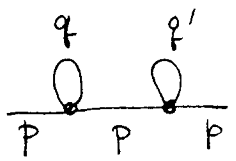
Vertex Functions

So far we've been able to reduce the # of diagrams to be considered by (a) showing that vac. parts don't contribute to $G_N(x_1, \ldots, x_N)$

(b) showing that disconnected parts need not be considered.

There's still another set of graphs that can be handled easily

Consider the 2^{nd} order cont. to G_2

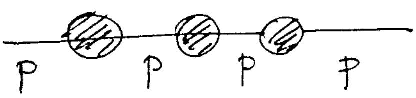


$$\left(-\frac{\lambda}{4!}\right)^2 \frac{1}{2!} (4\times 3)\cdot(4\times 3) \int_{q_1} G_0(q_1) \int_{q_2} G_0(q_2) \times$$

$$\times\, G_0^3(p)$$

Obviously this graph can be ~~[illegible]~~ split in two, just by cutting the middle line

In general



$$G_0^4(p)\left(\Sigma(p)\right)^3$$

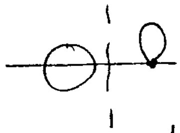
$$\Rightarrow \quad G_2(p) = G_0(p) + G_0(p)\,\Sigma(p)\,G_0(p) + G_0^3(p)\,\Sigma^2(p) + \cdots$$

$$G_2(p) = G_0(p)\sum_{n=0}^{\infty}\left(\Sigma(p)\,G_0(p)\right)^n = \frac{G_0(p)}{1-\Sigma(p)\,G_0(p)}$$

$$\Rightarrow \quad G_2^{-1}(p) = G_0^{-1}(p) - \Sigma(p)$$

$$G_2(p) = G_0(p) + G_0(p)\,\Sigma(p)\,G_2(p) \qquad \text{Dyson's equation.}$$

when $\Sigma(p)$ represents the set of all possible connected, one-particle irreducible graphs with their external legs amputated



one particle reducible

one particle irreducible.

The one particle irreducible two point function $\Sigma(p)$ is known as the mass operator or as the self energy (or two point vertex)

why? $\quad G_0^{-1}(p) = p^2 + m_0^2 \Rightarrow G_0^{-1}(0) = m_0^2$

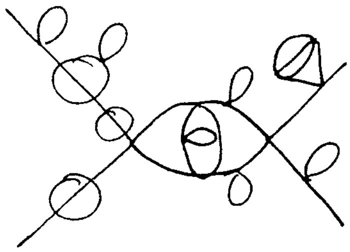
$$\Rightarrow \quad G_2^{-1}(p) = G_0^{-1}(p) - \Sigma(p) \Rightarrow G_2^{-1}(0) = m_0^2 - \Sigma(0) = m^2$$

thus $\Sigma(0)$ renormalizes the mass.

This result is infact general

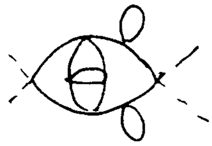
$$[G_c^2(p)]^{-1} = G_0^{-1}(p) - \Sigma(p)$$

<u>General Vertex Function:</u>



is not 1PI

but



is 1PI

We need a generating functional of 1PI vertex functions.

So far we've considered $F[J]$ which is a function of the external sources. In many cases however this is inconvenient since as $J \to 0$ we may still have $\langle \phi \rangle \neq 0$ (symmetry breaking)

We'd like to produce a Legendre transform from J to $\langle \phi \rangle$

Let $\langle \phi(i) \rangle = \bar{\Phi}(i) = \dfrac{\delta F}{\delta J(i)}$

and define the Legendre transf $\Gamma(\bar{\Phi})$ s.t.

