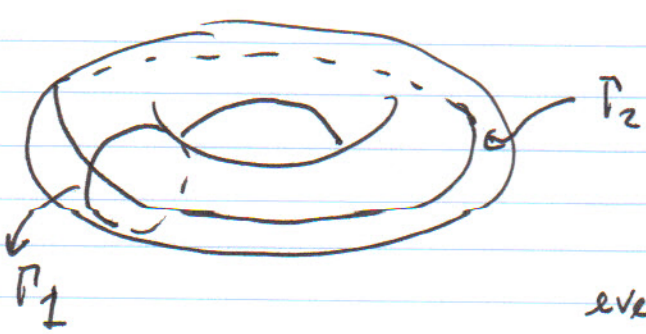
Topological Field Theories

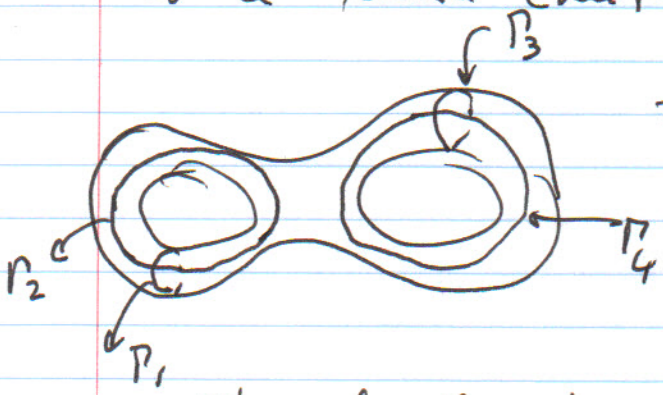
Let us consider a simple problem: a $\mathbb{Z}_2$ gauge theory in a deconfined phase, i.e. $D > 2$ and $K = \frac{1}{g^2} \to \infty$ (or $K > K_c$).

In the limit $K \to \infty$ all plaquettes must satisfy $\sigma\sigma\sigma\sigma = 1$. Let us imagine that the theory is defined on a <u>torus</u> in space × time $\Rightarrow$ i.e. the space-time is $T^2 \times \mathbb{R}$.

We can fix the gauge $\sigma = 1$ on all the time-like links. The condition that all configurations be <u>flat</u> ($\sigma\sigma\sigma\sigma = 1$) (no curl!) can be satisfied by taking $\sigma = 1$ also along the ~~spatial~~ spacial links. However on a torus the products $\prod_{\Gamma} \sigma$ (Wilson loops) along non-contractible loops are gauge invariant and can take either value ± 1 even for flat configs.



$\Rightarrow$ on a torus the vacuum is 4-fold degenerate ($D = 2+1$). In $D = d+1$ the degeneracy is $G = 2^d$. Also, even if $d=2$ but the space is a more complex closed (smooth) surface



the degeneracy is bigger: $G = (2^d)^g$ where g is the <u>genus</u> (the # of handles)

Thus for the sphere $g = 0 \Rightarrow G = 1$ non-degenerate

2-torus $4^1 = 4$

2 handles ($d=2$) $4^2 = 16$ etc.

$\Rightarrow$ the # of degenerate states grows with the topology of the surface. Moreover as quantum states, they are linearly independent $\Rightarrow$ a topological phase (vacuum) supports a finite-dimensional Hilbert space determined by the global properties of the surface.

All discrete gauge theories have phases of this type.

This property is common to all topological field theories.

Another example is the Chern-Simons gauge theory.

It's a gauge theory in 2+1 dimensions that breaks parity and time reversal. As such the Lagrangian ~~it~~ must be first must be odd under $t \to -t$ and $x_1 \to -x_1$, $x_2 \to x_2$ (or vice versa). The simplest choice is

$$\mathcal{L} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda \equiv \frac{k}{4\pi} A dA$$

for the abelian case, and

$$\mathcal{L} = \frac{k}{4\pi} \operatorname{tr}\left(A dA + \frac{2}{3} A^3\right)$$

for the non-abelian case. (Witten 89')

This theory is locally gauge invariant upto ~~boundary~~ boundary terms. It is also topological in that if $S = \int d^3x \sqrt{g}\, \mathcal{L}$, where $g^{\mu\nu}$ is the metric tensor $\Rightarrow \frac{\delta S}{\delta g^{\mu\nu}} = 0$ (or rather, $\frac{\delta Z}{\delta g_{\mu\nu}} = 0$)

Since $\frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu} \Rightarrow T_{\mu\nu} = 0$ "no energy"

