14
Quantum Field Theory and Statistical Mechanics

Until now we followed the standard approach of defining the quantum field theory in terms of its perturbative Feynman diagrammatic expansion around the free field theory. There we saw that each Feynman diagram needs to be regularized in the UV. In other words, the theory needs a *definition* in the UV. In this chapter we will take a detour into Statistical Mechanics which has a natural UV definition (as it is often defined on a lattice). We will see that the perspective from Statistical Mechanics offers a way to define the quantum field theory without the use of perturbation theory. In chapter 15 we will introduce the framework of the Renormalization Group which will make precise how to define a continuum field theory. We will return to renormalized perturbation theory in Chapter 16.

We have seen in earlier chapters that there is a connection between quantum field theory in D space-time (at $T = 0$) and classical statistical mechanics in D Euclidean dimensions. We will now explore this correspondence further. This correspondence has profound consequences for the physics of both quantum field theory and statistical mechanics, particularly in the vicinity of a continuous phase transition.

In Section 2.8 we used heuristic arguments to show that the partition function of an Ising model in D dimensions can be represented by the path integral of a continuum ϕ^4 scalar field theory in a D-dimensional Euclidean space-time. There are, however, some important differences. Since the configurations of the Ising model cannot vary on scales smaller than the lattice spacing, the expectation values of the physical observables do not have the UV divergencies discussed in Chapter 11. In other words, the Ising model can be regarded as a Euclidean scalar field theory with a lattice regularization. In addition, the microscopic degrees of freedom of the Ising model, the spins, obey the constraint $\sigma^2 = 1$. Instead, the fields of a scalar field theory are unconstrained but in the path integral its configurations are weighed by the potential $V(\phi) = \frac{g}{4!}\phi^4$.

The purpose of this chapter is not to present a complete theory of phase transitions, for which there are many excellent texts, such as the books by Cardy (Cardy, 1996), Goldenfeld (Goldenfeld, 1992), and Amit (Amit, 1980), among many other excellent books. Our goal is to present the perspective This perspective, that quantum field theories can be defined in terms of regularized theories near a phase transition, was formulated and developed by K. G. Wilson (Wilson, 1983). The connection between Statistical Mechanics and Quantum Field Theory is the key to this framework. To this end, in this chapter we will revisit the Ising model and discuss its phase transition in some detail. We will see that its behavior near its phase transition provides an example of the definition of a quantum field theory.

14.1 The Ising Model as a path integral

Let us consider a typical problem in equilibrium statistical mechanics. For the sake of simplicity, we will consider the Ising model in D dimensions. The arguments which we will give below are straightforward to generalize to other cases.

Let us consider a D-dimensional hypercubic lattice of unit spacing. At each site $\boldsymbol{r}$ there is an Ising spin variable $\sigma(\boldsymbol{r})$, each of them taking two possible values, ± 1. We will refer to the state of this lattice (or *configuration*) $[\sigma]$ to a particular set of values of this collection of variables. Let $H[\sigma]$ be the classical energy ($j = 1, ..., D$)

$$H[\sigma] = -J \sum_{\boldsymbol{r}, j=1,...D} \sigma(\boldsymbol{r})\sigma(\boldsymbol{r} + \hat{\boldsymbol{e}}_j) - h \sum_{\boldsymbol{r}} \sigma(\boldsymbol{r}) \tag{14.1}$$

where J is the exchange constant, h represents the Zeeman coupling to an external uniform magnetic field of the Ising spins, and T is the temperature. The partition function Z is (upon setting $k_B = 1$ the sum over the configuration space $[\sigma]$ of the Gibbs weight for each configuration,

$$Z = \sum_{[\sigma]} e^{-H[\sigma]/T} \tag{14.2}$$

We will assume that the system is on a hypercubic lattice and obeys periodic boundary conditions, and hence that the Euclidean space-time is a torus in D dimensions. We will also assume that the interactions are ferromagnetic and hence $J > 0$. Clearly the partition function is a function of the ratios J/T and h/T. The discussion that follows can be easily extended to any other lattice provided the interactions are ferromagnetic and for antiferromagnetic ($J < 0$) interactions if the lattice is bipartite. We will not consider the interesting problem of frustrated lattices.

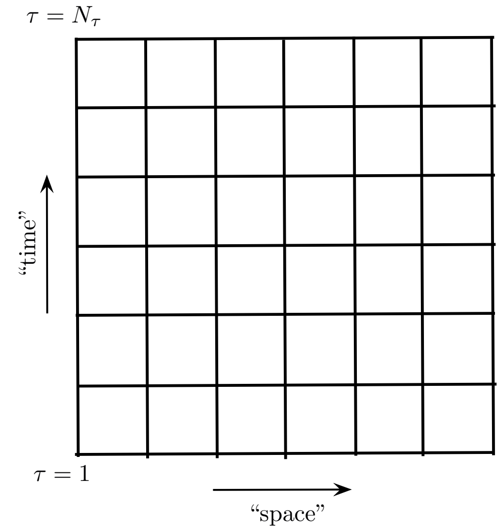

Figure 14.1 The Euclidean space-time lattice

Let us regard one of the spacial dimensions, labelled by D, as "time" or, more precisely, as imaginary time. Hence, we pick the dimension D as the "time dimension". Then we have $D-1$ "space dimensions" and one "time dimension." This is clearly an arbitrary choice given the symmetries of this problem. Thus, each hyperplane of the hypercubic lattice will be labeled by an integer-valued variable $\tau = 1, \ldots, N_\tau$, where N_τ is the linear size of the system along the direction τ (see Fig.14.1).

In this picture, we can regard each configuration $[\sigma]$ as the imaginary time evolution of an initial configuration at some initial imaginary time $\tau = 1$, i.e. the configuration on the first "row" or hyperplane, in time τ, as we go from row to row along the "time" direction, all the way to $\tau = N$. The integers τ will be regarded as a discretized imaginary time. The partition function is thus interpreted as a sum over the histories of these spin configurations and, hence, it is a path integral with a discretized imaginary (Euclidean) time.