$$\Gamma[\bar{\Phi}] = \sum_i \bar{\Phi}(i) J(i) - F[J]$$

$$\Rightarrow \frac{\delta \Gamma[\bar{\Phi}]}{\delta \bar{\Phi}(i)} = - \sum_j \frac{\delta F}{\delta J(j)} \frac{\delta J(j)}{\delta \bar{\Phi}(i)} + \sum_j \bar{\Phi}(j) \frac{\delta J(j)}{\delta \bar{\Phi}(i)} + \sum_j J(j)\, \delta(i,j)$$

$$= - \sum_j \bar{\Phi}(j) \cancel{\frac{\delta J(j)}{\delta \bar{\Phi}(i)}} + \cancel{\sum_j \bar{\Phi}(j) \frac{\delta J(j)}{\delta \bar{\Phi}(i)}} + J(i)$$

$$\Rightarrow \boxed{\frac{\delta \Gamma[\bar{\Phi}]}{\delta \bar{\Phi}(i)} = J(i)} \qquad \text{compare} \qquad \boxed{\frac{\delta F}{\delta J(i)} = \bar{\Phi}(i)}$$

However if $J \to 0$ still $\left.\dfrac{\delta F}{\delta J(i)}\right|_{J=0} = \bar{\Phi}_0(i)$ (~~the classical field~~)

then the symmetry is broken if $\phi_0(i) \neq 0$

Thus $\bar{\Phi}(i)$ satisfies $\left.\dfrac{\delta \Gamma[\bar{\Phi}]}{\delta \bar{\Phi}(i)}\right|_{J} = 0$ and it minimize the potential $\underset{\bar{\Phi}}{\Gamma}$ $\Rightarrow$ it is called the classical field.

Let's differentiate by $\bar\Phi(j)$

$$\Rightarrow \delta(i,j) = \frac{\delta^2 F}{\delta J(i)\,\delta\bar\Phi(j)} = \sum_k \frac{\delta^2 F}{\delta J(i)\,\delta J(k)}\;\frac{\delta J(k)}{\delta\bar\Phi(j)}$$

$$\& \quad J(k) = \frac{\delta\Gamma[\bar\phi]}{\delta\bar\Phi(k)} \quad\Rightarrow\quad \frac{\delta J(k)}{\delta\bar\phi(i)} = \frac{\delta^2\Gamma[\bar\Phi]}{\delta\bar\phi(k)\,\delta\bar\phi(i)}$$

$$\Rightarrow \delta(i,j) = \sum_k \frac{\delta^2 F}{\delta J(i)\,\delta J(k)}\;\frac{\delta^2\Gamma}{\delta\bar\Phi(k)\,\delta\bar\Phi(i)}$$

as $J\to 0$
$$\frac{\delta^2 F}{\delta J(i)\,\delta J(k)} \longrightarrow G_2^c(i,k)$$

$$\Rightarrow \Gamma^{(2)}(i,j) = \left.\frac{\delta^2\Gamma}{\delta\bar\phi(i)\,\delta\bar\Phi(j)}\right|_{J=0}$$ is the inverse of $G_2^c(i,k)$

$$\Rightarrow \Gamma^{(2)}(p) = [G_2(p)]^{-1} = p^2 + m_0^2 - \Sigma(p)$$

Thus $\Gamma^{(2)}(p)$ is ~~then~~ a sum of 1PI graphs.

$$\frac{\delta}{\delta J(\ell)}\,\delta(i,j) = 0 = \sum_k \frac{\delta^3 F}{\delta J(i)\,\delta J(k)\,\delta J(\ell)}\;\frac{\delta J(k)}{\delta\bar\Phi(j)} + \sum_k \frac{\delta^2 F}{\delta J(i)\,\delta J(k)}\;\frac{\delta^2 J(k)}{\delta J(\ell)\,\delta\bar\Phi(j)}$$

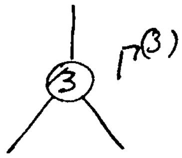
$$\frac{\delta^2 J(k)}{\delta J(\ell)\,\delta\bar\Phi(j)} = \sum_m \frac{\delta^3\Gamma}{\delta\bar\Phi(m)\,\delta\bar\Phi(k)\,\delta\bar\Phi(j)}\;\frac{\delta\bar\Phi(m)}{\delta J(\ell)}, \qquad \frac{\delta\bar\Phi(m)}{\delta J(\ell)} = \frac{\delta^2 F}{\delta J(\ell)\,\delta J(m)}$$