To see this let's look at the simple abelian theory.

and write it in components:

$$\mathcal{L} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - j^\mu A_\mu =$$

$$\equiv \left(\frac{k}{2\pi} \varepsilon_{ij} \partial_i A_j - j_0 \right) A_0 +$$

$$+ \frac{k}{4\pi} \varepsilon_{ij} A^i \partial^0 A^j + \vec{j} \cdot \vec{A}$$

$\Rightarrow$ Gauss' Law $\quad \vec{\nabla} \cdot \vec{E} - j_0 = 0 \iff \boxed{\frac{k}{2\pi} B = j_0}$

$\Rightarrow$ <u>charge nucleates flux</u> (flux attachment)

This is the origin of fractional statistics (as we will see)

Also $\Rightarrow \quad [A_1, A_2] = i \frac{2\pi}{k}$ are the commutation relations!

do vacuum degeneracy

and the Hamiltonian is $\quad H = - \vec{j} \cdot \vec{A}$

$\Rightarrow \vec{j} = 0 \iff H = 0 \qquad$ <u>no energy!</u>

<u>Gauge Invariance</u> (abelian theory for simplicity)

under a gauge transf. $\quad A_\mu \to A_\mu + \partial_\mu \Phi$

$\Rightarrow \quad S(A_\mu) = S(A_\mu + \partial_\mu \Phi)$ if

$$\int_\Omega d^3x \, (A_\mu + \partial_\mu \Phi) \, \varepsilon_{\mu\nu\lambda} \partial^\nu (A^\lambda + \partial^\lambda \Phi) = \int_\Omega d^3x \, A \, dA$$

$$+ \int_\Omega d^3x \, \partial_\mu \Phi \, \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$

$\Rightarrow$ the <u>change</u> is $\int_\Omega d^3x \; \partial_\mu \Phi \; F^{*\mu} \equiv$

$$\equiv \int_\Omega d^3x \; \partial_\mu (\Phi \; F^{*\mu}) - \int_\Omega d^3x \; \Phi \; \partial_\mu F^{*\mu}$$

$$\partial_\mu F^{*\mu} = \partial_\mu \; \varepsilon^{\mu\nu\lambda} \; \partial_\nu A_\lambda = 0 \qquad \text{(Bianchi Identity)}$$

$$\delta S = \int_\Omega d^3x \; \partial_\mu \left(\Phi \; \varepsilon^{\mu\nu\lambda} \; \partial_\nu A_\lambda \right)$$

$$\underset{\text{Gauss}}{=} \int_{\Sigma = \partial\Omega} dS_{\mu\nu} \; \Phi \; \varepsilon^{\mu\nu\lambda} \; \partial_\nu A_\lambda = \frac{1}{2} \int_\Sigma dS_{\mu\nu} \; F^{\mu\nu} \Phi$$

$$= \int_\Sigma dS_{\mu\nu} \; \Phi \; F^{*\mu}$$

i.e. a gauge transf. / $\Phi = \text{const}$ on $\Sigma = \partial\Omega$

$$\Rightarrow \delta S = \text{flux} \times \Phi$$

On a closed surface $\Sigma = \emptyset$ but we have to demand that the theory be gauge invariant even under <u>large gauge transformations</u> which wind around the surface. The action may change but $e^{i\Delta S} \equiv 1$ for that to be unobservable

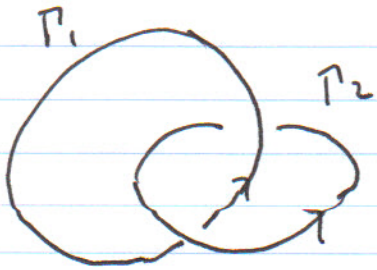
This leads to the requirement that the "coupling constant" $\boxed{k \in \mathbb{Z}}$ so that the weights are single valued. The same requirement applies to the non abelian case with action

$$S = \frac{k}{4\pi} \int_{\Omega} d^3x \left(A F^* + \frac{2}{3} A \wedge A \wedge A \right)$$

This theory is <u>topological</u> in the sense that if we change the space to a curved one with metric $g^{\mu\nu}$, the partition function is independent of $g^{\mu\nu}$ $\Rightarrow$ the observables do not depend on metric properties (i.e. distance) and are <u>topological invariants</u>.

