$ at $k^2 = K^2$ and $\Gamma_R^{(2)}(k, g_R)\Big|_{k^2=K^2} = K^2$

Both schemes are fine. However we've been thinking of a theory with a cutoff and therefore the renorm. functions are functions of Λ

If we regularize without a cutoff, for example if we use D-R, then the functions are non-trivial functions of ε and they are polynomials in $\frac{1}{\varepsilon}$ such that all the poles in ε are cancelled and the Γ_R's are finite as $d \to 4$. This alternative scheme is called minimal subtraction (t'Hooft-Veltman)

Within this scheme one does not need to specify renormalization conditions. One simply cancels the divergent contributions.

Renormalization constants to two loops: (massless theory)

(1) At criticality we have $m_R = 0 \Rightarrow m_0 \neq 0$. In this framework the 1PI vertex functions depend on K. We still have two other param λ, Λ. and m_0^2 is determined by requiring that $m_R^2 = 0$

The Γ_R's depend only on g_R and K $(\Lambda \to \infty)$

(massive thy, $m_0^2, \hat{\lambda}, \Lambda \Rightarrow m_R^2, g$)

(2) The constants are also functions of K (i.e. of the choice of renormalization point)

(3) The IR divergent graphs in $\Gamma^{(2)}$ are cancelled by mass renormalization. The quadratically divergent graphs Λ^2 are IR finite. (the $\ln \Lambda$ terms are cancelled) $\Rightarrow$ it is possible to define

$\Gamma_R^{(2)}(0, g_R) = 0$

Then we first determine T_c: $m_c^2 = m_c^2(\lambda, \Lambda)$

$$0 = m_c^2 + \frac{\lambda}{2} D_1(m_c^2, \Lambda) - \frac{\lambda^2}{4} D_1(m_c^2, \Lambda) D_2(m_c^2, \Lambda) - \frac{\lambda^2}{6} D_3(0, m_c^2, \Lambda)$$

(Z_ϕ is divided out)

$$m_c^2 = \sum_n m_i^2(\Lambda) \lambda^n$$

recall
$$m_0^2 = m_R^2 - \frac{\lambda}{2} D_1(m_R^2, \Lambda) + \frac{\lambda^2}{6} D_3(0, m_R^2, \Lambda)$$

$m_R \to 0$
$$m_c^2 = -\frac{\lambda}{2} D_1(0, \Lambda) + \frac{\lambda^2}{6} D_3(0, 0, \Lambda)$$

$$D_1(0, \Lambda) = \int_q^\Lambda \frac{1}{q^2} \quad ; \quad D_3(0,0,\Lambda) = \int_{q_1 q_2}^\Lambda \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2}$$

Both integrals diverge like Λ^2 at $d=4$ and hence are IR finite.

Coupling constant:

$$\lambda = g_R + \lambda_2 g_R^2 + \lambda_3 g_R^3 + \dots$$

$$Z_\phi = 1 + Z_2 g_R^2 + \dots \qquad (Z_1 = 0)$$

We have

$$Z_\phi \left[1 - \frac{\lambda^2}{6} \frac{\partial}{\partial k^2}\right.$$  $$\left.\Big|_{k^2 = K^2} \right] = 1$$

$$Z_\phi^2 \Big[\lambda - \frac{3}{2}\lambda^2$$  $$\Big|_{S.P.} + \frac{3}{4}\lambda^3$$  $$\Big|_{S.P.} + 3\lambda^3$$  $$\Big|_{S.P.} \Big] = g_R$$

$\downarrow$ $I(K,0,\Lambda)$ $\qquad$ $\downarrow$ $I^2(K,0,\Lambda)$ $\qquad$ $\downarrow$ $I_4(K,0,\Lambda)$

$$I(K,0,\Lambda) = \int_q^\Lambda \frac{1}{q^2 (P-q)^2} \Big|_{P^2 = K^2} = I_{SP}$$