14.2 The Transfer Matrix

Let us now show that the partition function can be written in the form of a *trace* of a matrix, known as the *Transfer Matrix* (see, e.g. (Schultz et al., 1964)),

$$Z = \sum_{[\sigma]} e^{-H[\sigma]/T} \equiv \text{tr}\, \hat{T}^N \tag{14.3}$$

where N is the number of rows or, in general, hyperplanes. To this end we define a complete set of states $\{|\sigma\rangle_\tau\}$ on each row (or hyperplane) labelled by $\tau = 1, ..., N$. Since each row contains N^{D-1} sites and, for the Ising model, we have two states per site (the two spin projections), the number of states in this basis is $2^{N^{D-1}}$. This concept is easily generalizable to systems with other types degrees of freedom.

Let us find an explicit expression for the matrix $\hat{T}$. In particular, we will look for a transfer matrix $\hat{T}$ with the factorized form

$$\hat{T} = \hat{T}_1^{1/2}\, \hat{T}_2\, \hat{T}_1^{1/2} \tag{14.4}$$

It turns out that there exists a large number of interesting problems in equilibrium classical statistical mechanics for which the transfer matrix $\hat{T}$ can be *chosen* to be a *hermitian* matrix. From the point of view of classical statistical mechanics, this follows (in part) from the fact that the Boltzmann weights are positive real numbers. This condition is however not sufficient. We will show bellow that there is a sufficient condition known as *reflection positivity*, which is the Euclidean equivalent of unitarity in quantum mechanics and quantum field theory.

For the particular case of the Ising model, it is possible to write the matrices $\hat{T}_1$ and $\hat{T}_2$ in terms of a set of real Pauli matrices $\hat{\sigma}_1(\boldsymbol{r})$ and $\hat{\sigma}_2(\boldsymbol{r})$ defined on each site $\boldsymbol{r}$ of the row which act on the states defined on each row. For the case at hand, after some algebra we find

$$\hat{T}_2 = \exp\left\{\frac{J_s}{T}\sum_{\boldsymbol{r},j}\hat{\sigma}_3(\boldsymbol{r})\hat{\sigma}_3(\boldsymbol{r}+\hat{\boldsymbol{e}}_j) + \frac{h}{T}\sum_{\boldsymbol{r}}\hat{\sigma}_3(\boldsymbol{r})\right\} \tag{14.5}$$

$$\hat{T}_1 = \left[\frac{1}{2}\ \sinh\left(\frac{2J_\tau}{T}\right)\right]^{N_s/2} \exp\left\{b\sum_{\boldsymbol{r}}\hat{\sigma}_1(\boldsymbol{r})\right\} \tag{14.6}$$

where N_s is the number of sites in each hyperplane (row), and $j = 1, \ldots D-1$. Here we have assumed that J_s and J_τ, the coupling constants along the "space" and "time" directions are not necessarily equal to each other, and the parameter b is given by

$$e^{-2b} = \tanh(J_\tau/T) \tag{14.7}$$

The correlation functions of the Ising model, for spins located at sites

$R = (\boldsymbol{R}, \tau)$ and $R' = (\boldsymbol{R}', \tau')$,

$$\left\langle \sigma(R)\ \sigma(R') \right\rangle = \frac{1}{Z} \sum_{[\sigma]} \sigma(R)\sigma(R') e^{-H[\sigma]} \tag{14.8}$$

can be expressed in terms of the transfer $\hat{T}$ in the suggestive form

$$\begin{aligned} \left\langle \sigma(R)\ \ \sigma(R') \right\rangle &= \frac{1}{Z}\ \mathrm{tr}\left(\hat{T}^{\tau} \hat{\sigma}_3(\boldsymbol{R}) \hat{T}^{\tau' - \tau} \hat{\sigma}_3(\boldsymbol{R}') \hat{T}^{\tau - \tau'} \right) \\ &\equiv \langle T[\hat{\sigma}_3(\boldsymbol{R}, \tau)\ \hat{\sigma}_3(\boldsymbol{R}', \tau')] \rangle \end{aligned} \tag{14.9}$$

where T is the imaginary-time time ordering symbol. Here we introduced the (pseudo) Heisenberg representation of the spin operators

$$\hat{\sigma}_3(\boldsymbol{R}, \tau) \equiv \hat{T}^{\tau} \hat{\sigma}_3(\boldsymbol{R}) \hat{T}^{-\tau} \tag{14.10}$$

Let $\{\lambda_n\}$ be the eigenvalues of the eigenstates $|n\rangle$ of the (hermitian) transfer matrix $\hat{T}$

$$\hat{T}|n\rangle = \lambda_n |n\rangle \tag{14.11}$$

Since the transfer matrix is hermitian, its eigenvalues are real numbers. Moreover since the transfer matrix is also real and symmetric, its eigenvalues are *positive* real numbers, $\lambda_n > 0$ (for all n).

14.3 Reflection Positivity

Let us now consider the case in which the correlation function is computed along the imaginary time coordinate only, and the spatial coordinates are set to be equal, $\boldsymbol{R} = \boldsymbol{R}'$. By following the line of argument that we used in Section 5.7 for the computation of the correlators in quantum field theory, we can formally compute the correlation function as a sum over the matrix elements of the operators on the eigenstates of the transfer matrix. In the thermodynamic limit, $N_\tau \to \infty$ and $N_s \to \infty$ the result simplifies to the following expression (compare with Eq.(5.198))

$$\begin{aligned} \langle \sigma(\boldsymbol{R}, \tau)\sigma(\boldsymbol{R}, \tau') \rangle &= \lim_{N_t \to \infty} \frac{1}{Z} \mathrm{tr}\Big(T[\hat{\sigma}_3(\boldsymbol{R}, \tau)\hat{\sigma}_3(\boldsymbol{R}, \tau')] \Big) \\ &= \sum_n |\langle G|\hat{\sigma}_3|n\rangle|^2 \left(\frac{\lambda_n}{\lambda_{\max}} \right)^{\tau' - \tau} \end{aligned} \tag{14.12}$$

where $|G\rangle$ is the eigenstate of the transfer matrix with *largest* eigenvalue $\lambda_{\max}$. A comparison with the analogous expression of Eq.(5.198) suggests that the quantity ξ defined by

$$\xi^{-1} = \ln\left(\frac{\lambda_{\max}}{\lambda_{n_0}}\right) \tag{14.13}$$

be the *correlation length* of the system. Here $|n_0\rangle$ is the eigenstate of the transfer matrix with the largest possible eigenvalue, λ_{n_0}, that is mixed with the state $|G\rangle$ by the spin operator $\hat{\sigma}_3(\boldsymbol{r})$.