<u>Example:</u>

The Wilson loop operator for <u>two</u> contours Γ_1, Γ_2



$$\langle W_{\Gamma_1 \cup \Gamma_2} \rangle_{CS} = \left\langle e^{i \oint_{\Gamma_1 \cup \Gamma_2} dx_\mu A^\mu} \right\rangle$$

In the abelian case we can do easily:

$$\left\langle e^{i\oint_\Gamma dx_\mu A_\mu} \right\rangle = \left\langle e^{i\int d^3x\, J_\mu A_\mu} \right\rangle$$

$$J_\mu(x) = \delta(x - z(t))\,\frac{dz_\mu}{dt} \qquad \left(t \text{ parametrizes the curve } \Gamma\right)$$

$$\Rightarrow \left\langle e^{i\int d^3x\, J_\mu A_\mu} \right\rangle = e^{-\frac{i}{2}\int d^3x \int d^3x'\, J_\mu(x)\, G_{\mu\nu}(x-x')\, J_\nu(x')}$$

$$\equiv e^{iI}$$

with $G_{\mu\nu}(x-x') = \langle A_\mu(x)\, A_\nu(x') \rangle = \frac{2\pi}{k}\, G_0(x-x')\, \varepsilon_{\mu\nu\lambda}\, \nabla_\lambda \delta(x-x')$

and $-\nabla^2 G_0(x-x') = \delta^3(x-x')$

$$I = \frac{\pi}{k}\int d^3x \int d^3x'\, J_\mu(x)\, J_\nu(x')\, G_0(x-x')\, \varepsilon_{\mu\nu\lambda}\, \nabla_\lambda\, \delta^3(x-x')$$

$$\equiv \frac{\pi}{k}\oint_\Gamma dx_\mu \oint_\Gamma dx'_\nu\, \nabla_\lambda G_0(x-x')\, \varepsilon_{\mu\nu\lambda}$$

$\Gamma \equiv \Gamma_1 \cup \Gamma_2$

"magnetostatics"

since $\nabla_\mu J_\mu = 0 \Rightarrow J_\mu = \varepsilon_{\mu\nu\lambda}\, \nabla_\nu B_\lambda$, $\nabla_\lambda B_\lambda = 0$

and $B_\lambda = \varepsilon_{\lambda\rho\sigma}\, \nabla_\rho \phi_\sigma$

$$\Rightarrow J_\mu = -\nabla^2 \phi_\mu$$

$$\phi_\mu(x) = \int d^3x'\, G_0(x-x')\, J_\mu(x')$$

$$B_\mu = \varepsilon_{\mu\nu\lambda} \nabla_\nu \phi_\lambda = \int d^3x' \, \varepsilon_{\mu\nu\lambda} \nabla_\nu^x G_0(x-x') J_\lambda(x')$$

$$= \int d^3x' \, G_0(x-x') \, \varepsilon_{\mu\nu\lambda} \nabla_\nu^{x'} J_\lambda(x')$$

$$= \oint_\Gamma \varepsilon_{\mu\nu\lambda} \nabla_\nu G_0(x-x') \, dx_\lambda$$

$$\Rightarrow I = \frac{\pi}{k} \oint_\Gamma dx_\mu \oint_\Gamma dx'_\nu \, \varepsilon_{\mu\nu\lambda} \nabla_\lambda G_0(x-x')$$

$$= \frac{\pi}{k} \int d^3x \, J_\mu B_\mu$$

flux!

$$= \frac{\pi}{k} \oint_\Gamma dx_\mu B_\mu = \frac{\pi}{k} \int_\Sigma dS_\mu \, \varepsilon_{\mu\nu\lambda} \nabla_\nu B_\lambda$$

$$\Gamma = \partial\Sigma \qquad = \frac{\pi}{k} \int_\Sigma dS_\mu J_\mu$$

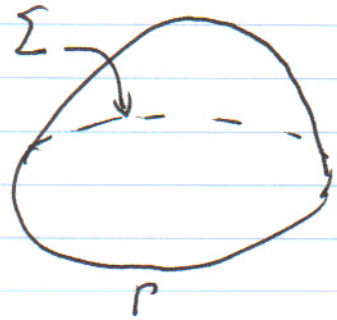
n_Γ = integer that counts the # of times J_μ pierces Σ

Gauss invariant (Hopf)