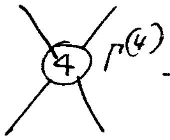
$$\cancel{\sum_K G_3^{(c)}(i,k,\ell)\,\frac{\delta^2\Gamma}{\delta\phi}}$$

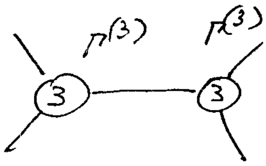
$$J\to 0 \qquad 0 = \sum_K G_3^{(c)}(i,k,\ell)\,\Gamma^{(2)}(k,i) + \sum_{k,m} G_2^{(c)}(i,k)\,G_2^{(c)}(\ell,m)\,\Gamma^{(3)}(m,k,i)$$

$$\Rightarrow \Gamma^{(2)} = [G_2]^{-1}$$

Pictorially

$G_3^{(c)} = -$ 

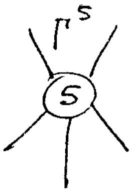
$G_4^{(c)} = -$ 

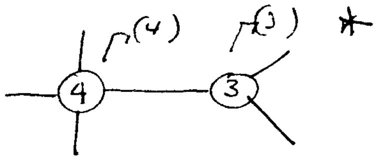
$+$ 

$*$

$*$ = one P.R. by body cut

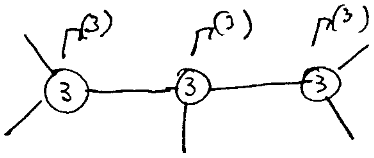
(all possible ways.)

$G_5^{(c)} = -$ 

$+$ 

$*$ $+$

all pos. ways.

$-$ 

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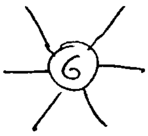
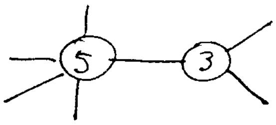
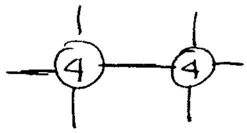
$G_6^{(c)} = -$ 
$+$ 
$*$ $+$ 
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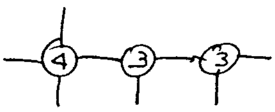
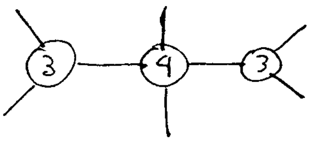
$+$ 
$*$ $+$ 
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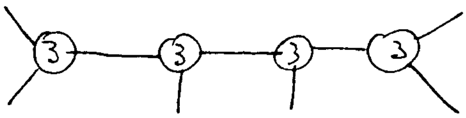
$+$ 
$*$

etc.

$$\Rightarrow \boxed{G_3^{(c)}(i_1,i_2,i_3) = - G_2^{(c)}(i_1,j_1)\, G_2^{(c)}(i_2,j_2)\, G_2^{(c)}(i_3,j_3)\, \Gamma^{(3)}(j_1,j_2,j_3)}$$

Note $G_2^{(c)}(i_1,i_2) = G_2^{(c)}(i_1,j_1)\, G_2^{(c)}(i_2,j_2)\, \Gamma^{(2)}(j_1,j_2)$

since $G_2 = \left(\Gamma^{(2)}\right)^{-1}$

Thus $\Gamma^{(3)}$ is the one P.I 3 point vertex. (note the − sign)

Clearly a graph may be reducible ~~by~~ only either by a cut of an external line or via a body cut

Define $\Gamma^{(N)}(1,\dots,N) = \left.\dfrac{\delta^N \Gamma(\bar\Phi)}{\delta\bar\Phi(1)\cdots\delta\bar\Phi(N)}\right|_{J=0}$

$$\Rightarrow (N>2) \qquad G_N^{(c)}(1,\dots N) = - G_2^{(c)}(1,1')\cdots G_2^{(c)}(N,N')\,\Gamma^{(N)}(1',\dots,N')$$

$$+ \; Q^{(N)}(1,\dots,N)$$

where the 1st terms are 1PR only via cuts of the external legs and the 2nd by body cuts (for the r-point function in a ϕ^r theory this term does not exist)