From the result of Eq.(14.12) it is apparent that the correlation function is real and positive. A similar result can be derived for the correlation function on two general coordinates (not necessarily along a given axis). Hence, if the matrix $\hat{T}$ is hermitian the correlation functions are positive.

There is a natural and important interpretation of this result in terms of a Hilbert space which will allow us to find a more precise connection between classical statistical mechanics and quantum field theory. The positivity requirement for the correlators can be relaxed to the condition of *reflection positivity* which states the following. Let $A_r[\sigma]$ be some arbitrary local operator localized near $r = (\boldsymbol{r}, \tau)$ and let us define the operation of the *reflection* across a hyperplane $\mathcal{P}$. Let $A_{\mathcal{P}r}[\sigma]$ be the same operator after reflection accross the hyperplane $\mathcal{P}$. Then the positivity requirement

$$\langle A_r[\sigma]\; A_{\mathcal{P}_r}[\sigma]\rangle \geq 0 \tag{14.14}$$

implies that the transfer matrix $\hat{T}$, defined in a direction *normal* to the hyperplane $\mathcal{P}$, must be a hermitian operator for all its eigenstates to have *positive norm.* This is the analog of unitarity in quantum field theory, i.e. the requirement that all the states in the spectrum must have positive norm which is a requirement for the probabilistic interpretation of quantum mechanics and of quantum field theory. This condition is clearly obeyed in the ferromagnetic Ising model and in large number of systems of physical interest. There are, however, many interesting problems in classical statistical mechanics which do not satisfy the condition of reflection positivity.

14.4 The Ising Model in the limit of extreme spatial anisotropy

Let us now go back to the Ising Model and consider the spatial symmetries of the correlation functions. If the coupling constants are equal, $J_\tau = J_s$, then the system is invariant under the point group symmetries of the hypercubic lattice. This means that the correlation functions have to have these symmetries as well. Now, if the correlation length is very large, much larger than the lattice constant, it should be possible to approximate the point-group symmetry by the symmetries of D-dimensional Euclidean rotations.

Hence, in this limit which as we will see means that the system is close to a continuous phase transition, the hypersurfaces of equal correlation, on which the correlators take constant values, are, to an excellent approximation, the boundary of a D-dimensional hypersphere, S_D.

However, this line of reasoning also means that it should be possible to change the system by smoothly deforming it away from isotropy to a spatially anisotropic system, by changing J_τ and J_s away from the isotropy condition, without changing its properties in any essential way. Thus if we shear the lattice, by increasing the coupling J_τ and decreasing the coupling J_s in a suitable way, all that must change is that the hypersurfaces of equal correlation must become hyper-ellipsoids. In such process of deformations, the correlation length increases in lattice units along the direction we agreed to call time, and decreases along the orthogonal, space directions. We can now imagine compensating this deformation by adding additional hyperplanes so that the hypersurfaces of equal correlation become, to a desired degree of approximation, spherical once again (see Fig.14.2).

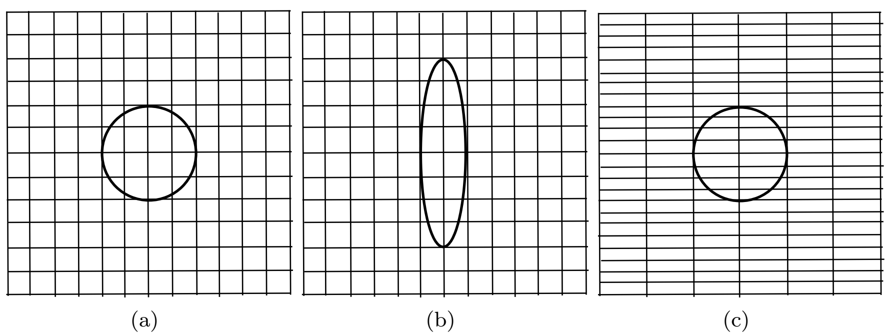

Figure 14.2 Curves of constant correlation and the time continuum limit: a) isotropic lattice with $J_\tau = J_s$, b) anisotropic lattice with $J'_\tau > J_\tau$ and $J'_s < J_s$ such that the correlation length is doubled along the (vertical) time axis and halved along the (horizontal) space axis, c) lattice with twice as many rows on the time direction such that the correlation lengths along the two orthogonal directions are now equal to each other.

If this process of deformation is repeated *ad infinitum*, the coupling constant along the time direction becomes very large, $J_\tau \to \infty$, and the coupling constant along the space directions becomes very small, $J_s \to 0$. In this limit, the time direction becomes continuous. This limit also requires that we tune down the magnetic field $h \to 0$.