 $= I \sim \Lambda^{d-4} = \Lambda^{-\epsilon}$

$$I_4(K,0,\Lambda) = \int_{q_1 q_2}^\Lambda \frac{1}{q_1^2 q_2^2 (k_1+k_2-q_1)^2 (k_3+q_1-q_2)^2} \Big|_{S.P.} = I_{4SP}$$

$$\Rightarrow \quad Z_2 = \frac{1}{6} \frac{\partial}{\partial k^2} D_3(k,\Lambda) \Big|_{k^2 = K^2}$$

$\boxed{L17}$

$$(1 + 2 Z_2 g_R^2)\left(g_R + \lambda_2 g_R^2 + \lambda_3 g_R^3 - \frac{3}{2}(g_R + \lambda_2 g_R^2)^2 I_{SP} + \frac{3}{4} g_R^3 I_{SP}^2 + \right.$$

$$\left. + 3 g_R^3 I_{4SP} \right) = g_R$$

Expanding we get:

$$\lambda_2 = \frac{3}{2} I_{S.P.}$$

$$\lambda_3 = \frac{15}{4} I_{S.P.}^2 - 3 I_{4\,S.P.} - 2 Z_2$$

Thus if we want to calculate in the massless theory only 3 integrals are necessary $I_{S.P.}$, $I_{4.S.P}$ and $\frac{\partial}{\partial k^2} D_3 \big|_{S.P.}$

Notice that although D_3 diverges like Λ^2 (as $k\to 0$) $\frac{\partial D_3}{\partial k^2}$ diverges logarithmically.

Counterterms

Let us find the generating function of the renormalized connected green's functions,

$$G_{CR}^{(N)}(k_i; m_R^2; g_R, \Lambda) = Z_\phi^{-N/2}\, G_c^{(N)}(k_i; m_0^2; \lambda, \Lambda)$$

which is finite as $\Lambda \to \infty$.

$$Z_\phi^{-N/2} \frac{\delta^N F\{J\}}{\delta J(k_1)\cdots\delta J(k_N)}\bigg|_{J=0} = \frac{\delta^N F\{Z_\phi^{+1/2} J\}}{\delta J(k_1)\ldots\delta J(k_N)}\bigg|_{J=0}$$

and a Hamiltonian density

$$\mathcal{H}' = \frac{1}{2}(\nabla_\mu \phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J'\phi$$

$$J' = Z_\phi^{-1/2} J$$

Since ϕ is an integration variable we can define a new ("renormalized") field $\phi_R = Z_\phi^{-1/2}\phi$

Then

$$\mathcal{H}' = \frac{1}{2} Z_\phi (\nabla_\mu \phi_R)^2 + \frac{1}{2} Z_\phi m_0^2 \phi_R^2 + \frac{\lambda}{4!} Z_\phi^2 \phi_R^4 - J \phi_R$$

Define now a renormalized Hamiltonian

$$\mathcal{H}_R = \frac{1}{2} (\nabla_\mu \phi_R)^2 + \frac{m_R^2}{2} \phi_R^2 + \frac{g_R}{4!} \phi_R^2 - J \phi_R$$

and

$$\mathcal{H}' = \mathcal{H}_R + \frac{1}{2} (Z_\phi - 1)(\nabla_\mu \phi_R)^2 + \frac{1}{2} (Z_\phi m_0^2 - m_R^2) \phi_R^2 + \frac{1}{4!} (\lambda Z_\phi^2 - g_R) \phi_R^4$$

These three terms are known as counterterms since this modified Hamiltonian $\mathcal{H}_R$ leads to finite results as a result of the cancellations induced by these counterterms. What matters here is that these are the most general operators, compatible with the symmetry, with dimension less or equal than d_c. (at $d = d_c$). It is also essential that the counterterms have the <u>same</u> structure as the bare Lagrangian does. Otherwise additional operators should have been added to the bare Lagrangian to insure renormalizability. There is a general procedure to generate the counterterms devised by Bogoliubov.