Momentum space:

L5

$$G_2^{(c)}(k_1,k_2) = \delta^d(k_1+k_2)\, G_2(k_1)\, (2\pi)^d$$

$$\Rightarrow \text{F.T.} \Rightarrow \quad G_2(k) = \left[\Gamma^{(2)}(k)\right]^{-1}$$

with $\Gamma^{(2)}(k_1,k_2) = (2\pi)^d\, \delta^d(k_1+k_2)\, \Gamma^{(2)}(k_1)$

and $G_N^{(c)}(k_1,\dots,k_N) = - G_2^{(c)}(k_1)\cdots G_2^{(c)}(k_N)\,\Gamma^{(N)}(k_1,\dots,k_N) + Q^N(k_1\dots k_N)$

The effective potential

Let $v = \bar{\Phi} = \langle \phi \rangle$. Then, with the above def. for $\Gamma^{(N)}$ we can write

$$\Gamma\{\bar{\Phi}\} = \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \cdots \int dx_N \, \Gamma^{(N)}(x_1, \dots, x_N | v) \, [\bar{\Phi}(x_1) - v(x_1)] \cdots [\bar{\Phi}(x_N) - v]$$

if $J \to 0$ then the sum starts at $N=2$.

and $v = \lim_{J \to 0} \bar{\phi}$

$$\Rightarrow \left.\begin{array}{l} \dfrac{\delta \Gamma}{\delta \bar{\Phi}} = J \underset{J\to 0}{=} 0 \\ \Gamma^{(2)} \geq 0 \end{array}\right\} \Rightarrow \Gamma \text{ is a minimum.}$$

But

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(z-a)^n \qquad \text{around } a$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \, z^n \qquad \text{around } 0$$

$$\Rightarrow \Gamma\{\bar{\Phi}\} = \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \cdots dx_N \, \underbrace{\Gamma^{N}(x_1, \dots, x_N)}_{\text{in the symmetric theory!}} \, \bar{\Phi}(x_1) \cdots \bar{\Phi}(x_N)$$

$\Rightarrow$ the classical field $\bar{\Phi}$ is det. by $\dfrac{\delta \Gamma}{\delta \bar{\Phi}} = 0$

and if $\bar{\Phi} \neq 0 \Rightarrow$ the symmetry is spontaneously broken.

Moreover for $\Phi = \text{const} = \bar{\Phi}$

$$\Rightarrow \Gamma = \sum_{N=2}^{\infty} \frac{1}{N!} \left[\int dx_1 \cdots dx_N \; \Gamma^{(N)}(x_1, \dots, x_N) \right] \bar{\Phi}^N$$

or

$$\Gamma^N(x_1, \dots, x_N) = \int \frac{d^dk_1}{(2\pi)^d} \cdots \frac{d^dk_N}{(2\pi)^d} \; \Gamma(k_1, \dots, k_N) \, e^{-i k_j \cdot x_j}$$

$$\Gamma(k_1, \dots, k_N) = (2\pi)^d \, \delta^d(\Sigma k_j) \; \bar{\Gamma}(k_1, \dots, k_N)$$

$$\Rightarrow \Gamma(\bar{\Phi}) = \sum_{N=2}^{\infty} \frac{1}{N!} \; \bar{\Gamma}^{(N)}(0, \dots, 0) \; \bar{\Phi}^N \underbrace{(2\pi)^d \, \delta^d(0)}_{= V}$$

$$\Rightarrow \Gamma(\bar{\Phi}) = (2\pi)^d \, \delta^d(0) \; \underbrace{U(\bar{\Phi})}_{\text{"potential}}$$

$$U(\bar{\Phi}) = \sum_{N=2}^{\infty} \frac{1}{N!} \; \bar{\Gamma}^{(N)}(0, \dots, 0) \; \bar{\Phi}^N$$