In this limit, the transfer matrix takes a much simpler form

$$\hat{T}_1^{1/2}\,\hat{T}_2\,\hat{T}_1^{1/2} \approx e^{-\epsilon\hat{H}+O(\epsilon^2)} \tag{14.15}$$

where $\hat{H}$ is a local hermitian operator and ϵ is a suitably chosen small parameter. Also, since $\tanh b = e^{-2J_\tau/T}$, the limit of b small is the limit of $\frac{J_\tau}{T}$ large. Thus we write

$$\frac{J_s}{T} = g\,e^{-2J_\tau/T}, \qquad \frac{h}{T} = \bar{h}\,e^{-2J_\tau/T} \tag{14.16}$$

By choosing

$$\epsilon = e^{-2J_\tau/T} \tag{14.17}$$

we find that the operator $\hat{H}$ of Eq.(14.15) is given by

$$\hat{H} = -\sum_{\boldsymbol{r}} \hat{\sigma}_1(\boldsymbol{r}) - g\sum_{\boldsymbol{r},j} \hat{\sigma}_3(\boldsymbol{r})\hat{\sigma}_3(\boldsymbol{r}+\hat{\boldsymbol{e}}_j) - \bar{h}\sum_{\boldsymbol{r}} \hat{\sigma}_3(\boldsymbol{r}) \tag{14.18}$$

where g will be regarded as a coupling constant and $\bar{h}$ as a uniform longitudinal magnetic field.

In this picture "time" is continuous. The equivalence of the sequence of deformed systems holds provided that we require that

$$N_\tau e^{-2J_\tau/T} = \text{fixed} = \bar{\beta} \tag{14.19}$$

Thus, as $\frac{J_\tau}{T} \to \infty$ we should let $N_\tau \to \infty$. The fixed number $\bar{\beta}$ should be infinite if the original system is thermodynamically large along that direction. Thus

$$Z = \text{tr}\,\hat{T}^{N_\tau} \underset{N_\tau\to\infty}{\simeq} \text{tr}\,e^{-\bar{\beta}\hat{H}} \tag{14.20}$$

and we recognize $\bar{\beta}$ as the effective inverse temperature of a *quantum mechanical* system in $D-1$ dimensions whose Hamiltonian is given by Eq.(14.18). This quantum spin system is known as the Ising model in a transverse field (and a longitudinal field h). The reader should recognize that we have done is the reverse of the procedure that we did before to define a path integral: instead of discretizing the time variable we have made it continuous.

Further taking the parameter $\bar{\beta} \to \infty$ (to restore full symmetry) amounts to take the effective temperature of the equivalent system to zero. Hence, in the thermodynamic limit we get

$$Z = \lim_{N_\tau \infty} \text{tr}\hat{T}^{N_\tau} \to \lambda_{\max}^{N_\tau} = e^{-\bar{\beta}E_0} \tag{14.21}$$

where $\lambda_{\max}$ is the largest eigenvalue of the transfer matrix and E_0 is the lowest eigenvalue of $\hat{H}$, i.e. its ground state.

We conclude that, if we know the ground state energy of $\hat{H}$, or equivalently the largest eigenvalue of the transfer matrix $\hat{T}$, we know the partition function Z in the thermodynamic limit. This observation is the key to (one of the very many methods) to solve the Ising model in $D = 2$ dimensions.

In summary, we showed that the classical statistical mechanics of the Ising model in D dimensions is equivalent to the quantum mechanics of the Ising model in a transverse field in $D - 1$ space dimensions (Fradkin and Susskind, 1978). This mapping also holds to any classical models, with global or local symmetries, with real and positive Gibbs weights and, satisfy reflection positivity. This mapping is simply the lattice version of the path integral of the quantum theory.

14.5 Symmetries and Symmetry Breaking

14.5.1 Global $\mathbb{Z}_2$ Symmetry

Let us now discuss the symmetries of this system which are the same in all dimensions. The classical Ising model has (if the external field is set to zero, $h = 0$) the global spin-flip symmetry $[\sigma] \to [-\sigma]$. Thus, the classical Hamiltonian $H[\sigma]$ is invariant under the operations

$$\begin{aligned} I &: [\sigma] \mapsto [\sigma], && \text{(the identity)} \\ R &: [\sigma] \mapsto [-\sigma], && \text{(global spin flip)} \end{aligned} \tag{14.22}$$

These operations form a group since

$$I \star I = I, \qquad R \star R = I, \qquad I \star R = R \star I = R \tag{14.23}$$

where $\star$ denotes the composition of the two operations. This set of operations define the group $\mathbb{Z}_2$ of permutations of two elements.

The equivalent quantum problem has, as it should, exactly the same symmetries. The operators $\hat{I}$ and $\hat{R}$ act on the quantum states defined on the rows (or hyperplanes) precisely in the same way as in Eq.(14.23). We can construct these symmetry operators form the underlying operators of the lattice model quite explicitly. Let $\hat{I}(\boldsymbol{r})$ and $\hat{R}(\boldsymbol{r}) \equiv \hat{\sigma}_1(\boldsymbol{r})$ be the 2×2 identity matrix and the σ_1 Pauli matrix acting on the spin state at site $\boldsymbol{r}$. Then we can write for $\hat{I}$ and $\hat{R}$ the expressions

$$\hat{I} = \prod_{\boldsymbol{r}} \otimes \, \hat{I}(\boldsymbol{r}), \qquad \hat{R} = \prod_{\boldsymbol{r}} \otimes \, \hat{\sigma}_1(\boldsymbol{r}) \tag{14.24}$$

where $\otimes$ denotes the tensor product. These operators satisfy the obvious properties

$$\begin{aligned} \hat{I}^{-1} &= \hat{I}, & \hat{R}^{-1} &= \hat{R} \\ \hat{I}\hat{\sigma}_1(\boldsymbol{r})\hat{I}^{-1} &= \hat{\sigma}_1(\boldsymbol{r}), & \hat{R}\hat{\sigma}_1(\boldsymbol{r})\hat{R}^{-1} &= \hat{\sigma}_1(\boldsymbol{r}) \\ \hat{I}\hat{\sigma}_3(\boldsymbol{r})\hat{I}^{-1} &= \hat{\sigma}_3(\boldsymbol{r}), & \hat{R}\hat{\sigma}_3(\boldsymbol{r})R^{-1} &= -\hat{\sigma}_3(\boldsymbol{r}) \end{aligned} \tag{14.25}$$

From these properties it follows that, if $\bar{h} = 0$, the Hamiltonian is invariant under the $\mathbb{Z}_2$ symmetry,

$$[\hat{I}, \hat{H}] = [\hat{R}, \hat{H}] = 0 \tag{14.26}$$

14.5.2 Qualitative Behavior of the Ground State and Spontaneous Symmetry Breaking

We can write $\hat{H}$ as a sum of two terms $\hat{H} = \hat{H}_0 + \hat{V}$, with

$$\hat{H}_0 = -\sum_{\boldsymbol{r}} \hat{\sigma}_1(\boldsymbol{r}), \qquad \hat{V} = -g\sum_{\mathrm{r},j} \hat{\sigma}_3(\boldsymbol{r})\hat{\sigma}_3(\boldsymbol{r} + \hat{\boldsymbol{e}}_j) \tag{14.27}$$

where $j = 1, \dots, D-1$.

