Topological Field Theories

Let us consider a simple problem: a $\mathbb{Z}_2$ gauge theory in a deconfined phase, i.e. $D > 2$ and $k = \frac{1}{g^2} \to \infty$ (or $k > k_c$).

In the limit $k \to \infty$ all plaquettes must satisfy $\mathbb{P}_{0000} = 1$. Let us imagine that the theory is defined on a torus in space $x$ time $\Rightarrow$ i.e. the space-time is $T^2 \times \mathbb{R}$.

We can fix the gauge $\sigma = 1$ on all the time-like links. The condition that all configurations be flat ($\mathbb{P}_{0000} = 1$) (no curl!) can be satisfied by taking $\sigma = 1$ also along the special links that are Wilson loops.

However, on a torus the products $\prod_1^n \sigma$ along non-contractible loops are gauge invariant and can take either value $\pm 1$ even for flat configs.
\( \Rightarrow \) on a torus the vacuum is \( 4 \)-fold degenerate (\( D = 2+1 \)). In \( D = d+1 \) the degeneracy is \( 2^d \). Also, even if \( d=2 \) but the space is a more complex closed (smooth) surface the degeneracy is bigger:

\[ G = 2^{g_{\text{genus}}} \]

Thus for the sphere \( g = 0 \Rightarrow G = 1 \) non-degenerate 2-torus \( 4^1 = 4 \)

2 handles (\( d=2 \)) \( 4^2 = 16 \) etc.

\( \Rightarrow \) the \# of degenerate states grows with the topology of the surface. Moreover as quantum states, they are linearly independent \( \Rightarrow \) a topological phase (vacuum) supports a finite-dimensional Hilbert space determined by the global properties of the surface.

All discrete gauge theories have phases of this type.
This property is common to all topological field theories.

Another example is the Chern-Simons gauge theory. It is a gauge theory in $2+1$ dimensions that breaks parity and time reversal. As such, first must be odd under $t \rightarrow -t$ and $x_1 \rightarrow -x_1, x_2 \rightarrow x_2$ (or vice versa). The simplest choice is

$$L = \frac{k}{4\pi} \varepsilon_{\mu
u\lambda} \partial^\mu A^\nu \partial^\lambda A^\lambda \equiv \frac{k}{4\pi} \mbox{AdA}$$

for the abelian case, and

$$L = \frac{k}{4\pi} \mbox{Tr} (\mbox{AdA} + \frac{2}{3} A^3)$$

for the non-abelian case. (written 89)

This theory is locally gauge invariant up to boundary terms. It is also topological in that

if $S = \int d^3 x \sqrt{g} L$, where $g_{\mu\nu}$ is the metric tensor => $\frac{\delta S}{\delta g_{\mu\nu}} = 0$ (or rather, $\frac{\delta S}{\delta g_{\mu\nu}} = 0$)

since $\frac{\delta S}{\delta g_{\mu\nu}} = T_{\mu\nu} \Rightarrow T_{\mu\nu} = 0$ "no energy"

To see this let's look at the simple abelian thing.
and write it as follows:

\[ L = \frac{k}{2\pi} \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - j^\mu A^\mu = \]

\[ = \left( \frac{k}{2\pi} \varepsilon_{\mu\nu\lambda} \frac{E^\lambda}{E_0} \partial_\nu A_0 - j_0 \right) A_0 + \]

\[ + \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} A^\nu \partial^\mu A^\lambda + \vec{j} \cdot \vec{A} \]

\[ \rightarrow \text{ Gauss' Law } \nabla \cdot \vec{E} - j_0 = 0 \Rightarrow \left[ \frac{k}{2\pi} \right] B = j_0 \]

\[ \rightarrow \text{ charge induces flux (flux attachment) } \]

This is the origin of fractional statistics (as we will see)

Also \[ [A_1, A_2] = i \frac{2\pi}{k} \] are the commutation relations.

and the Hamiltonian is \[ H =-\vec{j} \cdot \vec{A} \]

\[ \rightarrow \vec{j} = 0 \Rightarrow H = 0 \text{ no energy!} \]

Gauge Invariance (Abelian gauge for spin 1/2)

Under a gauge transformation \[ A_\mu \rightarrow A_\mu + \partial_\mu \phi \]

\[ \Rightarrow S(A_\mu) = S(A_\mu + \partial_\mu \phi) \text{ of } \]

\[ \oint d^3x \left( A_\mu + \partial_\mu \phi \right) \varepsilon_{\mu\nu\lambda} (A^\nu + \partial^\nu \phi) = \oint d^2x \vec{A} \cdot d\vec{A} \]

\[ \oint d^3x \partial_\mu \phi \varepsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \]

\[ \oint_\partial d^3x \]
\( \text{the change \ is} \ \int d^3x \ \partial_\mu \bar{F}^\mu = \)
\[ \int d^3x \ \partial_\mu (\bar{F}^\mu F_{\star}^\mu) - \int d^3x \ \bar{F}^\mu F_{\star}^\mu \]

\( \partial_\mu F_{\star}^\mu = \partial_\mu \epsilon^{\mu \nu \lambda} \partial_\nu A_\lambda = 0 \) (Bianchi Identity)

\[ \delta S = \int d^3x \ \partial_\mu (\bar{F} \epsilon^{\mu \nu \lambda} \partial_\nu A_\lambda) \]
\[ = \int d\Sigma \ \bar{F} \epsilon^{\mu \nu \lambda} \partial_\nu A_\lambda = \frac{1}{2} \int d\Sigma \ \epsilon^{\mu \nu \lambda} \bar{F} \]