<u>effective potenti</u>

Note that the $\bar{\Gamma}^{(N)}(0, \dots, 0)$'s <u>are computed in the symmetric theory</u>

In this framework there will be symmetry breaking if U has a minimum at $\bar{\Phi} \neq 0$

If we identify $J(x) \equiv H$ the external phys. field

$$\Rightarrow \cancel{\frac{\delta \Gamma}{\delta \bar{\Phi}} \equiv \frac{d\Gamma}{d\bar{\Phi}} = J = H} \qquad \frac{\delta \Gamma}{\delta \bar{\Phi}} = J(x)$$

$$\Rightarrow \cancel{\frac{dU}{d\bar{\Phi}}} \quad \boxed{\frac{dU}{d\bar{\Phi}} = H}$$

$$\Rightarrow \quad H = \sum_{N=1}^{\infty} \frac{1}{N!} \; N \; \bar{\Gamma}^{(N)}(0,\dots,0)\, \bar{\Phi}^{N-1}$$

$$\Rightarrow \boxed{H = \sum_{N=0}^{\infty} \frac{1}{N!} \bar{\Gamma}^{(N+1)}(0,\dots,0) \; \bar{\Phi}^{N}}$$ is the equation of state

Thus we first must compute the effective potential and ~~Φ~~ determine from it the vacuum (gnd. state). Next one computes the full vertex functions, either in the symmetric or ~~sym~~ broken symmetry state, ~~by expanding in $\Gamma\{\bar\Phi\}$~~ by identifying in $\Gamma\{\bar\phi\}$ the coeff. of $(\bar\phi(x_1) - v)\;(\bar\phi(x_2) - v)\dots$ where ~~the~~ v is the classical field which minimizes $V(\bar\Phi)$, i.e.

$$\Gamma^{(N)}(1,\dots,N; v) = \left.\frac{\delta^N \Gamma\{\bar\phi\}}{\delta\bar\phi(1)\cdots\delta\bar\phi(N)}\right|_{\bar\phi = v}$$

* Lecture 14 (2/16)

Ward Identities

I want to discuss now the ~~effects~~ consequences of the existence of a continuous symmetry (say $O(2)$);

Let $\vec{\phi}(x) = \begin{bmatrix} \pi(x) \\ \sigma(x) \end{bmatrix}$

$$\mathcal{L}(\phi) = \frac{1}{2}\left[(\nabla\vec\phi)^2 + m_0^2 \vec\phi^{\,2}\right] + \frac{\lambda}{4!}\left(\vec\phi^{\,2}\right)^2$$

and $\vec\phi\,' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec\phi \equiv T\vec\phi$ is an invariance

of $\theta = $ const. $\quad \mathcal{L}(\vec\phi\,') = \mathcal{L}(\vec\phi)$

For θ infinitesimal (ϵ)

$$T = \mathbb{1} + \epsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The generating functional is invariant if the sources are rotated accordingly $\vec{J}' = T\vec{J}$ since T is orthogonal $\Rightarrow$ $\vec{J}\cdot\vec{\Phi}$ is invariant. Measure is invariant

$$\Rightarrow \quad \vec{J}' = \vec{J} + \epsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} J_\pi \\ J_\sigma \end{pmatrix} \Rightarrow$$

$$\begin{cases} J'_\pi = J_\pi - \epsilon J_\sigma \\ J'_\sigma = J_\sigma + \epsilon J_\pi \end{cases} \quad \text{etc.} \qquad \Rightarrow \delta J_\pi = -\epsilon J_\sigma$$

$$\delta J_\sigma = +\epsilon J_\pi$$

Since F is invariant, we have

$$\delta F = \int d^dx \left[\frac{\delta F\{\vec{J}\}}{\delta J_\sigma(x)} \delta J_\sigma(x) + \frac{\delta F\{\vec{J}\}}{\delta J_\pi(x)} \delta J_\pi(x) \right] = 0$$

i.e.