1. The symmetric phase, $g \ll 1$

Let us consider first the limit $g \ll 1$, which corresponds to the high temperature limit in the classical model in one higher dimension. If $g \ll 1$ we can study the properties of the ground state in perturbation theory in powers of g. The unperturbed ground state $|\Psi_0\rangle_0$ is

$$|\Psi_0\rangle_0 = \prod_{\boldsymbol{r}} \otimes\, |+, \boldsymbol{r}\rangle \equiv |+\rangle \tag{14.28}$$

Here $|+,\rangle$ is the eigenstate of $\hat{\sigma}_1(\boldsymbol{r})$ with eigenvalue $+1$. Since it is an eigenstate of σ_1 at every site $\boldsymbol{r}$, this ground state is non-degenerate and is invariant under global $\mathbb{Z}_2$ transformations

$$\hat{R}|\Psi_0\rangle_0 = |\Psi_0\rangle_0 \tag{14.29}$$

In this state,

$${}_0\langle\Psi_0|\hat{\sigma}_3(\boldsymbol{r})|\Psi_0\rangle_0 = 0 \tag{14.30}$$

since $\hat{\sigma}_3(\boldsymbol{r})$ is off-diagonal in the basis of eigenvectors of $\hat{\sigma}_1$. Similarly the equal-time correlation function, which is equivalent to the correlation function on a fixed row in the classical system in D dimensions, also vanishes in this state

$${}_0\langle\Psi_0|\hat{\sigma}_3(\boldsymbol{r})\ \hat{\sigma}_3(\mathbf{R}')|\Psi_0\rangle_0 = 0 \tag{14.31}$$

In perturbation theory in powers of g the ground state gets corrected order-by-order by virtual processes that consist of σ_1 spin flips. Using Brillouin-Wigner perturbation theory, the perturbed ground state, to lowest order, is given by

$$|\Psi\rangle_0 = |\Psi_0\rangle_0 + \hat{P}\frac{\hat{V}}{E_0 - \hat{H}_0}|\Psi\rangle_0 + \dots \tag{14.32}$$

where $\hat{P}$ projects the state $|\Psi_0\rangle_0$ out of the sum. For instance, for $D = 2$, the state $| + + \dots +\rangle$ gets perturbed by pairs of σ_1 spin flips

$$\hat{V}|\Psi_0\rangle = -g\sum_R \mathcal{A}(R)| + +\dots + - - +\dots+\rangle \tag{14.33}$$

where the flipped pair of spins are located at R, and $\mathcal{A}$ is a easily calculable prefactor.

At higher orders in g, we will get more pairs in flipped states. Moreover, at higher orders the individual spins of a pair get separated from each other. Hence in this picture we can regard the individual spin flips as "particles" that are created in pairs from the vacuum (the ground state). The perturbative expansion can then be pictured as the a set of worldlines of these pairs of particles which are, eventually, also annihilated in pairs and must be regarded as a set of *loops*. This picture of the expansion in powers of g is equivalent to the high temperature expansion of classical statistical mechanics of the Ising model which is also a theory of loops on the D-dimensional lattice.

What does this structure of the ground state imply for the correlation function?

$$C(\boldsymbol{R}, \boldsymbol{R}') = \frac{\langle\Psi_0|\hat{\sigma}_3(\boldsymbol{R})\ \hat{\sigma}_3(\boldsymbol{R}')|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{14.34}$$

Since $\sigma_3^2 = I$, we can write the product of two such spin operators at points $\boldsymbol{R}$ and $\boldsymbol{R}'$ as

$$\hat{\sigma}_3(\boldsymbol{R})\ \hat{\sigma}_3(\boldsymbol{R}') = \prod_{(\boldsymbol{r},\boldsymbol{r}')\in\Gamma_{\boldsymbol{R},\boldsymbol{R}'}} \hat{\sigma}_3(\boldsymbol{r})\ \hat{\sigma}_3(\boldsymbol{r}') \tag{14.35}$$

where $\boldsymbol{r}$ and $\boldsymbol{r}'$ are pairs of neighboring sites on an *arbitrary* path $\Gamma_{\boldsymbol{R},\boldsymbol{R}'}$ that goes from $\boldsymbol{R}$ to $\boldsymbol{R}'$. Thus, we can also write the correlation function as

$$C(\boldsymbol{R}, \boldsymbol{R}') = \frac{\langle\Psi_0|\prod_{(\boldsymbol{r},\boldsymbol{r}')\in\Gamma_{\boldsymbol{R},\boldsymbol{R}'}} \hat{\sigma}_3(\boldsymbol{r})\ \hat{\sigma}_3(\boldsymbol{r}')|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \tag{14.36}$$

This expression only picks up a non-zero contribution in perturbation theory in g if we go to an order n sufficiently high so that $n \gtrsim |\boldsymbol{R} - \boldsymbol{R}'|$, where $|\boldsymbol{R} - \boldsymbol{R}'|$ is the number of bonds on the shortest path $\Gamma_{\boldsymbol{R},\boldsymbol{R}'}$. This argument shows that the correlation function *decays exponentially fast* with distance as

$$C(\boldsymbol{R}, \boldsymbol{R}') \approx \text{const } \times e^{-|\boldsymbol{R}-\boldsymbol{R}'|/\xi} \tag{14.37}$$

where ξ is the correlation length. To lowest order in g, we find

$$\xi^{-1} \simeq \ln(1/g + \dots) \tag{14.38}$$

It turns out that the expansion in powers of g has a finite radius of convergence in all space dimensions, $D - 1$. Within the radius of convergence, this approximation is a strict upper bound for the correlation function $C(\boldsymbol{R}, \boldsymbol{R}')$. Hence, for all values of g small enough to be within the non-vanishing and finite radius of convergence, this system is in the a phase that is smoothly related to the unperturbed state found at $g = 0$. In this *entire phase*, the correlation functions decay exponentially with distance, characterized by a finite correlation length ξ.