Gauss
\[ \Sigma = \partial \Sigma \]

\( \bar{F} \epsilon^{\mu \nu \lambda} \partial_\nu A_\lambda \]

i.e. a gauge trans. \( \bar{F} = \text{const} \) \( \Sigma = \partial \Sigma \)

\[ \Rightarrow \delta S = \text{flux} \times \bar{F} \]

On a closed surface \( \Sigma = \partial \Sigma \) but we have to demand that the theory be gauge invariant even under large gauge transformations which wind around the surface. The action may change but \( e^{i \delta S} = 1 \) for that to be unobservable
This leads to the requirement that the "coupling constant" $k \in \mathbb{Z}$ so that the weights are integer valued. The same requirement applies to the non-abelian case with action

$$S = \frac{k}{4\pi} \int d^3x \left( A F^* + \frac{2}{3} A \wedge A \wedge A \right)$$

This theory is \textit{topological} in the sense that if we change the space to a curved one with metric $g_{\mu\nu}$, the partition function is independent of $g_{\mu\nu}$: the observables do not depend on metric properties (i.e., distance) and are \textit{topological} invariants.

\textbf{Example:}

The Wilson loop operator for \textit{two} contours $\Gamma_1, \Gamma_2$

$$\langle W_{\Gamma_1, \Gamma_2} \rangle = \langle e^{i \oint_{\Gamma_1} A^\mu dx^\mu} e^{i \oint_{\Gamma_2} A^\mu dx^\mu} \rangle$$
In the abelian case we can do easily:

\[ \langle e^{i \phi} d^3 x A_\mu \rangle = \langle e^{i \oint d^3 x J_\mu A_\mu} \rangle \]

\[ J_\mu(x) = \delta(x - z(t)) \frac{dz_\mu}{dt} \quad (t \text{ parametrizes the curve } \Gamma) \]

\[ \Rightarrow \langle e^{i \int d^3 x J_\mu A_\mu} \rangle = e^{-\frac{\xi}{2} \int d^3 x \int d^3 x' J_\mu(x) G_{\mu \nu}(x-x') J_\nu(x')} \equiv e^{i \mathcal{I}} \]

with

\[ G_{\mu \nu}(x-x') = \langle A_\mu(x) A_\nu(x') \rangle = \frac{2 \pi}{k} G_0(x-x') \varepsilon_{\mu \nu \lambda} \nabla \delta(x-x') \]

and

\[ - \nabla^2 G_0(x-x') = \delta^3(x-x') \]

\[ \mathcal{I} = \frac{\pi}{k} \int d^3 x \int d^3 x' J_\mu(x) J_\nu(x') G_0(x-x') \varepsilon_{\mu \nu \lambda} \nabla_\lambda \delta^3(x-x') \]

\[ \equiv \frac{\pi}{k} \oint \int_\Gamma d^3 x \oint_\Gamma d^3 x' \nabla G_0(x-x') \varepsilon_{\mu \nu \lambda} \]

\[ \Pi \equiv \Pi_0 \Pi_2 \quad \text{"magnetostatics"} \]

since \( \nabla_\mu J_\mu = 0 \Rightarrow J_\mu = \varepsilon_{\mu \nu \lambda} \nabla_\nu B_\lambda \quad \nabla_\lambda B_\lambda = 0 \)

and

\[ B_\lambda = \varepsilon_{\lambda \rho \sigma} \nabla_\rho \phi_0 \]

\[ \Rightarrow J_\mu = - \nabla^2 \phi_0 \]

\[ \phi_0(x) = \int d^3 x' G_0(x-x') \ast J_\mu(x') \]
\[ B_\mu = \varepsilon_{\mu \nu \lambda} \nabla_\nu \phi_\lambda = \int d^3x' \varepsilon_{\mu \nu \lambda} \nabla'_\nu G_0(x-x') \nabla_\nu J_\lambda(x') \]
\[ = \int d^3x' G_0(x-x') \varepsilon_{\mu \nu \lambda} \nabla'_\nu J_\lambda(x') \]
\[ = \oint \varepsilon_{\mu \nu \lambda} \nabla_\nu G_0(n-x') d\lambda \]
\[ \Rightarrow I = \frac{\pi}{k} \oint d\lambda \oint d\sigma \varepsilon_{\mu \nu \lambda} \nabla_\lambda G_0(x-x') \]

\[ = \frac{\pi}{k} \int d^3x \oint J_\mu B_\mu \]

\[ = \frac{\pi}{k} \oint d\sigma \omega B_\mu = \frac{\pi}{k} \sum \int d\sigma \varepsilon_{\mu \nu \lambda} \nabla_\nu B_\lambda \]

\[ \omega = \sigma \sum = \frac{\pi}{k} \sum \int d\sigma J_\mu \]

\[ n_\mu = \text{integer that counts the } \# \text{ of times } J_\mu \text{ pierces } \Sigma \]

\[ \text{Gauss invariant (Hopf)} \]

\[ e^{i \oint d\sigma \cdot A_\mu} = e^{i \frac{\pi}{k} n_\mu} \]

\[ \text{Linking } \# \]

\[ \Sigma \]

\[ \mu \]

\[ \text{Linking } H \]
Example: Compare two types of loops:

\[ W_1 \quad W_2 = W_1 e^{i\pi \frac{\varphi}{k}} \]

\[ \Rightarrow \text{wave function is multivalued!} \]

\[ \Psi(1, 2) = e^{i\varphi} \Psi(2, 1) \]

\[ \varphi = \pm \frac{\pi}{k} \Rightarrow \text{fractional statistics} \]