$$\delta F = \int d^dx \, \epsilon \left[\frac{\delta F\{\vec{J}\}}{\delta J_\sigma(x)} J_\pi(x) - \frac{\delta F\{\vec{J}\}}{\delta J_\pi(x)} J_\sigma(x) \right] = 0$$

or

$$\int d^dx \left[\bar{\Phi}_\sigma(x) J_\pi(x) - \bar{\Phi}_\pi(x) J_\sigma(x) \right] = 0$$

$$\boxed{\int d^dx \left[\bar{\Phi}_\sigma(x) \frac{\delta\Gamma\{\bar{\Phi}\}}{\delta\bar{\Phi}_\pi(x)} - \bar{\Phi}_\pi(x) \frac{\delta\Gamma\{\bar{\Phi}\}}{\delta\bar{\Phi}_\sigma(x)} \right] = 0}$$ W-I for the G.F.

which says that $\Gamma\{\bar{\Phi}\}$ is invariant under $\vec{\Phi} \to T\vec{\Phi}$. This identity is always valid.

$$\frac{\delta}{\delta \bar\Phi_\pi(y)}\left[\int dx^d \left\{ \frac{\delta \Gamma}{\delta \bar\Phi_\pi} \bar\Phi_\sigma - \frac{\delta \Gamma}{\delta \bar\Phi_\sigma} \bar\Phi_\pi \right\}\right] = 0$$

$$\int dx^d \left\{ \frac{\delta^2 \Gamma}{\delta \bar\Phi_\pi(y)\, \delta \bar\Phi_\pi(x)} \bar\Phi_\sigma(x) - \frac{\delta^2 \Gamma}{\delta \bar\Phi_\sigma(x)\, \delta \bar\Phi_\pi(y)} \bar\Phi_\pi(x) - \right.$$

$$\left. - \frac{\delta \Gamma}{\delta \bar\Phi_\sigma(x)} \delta^d(x-y) \right\} = 0$$

$$\int dx^d \left[\frac{\delta^2 \Gamma}{\delta \bar\Phi_\pi(x)\, \delta \bar\Phi_\pi(y)} \bar\Phi_\sigma(x) - \frac{\delta^2 \Gamma}{\delta \bar\Phi_\sigma(x)\, \delta \bar\Phi_\pi(y)} \bar\Phi_\pi(y) \right] = \frac{\delta \Gamma}{\delta \bar\Phi_\sigma(x)}$$

If the symmetry is broken, say $\bar\Phi = \begin{bmatrix} 0 \\ u \end{bmatrix}$,

$$\Rightarrow \quad u \int dx^d \frac{\delta^2 \Gamma}{\delta \bar\Phi_\pi(x)\, \delta \bar\Phi_\pi(y)} = J_\sigma$$

then if $\begin{pmatrix} J_\pi \\ J_\sigma \end{pmatrix} \to 0 \quad \frac{\delta^2 \Gamma}{\delta \bar\Phi_\pi(x)\, \delta \bar\Phi_\pi(y)} \to \Gamma^{(2)}_{\pi\pi}(x-y)$

$$\Rightarrow \quad \boxed{u \int dx^d\, \Gamma^{(2)}_{\pi\pi}(x-y) = 0} \quad \text{or} \quad \boxed{u \int dx^d\, \Gamma^{(2)}_{\pi\pi}(x-y) = H}$$

$J \to 0 \qquad\qquad \vec J = \begin{bmatrix} 0 \\ H \end{bmatrix}$

$$\Rightarrow \quad \lim_{\vec p \to 0} u\, \bar\Gamma^{(2)}_{\pi\pi}(p) = H$$

$$\Rightarrow \quad \bar\Gamma^{(2)}_{\pi\pi}(p) = \frac{H}{u} \quad \Rightarrow \quad (\text{if } u \neq 0)\ \bar\Gamma^{(2)}_{\pi\pi} \to 0 \text{ as } H \to 0$$

$\Rightarrow$ $G^{(c)}_{2,\pi\pi}(p)$ has a pole at zero momentum in the broken phase.
this is the famous Goldstone boson (and then.)