This result suggests that in this phase there is a finite energy (mass) gap $G \sim 2 + O(g)$ which can be easily checked by calculating the gap in perturbation theory. We will see below that the proportionality of the mass gap and the inverse correlation length becomes asymptotically exact near the phase transition, where this system becomes effectively Lorentz invariant, with a suitably defined "speed of light".

2. The broken symmetry phase, $g \gg 1$

For $g \gg 1$, which is equivalent to the low temperature regime of the classical problem in one extra dimension, the roles of the unperturbed and perturbation terms of the Hamiltonian get switched. Hence, we rewrite the Hamiltonian as

$$\hat{H}_0 = -g \sum_{\mathbf{r}_{ij}} \hat{\sigma}_3(\boldsymbol{r})\hat{\sigma}_3(\mathbf{r} + \hat{\boldsymbol{e}}_j), \qquad \hat{V} = -\sum_{\boldsymbol{r}} \hat{\sigma}_1(\boldsymbol{r}) \tag{14.39}$$

The unperturbed ground state now must be an eigenstate of $\hat{\sigma}_3(\boldsymbol{r})$. The leading-order ground state is *doubly degenerate*,

$$|\Psi_\uparrow\rangle_0 \equiv |\uparrow, \dots, \uparrow\rangle, \qquad |\Psi_\downarrow\rangle_0 \equiv |\downarrow, \dots, \downarrow\rangle \tag{14.40}$$

where, for all $\boldsymbol{r}$,

$$\hat{\sigma}_3(\boldsymbol{r})|\Psi_\uparrow\rangle_0 = |\Psi_\uparrow\rangle_0, \qquad \hat{\sigma}_3(\boldsymbol{r})|\Psi_\downarrow\rangle_0 = -|\Psi_\downarrow\rangle_0 \tag{14.41}$$

If the system is finite, these states will mix in perturbation theory. Indeed, if the lattice has N_s spatial sites, the mixing will occur to order N_s in perturbation theory. But, if we take the thermodynamic limit $N_s \to \infty$ *first*, then there is no mixing to any *finite* order in perturbation theory. The obvious exception is when the space-time dimension is $D = 1$, for which the number of sites in the "row" is just one, $N_s = 1$. Thus, for $D > 1$, or $D \geq 2$, these two states belong to two essentially decoupled pieces of the Hilbert space. In the classical system the property we just described is known as broken ergodicity, in the sense that the system only explores half of the total number of possible configurations.

This is an example of *spontaneous symmetry breaking*. Although the Hamiltonian is invariant under $\mathbb{Z}_2$ global transformations, *in the thermodynamic limit* the ground state is not $\mathbb{Z}_2$ invariant. It is possible to prove rigorously, that for $D \geq 2$, the expansion in powers in $1/g$ is convergent and that it has a finite radius of convergence. Therefore, the behavior of all quantities obtained in the limit $g \to \infty$ will also hold beyond some large but finite value of the coupling constant g. In other words, for large enough g, there is a stable phase with a spontaneously broken $\mathbb{Z}_2$ global symmetry. For instance, it is easy to see that in this phase the energy gap between the ground state and the first excited state is $G(g) = \frac{2}{g} + O(1/g^2)$.

To illustrate the concept of spontaneous symmetry breaking we will calculate, to lowest orders in $1/g$, the local spontaneous magnetization, the expectation value of the local spin operator,

$$m = \langle \Psi_\uparrow | \hat{\sigma}_3(\boldsymbol{r}) | \Psi_\uparrow \rangle \tag{14.42}$$

To lowest order in $1/g$ we have

$${}_0\langle \Psi_\uparrow | \hat{\sigma}_3(\boldsymbol{r}) | \Psi_\uparrow \rangle_0 = +1, \qquad {}_0\langle \Psi_\downarrow | \hat{\sigma}_3(\boldsymbol{r}) | \Psi_\downarrow \rangle_0 = -1 \tag{14.43}$$

Clearly, in general the expectation value is a function of the coupling constant g, $m = f(g)$. In general, the correlation function has the form

$$C(\boldsymbol{R}, \boldsymbol{R}') = \langle \Psi_\uparrow | \hat{\sigma}_3(\boldsymbol{R}) \hat{\sigma}_3(\boldsymbol{R}') | \Psi_\uparrow \rangle = m^2(g) + C_{\text{conn}}(\boldsymbol{R} - \boldsymbol{R}') \tag{14.44}$$

where the connected part, denoted by $C_{\text{conn}}(\boldsymbol{R} - \boldsymbol{R}')$, decays exponentially fast at long distances. Hence, as $|\boldsymbol{R} - \boldsymbol{R}'| \to \infty$, the correlation function approaches (exponentially fast) the constant value $m^2(g)$, which is the square of the expectation value of the order parameter (the local magnetization).

14.6 Solution of the two-dimensional Ising Model

The solution of the two-dimensional Ising model on a square lattice by Lars Onsager in 1944 is one of the triumphs of theoretical physics of the twentieth century (Onsager, 1944). We will see that it will also help us to understand how to define a quantum field theory.

The arguments given in the preceding section tell us that the Ising Model on a square lattice is equivalent to the one-dimensional quantum system with Hamiltonian

$$H = -\sum_{n=-\frac{L}{2}+1}^{L/2} \sigma_1(n) - g \sum_{n=-\frac{L}{2}+1}^{L/2} \hat{\sigma}_3(n)\hat{\sigma}_3(n) \tag{14.45}$$

where $L = N_s$ is the number of sites along the one-dimensional space. We will consider here a system with periodic boundary conditions.