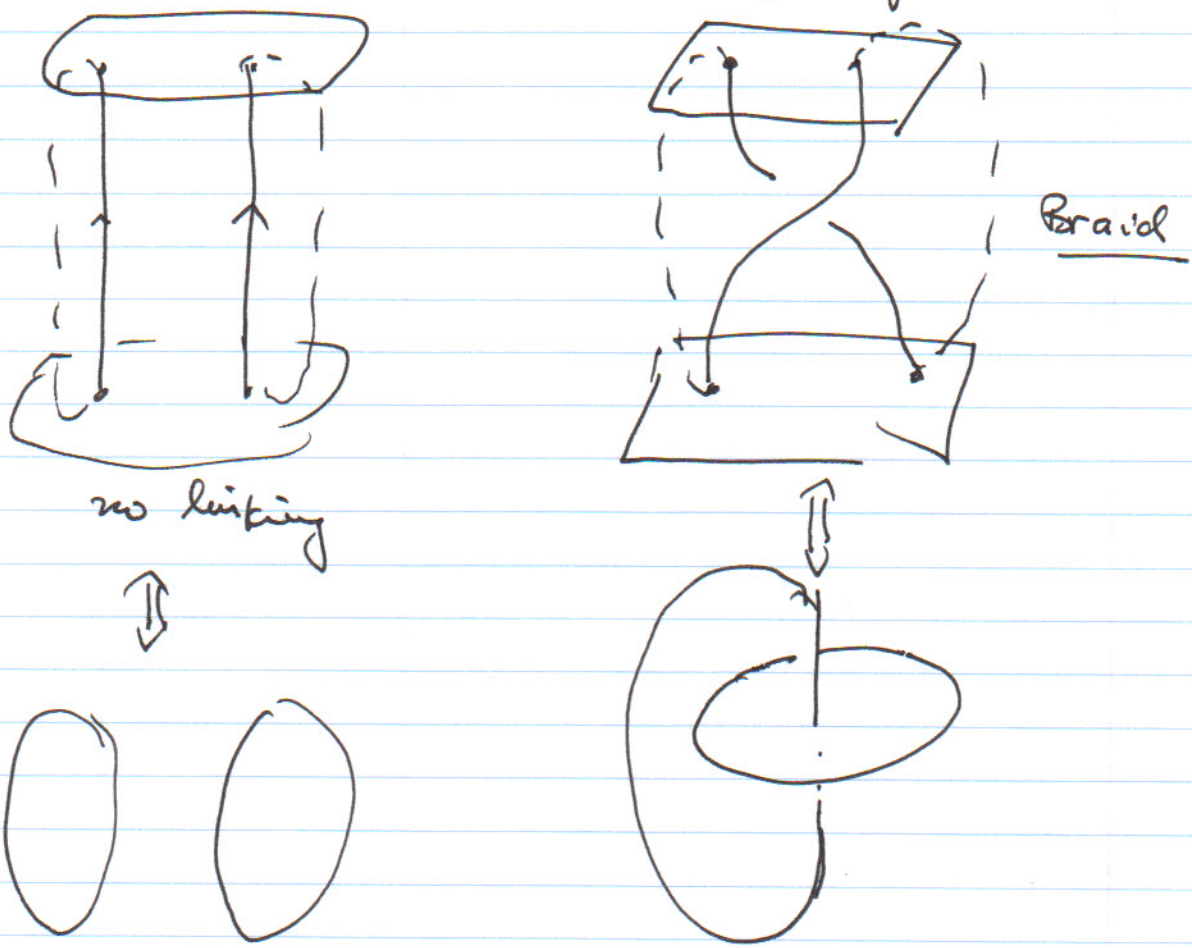
Linking #

$$\Rightarrow \left\langle e^{i \oint_\Gamma dx_\mu A_\mu} \right\rangle = e^{i \frac{\pi}{k} n_\Gamma}$$

Linking #

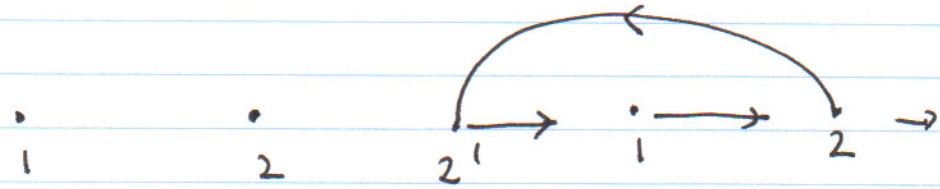


<u>Example:</u> Compare two configs. of loops



W_1 $\qquad\qquad$ $W_2 = W_1 \, e^{\pm i\frac{\pi}{k}}$

$\Rightarrow$ wave function is multivalued!



$$\Psi(1,2) = e^{i\varphi}\,\Psi(2,1)$$

$\varphi = \pm\frac{\pi}{k} \Rightarrow$ fractional statistics!