Thus we discover that there is an alternative: either

(a) ~~the~~ the theory is symmetric, i.e. ~~[illegible]~~ $u=0$

or

(b) the symmetry is spontaneously broken, $u \neq 0$ with $J \to 0$, and there are massless excitations (Nambu–Goldstone Boson

In the $O(N)$ case we have $\frac{N(N-1)}{2}$ generators

$$(L_{ij})_{kl} = -i\,[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]$$

If the symmetry is spontaneously broken, we have

$$\Phi = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u \end{bmatrix} \qquad \text{in general} \quad \Phi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_{N-1} \\ \sigma \end{bmatrix} = \begin{bmatrix} \vec{\pi} \\ \sigma \end{bmatrix}$$

Obviously the $N-1$ components have a symmetry $O(N-1)$. Thus the symmetry which is actually broken is

$$O(N)/O(N-1) \equiv S_N \quad (N\text{-dim sphere}) \text{ rather than } O(N)$$

$$\Rightarrow \qquad \Phi'_a = (e^{i\vec{\lambda}\cdot\vec{L}})_{ab}\,\Phi_b \qquad (\lambda_{ij} = -\lambda_{ji})$$

The broken generators are L_{in} and L_{ij} (with $i,j \neq n$) are the generators of the unbroken $O(N-1)$ symmetry. Let's act with an infinitesimal transf. with the L_{in}'s

$$\Phi'_a = (e^{i\lambda_{in}L_{in}})_{ab}\,\Phi_b$$

$$\delta\Phi_a = i\,\lambda_{in}(L_{in})_{ab}\,\Phi_b$$

$$\delta\Phi_a = \lambda_{in}\,[\delta_{ia}\delta_{nb} - \delta_{ib}\delta_{na}]\,\Phi_b$$

$$\delta \Phi_a = \lambda_{an} \Phi_n - \lambda_{bn} \delta_{na} \Phi_b \qquad \lambda_{nn} = 0$$

$$\Rightarrow \quad \delta\sigma = \cancel{\&} - \lambda_{bn} \pi_b$$

$$\delta \pi_a = \lambda_{an} \sigma \qquad \Rightarrow \qquad \boxed{\begin{aligned} \delta\sigma &= \lambda_{nb} \pi_b \\ \delta \pi_a &= -\lambda_{na} \sigma \end{aligned}}$$

Likewise

$$\delta J_\sigma = - \lambda_{nb} J_{\pi_b}$$

$$\delta J_{\pi_a} = + \lambda_{na} J_\sigma$$

$$\delta F = 0 = \int dx^d \left[\frac{\delta F}{\delta J_\sigma(x)} \delta J_\sigma + \frac{\delta F}{\delta J_{\pi_a}(x)} \delta J_{\pi_a}(x) \right]$$

$$0 = \int dx^d \left[- \lambda_{nb} \, \sigma(x) \, J_{\pi_b}(x) + \lambda_{na} J_\sigma(x) \pi_a(x) \right]$$

$$0 = \int dx^d \, \lambda_{na} \left[J_\sigma(x) \, \pi_a(x) - \sigma(x) \, J_{\pi_a}(x) \right]$$

$$\boxed{0 = \int dx^d \, \lambda_{na} \left[\frac{\delta \Gamma}{\delta \sigma(x)} \pi_a(x) - \frac{\delta \Gamma}{\delta \pi_a(x)} \sigma(x) \right]}$$

Since λ_{na} arbitrary $\Rightarrow$

$$0 = \int dx^d \left[\frac{\delta \Gamma}{\delta \sigma(x)} \pi_a(x) - \frac{\delta \Gamma}{\delta \pi_a(x)} \sigma(x) \right]$$

$$0 = \frac{\delta}{\delta \pi_b(y)} \int dx^d \left[\quad \right] \Rightarrow \quad 0 = \int dx^d \left[\frac{\delta^2 \Gamma}{\delta \sigma(x) \delta \pi_b(y)} \pi_a + \frac{\delta \Gamma}{\delta \sigma} \delta_{ab} \delta(x-y) - \frac{\delta^2 \Gamma}{\delta \pi_a(x) \, \delta \pi_b(y)} \sigma(x) \right]$$