14.6.1 The Jordan-Wigner Transformation

A naive look at the Hamiltonian leaves us with a puzzle. We have been raised to think that quadratic Hamiltonians are trivial. So, why the fuzz? Isn't $\hat{H}$ bilinear in $\hat{\sigma}$'s? The problem is that, in spite of the fact that H is a bilinear form in $\hat{\sigma}$'s, it is not a free theory. It is trivial to check that the equation of motion for $\hat{\sigma}_3(\boldsymbol{r})$ is not linear since, in general dimension D, the equation of motion is

$$i\partial_t \hat{\sigma}_3(\boldsymbol{r}) = [\hat{\sigma}_3(\boldsymbol{r}), \hat{H}] = -2i\hat{\sigma}_2(\boldsymbol{r}) = -2\hat{\sigma}_3(r)\hat{\sigma}_1(\boldsymbol{r}) \tag{14.46}$$

and

$$i\partial_t \hat{\sigma}_1(\boldsymbol{r}) = [\hat{\sigma}_1(\boldsymbol{r}), \hat{H}] = +g\hat{\sigma}_3(\boldsymbol{r})\hat{\sigma}_1(\boldsymbol{r}) \sum_{j=1,\dots,D-1} \left(\hat{\sigma}_3(\boldsymbol{r} + \boldsymbol{e}e_j) + \hat{\sigma}_3(\boldsymbol{r} - \boldsymbol{e}_j)\right) \tag{14.47}$$

The reason behind the non-linearity is the fact that the (equal-time) commutation relations

$$\begin{aligned} [\hat{\sigma}_3(\boldsymbol{r}), \hat{\sigma}_3(\boldsymbol{r}')] &= [\hat{\sigma}_1(\boldsymbol{r}), \hat{\sigma}_1(\boldsymbol{r}')] = 0 \\ [\hat{\sigma}_3(\boldsymbol{r}), \hat{\sigma}_1(\boldsymbol{r}')] &= 0, \qquad \boldsymbol{r} \neq \boldsymbol{r}' \\ \{\hat{\sigma}_l(\boldsymbol{r}), \hat{\sigma}_k(\boldsymbol{r})\} &= 2\delta_{lk}, \qquad l, k = 1, 3 \end{aligned} \tag{14.48}$$

Hence, the commutation relations are not canonical. Instead, they seem to describe objects which are bosons on different sites but fermions on the same site. Alternatively they can be regarded as bosons with hard cores. Indeed, the raising and lowering operators

$$\hat{\sigma}^{\pm} = \frac{1}{2}\left(\hat{\sigma}_3 \mp i\hat{\sigma}_2\right) \tag{14.49}$$

act on the states $|\pm\rangle$, the eigenstates of $\hat{\sigma}_1$, as

$$\begin{aligned} \hat{\sigma}^+|+\rangle &= 0, & \hat{\sigma}^+|-\rangle &= |+\rangle \\ \hat{\sigma}^-|+\rangle &= |-\rangle, & \hat{\sigma}^-|-\rangle &= 0 \end{aligned} \tag{14.50}$$

and can be regarded as the creation and annihilation operators of some oscillator, but with the hard core constraint that the boson occupation number $\hat{n} = \hat{\sigma}^+\hat{\sigma}^-$ should only have eigenvalues 0 and 1 since

$$\hat{\sigma}^+\hat{\sigma}^-|+\rangle = |+\rangle, \qquad \hat{\sigma}^+\hat{\sigma}^-|-\rangle = 0 \tag{14.51}$$

If $D = 2$, the quantum problem has $d = 1$ space dimensions. It turns out that there is a very neat and useful transformation which will enable us to deal with this problem. This is the Jordan-Wigner transformation (Jordan and Wigner, 1928; Lieb et al., 1961). The key idea behind this transformation is that, in one-dimension only, hard-core bosons are equivalent to fermions! Qualitatively this is easy to understand. If the particles live on a line, the bosons cannot exchange their positions by purely dynamical effects since the hard-core condition forbids that possibility. Similarly, one-dimensional fermions cannot change their relative ordering as a result of their dynamics due to the Pauli principle. Thus, the strategy is to show that our problem is secretly a *fermion problem*. From now on, we will restrict our discussion to one (space) dimension.

Let us consider the kink creation operator $\hat{K}(n)$

$$\hat{K}(n) = \prod_{j=-\frac{L}{2}+1}^{n} (-\hat{\sigma}_1(j)) \tag{14.52}$$

The operator flips all of the spins to the left of site $n+1$. Clearly, when acting on the "high-temperature" (or $g \ll 1$) ground state $|+\rangle$ we get

$$\hat{K}(n)|+\rangle = (-1)^n|+\rangle \tag{14.53}$$

i.e. it acts as a counting operator in the phase with $g \ll 1$. But, in the "low temperature" ground state (i.e. for $g \gg 1$), we get instead

$$\hat{K}(n)|\uparrow \ldots\ldots\ldots\ldots \uparrow\rangle = |\downarrow \ldots \underset{n}{\downarrow}\underset{n+1}{\uparrow} \ldots \uparrow\rangle \tag{14.54}$$

This state is called a *kink* (or domain wall), or a *topological soliton*. Clearly $\hat{K}_n$ disturbs the boundary conditions for $g > g_c$, but it does not for $g < g_c$, where

$$\langle\Psi_0|\hat{K}(n)|\Psi_0\rangle \neq 0 \qquad (g < g_c) \tag{14.55}$$

Hence, the high-temperature phase (disordered) is a condensate of kinks. The operator $\hat{K}$ is also known as disorder operator (Kadanoff and Ceva, 1971; Fradkin and Susskind, 1978), since it takes a non-zero expectation value in a disordered phase.