If $\Phi = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \Rightarrow$ $\quad \langle \sigma \rangle = u$

$$\delta_{ab} \frac{\delta \Gamma}{\delta \sigma(y)} = \int d^dx \; \frac{\delta^2 \Gamma}{\delta \pi_a(x)\, \delta \pi_b(y)} \; u$$

$$\Rightarrow \boxed{\delta_{ab}\, J_\sigma = u \lim_{p \to 0} \Gamma^{(2)}_{\pi_a \pi_b}(p)}$$

$\Rightarrow$ (1) $\Gamma_{\pi_a \pi_b}$ must be diagonal $\quad \Gamma^{(0)}_{\pi_a \pi_b} = \delta_{ab}\, \Gamma_{\pi\pi}(0)$

and all masses $m^2_{\pi_a} = m^2_{\pi_b}$ (degenerate multiplet)

(2) $J_\sigma \to 0 \Rightarrow u \lim_{p \to 0} \Gamma_{\pi_a \pi_a}(p) = 0$

$\Rightarrow u \neq 0$ all π excitations are massless $\Rightarrow$ $N-1$ Goldstone bosons.

or $u = 0$ and the theory is symmetric.

These results are valid order by order in P.T.

We can get, in fact, an ∞ set of identities. Example: $(N=2)$

$$\frac{\delta}{\delta\sigma(z)} \int dx \left[\sigma(x)\, \Gamma_{\pi\pi}(x,y) - \delta(x-y)\, \Gamma_\sigma(x) - \pi(x)\, \Gamma_{\sigma\pi}(x,y) \right] = 0$$

1st: $\int dx \left[\delta(x-z)\, \Gamma_{\pi\pi}(x,y) + \sigma(x)\, \Gamma^{(3)}_{\pi\pi\sigma}(x,y,z) - \delta(x-y)\, \Gamma_{\sigma\sigma}(x,z) \right.$

$$\left. - \pi(x)\, \Gamma_{\sigma\pi\sigma}(x,y,z) \right] = 0$$

$$\Rightarrow \Gamma_{\pi\pi}(z,y) - \Gamma_{\sigma\sigma}(y,z) + u \int dx\; \Gamma_{\pi\pi\sigma}(x,y,z) = 0$$

$$\Rightarrow \boxed{\Gamma^{(2)}_{\sigma\sigma}(p) - \Gamma^{(2)}_{\pi\pi}(p) = u\, \Gamma_{\pi\pi\sigma}(0, p, -p)}$$

$\Rightarrow u = 0 \Rightarrow \Gamma_{\sigma\sigma}(p) = \Gamma_{\pi\pi}(p)$ (symmetric) $\wedge$ $u \neq 0 \Rightarrow \Gamma^{(2)}_{\sigma\sigma}(0) = u\, \Gamma^{(3)}_{\pi\pi\sigma}(0,0,0)$

Also (taking 2 more derivatives $\frac{\delta}{\delta\sigma(t)}\frac{\delta}{\delta\sigma(w)}$)

$$\Gamma^{(4)}_{\pi\pi\sigma\sigma}(z,y,t,w) + \Gamma^{(4)}_{\pi\pi\sigma\sigma}(w,y,z,t) + \Gamma^{(4)}_{\pi\pi\sigma\sigma}(t,y,z,w) = \Gamma^{(4)}_{\sigma\sigma\sigma\sigma}(y,z,t,$$

$\Rightarrow$ the F.T. at a sym. point satisfies (e.g. all p's $= 0$)

$$3\,\Gamma^{(4)}_{\pi\pi\sigma\sigma}\Big|_{S.P.} = \Gamma_{\sigma\sigma\sigma\sigma}\Big|_{S.P.}$$

Thus rotational invariance is guaranteed.