We will now see that a clever combination of order (i.e. $\hat{\sigma}_3$) and disorder ($\hat{K}$) operators yields a Fermi field. Let us consider the operators

$$\hat{\chi}_1(j) = \hat{K}(j-1)\hat{\sigma}_3(j), \qquad \hat{\chi}_2(j) = i\hat{K}(j)\hat{\sigma}_3(j) \tag{14.56}$$

Since these operators are products of Pauli matrices, they trivially obey the following identities (for all j)

$$\hat{\chi}_1^\dagger(j) = \hat{\chi}_1(j), \qquad \hat{\chi}_2^\dagger(j) = \hat{\chi}_2(j), \qquad \hat{\chi}_1(j)^2 = \hat{\chi}_2(j)^2 = 1 \tag{14.57}$$

More generally, it is straightforward to show that they obey the algebra

$$\{\hat{\chi}_1(j), \hat{\chi}_1(j')\} = \{\hat{\chi}_2(j), \hat{\chi}_2(j')\} = 2\delta_{jj'}, \qquad \{\hat{\chi}_1(j), \hat{\chi}_2(j')\} = 0 \tag{14.58}$$

Therefore, the fields $\hat{\chi}_1(j)$ and $\hat{\chi}_2(j)$ are self-adjoint (i.e. real) fermions, known as Majorana fermions.

Let us define the complex (or Dirac) fermions $\hat{\psi}(j)$ and $\hat{\psi}^\dagger(j)$

$$\hat{\psi}^\dagger(j) \equiv \hat{K}(j-1)\hat{\sigma}^+(j), \qquad \hat{\psi}(j) \equiv \hat{K}(j-1)\hat{\sigma}^-(j) \tag{14.59}$$

where we used the projection operators $\hat{\sigma}^\pm$ (c.f. Eq.(14.49)). Equivalently, we can write

$$\hat{\psi} = \frac{1}{2}\left(\hat{\chi}_1 + i\hat{\chi}_2\right), \qquad \hat{\psi}^\dagger = \frac{1}{2}\left(\hat{\chi}_1 - i\hat{\chi}_2\right) \tag{14.60}$$

It is straightforward to check that the operators $\psi^\dagger(j)$ and $\psi(j)$ obey the canonical fermion algebra

$$\{\hat{\psi}(j), \hat{\psi}^\dagger(j')\} = \delta_{jj'}, \qquad \{\hat{\psi}(j), \hat{\psi}(j')\} = 0 \tag{14.61}$$

It is easy to invert the transformation of Eq.(14.59). We first observe that the local fermion number operator, $\hat{n}(j) = \hat{\psi}^\dagger(j)\hat{\psi}(j)$, is given by

$$\psi^\dagger(j)\psi(j) = \frac{1}{2}\left(1 + \hat{\sigma}_1(j)\right) \tag{14.62}$$

Hence,

$$-\hat{\sigma}_1(j) = -2\hat{\psi}^+(j)\hat{\psi}(j) + 1 \equiv 1 - 2\hat{n}(j) \tag{14.63}$$

Since the operator $\hat{n}(j)$ satisfies $\hat{n}^2(j) = \hat{n}(j)$, we can also write

$$-\hat{\sigma}_1(j) = e^{i\pi\hat{n}(j)} = e^{i\pi\hat{\psi}^\dagger(j)\psi(j)} \tag{14.64}$$

Therefore, the inverse Jordan-Wigner transformation is

$$\begin{aligned}\hat{\sigma}^{+}(j) &= e^{i\pi \sum_{l<j} \hat{\psi}^{\dagger}(l)\hat{\psi}(l)} \hat{\psi}^{\dagger}(j) \\ \hat{\sigma}^{-}(j) &= e^{i\pi \sum_{l<j} \hat{\psi}^{\dagger}(l)\hat{\psi}(l)} \hat{\psi}(j)\end{aligned} \tag{14.65}$$

and, similarly,

$$\begin{aligned}\hat{\sigma}_3(j) &= e^{i\pi \sum_{l<j} \hat{\psi}^{\dagger}(l)\hat{\psi}(l)} \quad (\hat{\psi}^{\dagger}(j) + \psi(j)) \\ \hat{\sigma}_2(j) &= e^{i\pi \sum_{l<j} \hat{\psi}^{\dagger}(l)\hat{\psi}(l)} \quad \frac{1}{i}(-\hat{\psi}^{\dagger}(j) + \psi(j))\end{aligned} \tag{14.66}$$

We can use these results to map the Hamiltonian of the spin system to a Hamiltonian for the equivalent Fermi system. For L even, we find

$$\begin{aligned}\hat{H} = -L + \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} 2\hat{\psi}^{\dagger}(j)\hat{\psi}(j) + g \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} (\hat{\psi}^{\dagger}(j) - \hat{\psi}(j))(\hat{\psi}^{\dagger}(j+1) + \hat{\psi}(j+1)) \\ + \text{boundary term}\end{aligned} \tag{14.67}$$

The boundary term is given by

$$\begin{aligned}-g\eta\hat{\sigma}_3\left(\frac{L}{2}\right)\hat{\sigma}_3\left(-\frac{L}{2}+1\right) = \\ -g\eta\hat{Q}\left(\hat{\psi}^{\dagger}\left(\frac{L}{2}\right) - \hat{\psi}\left(\frac{L}{2}\right)\right)\left(\hat{\psi}^{\dagger}\left(-\frac{L}{2}+1\right) + \hat{\psi}\left(-\frac{L}{2}+1\right)\right)\end{aligned} \tag{14.68}$$

where $\eta = \pm 1$ for periodic or anti-periodic boundary conditions for the spins. Here, Q is the operator

$$\hat{Q} = \hat{R} = \prod_{-\frac{L}{2}+1}^{\frac{L}{2}} (-\hat{\sigma}_1(j)) = e^{i\pi\hat{N}} \tag{14.69}$$

where $\hat{N}$ is the total number of fermions.

However, $\hat{N}$ does not commute with the Hamiltonian $\hat{H}$ and, hence, the total number of complex (Dirac) fermions is not conserved. This is apparent since the fermion Hamiltonian has terms that create and destroy fermions in pairs. Nevertheless the fermion *parity* operator $(-1)^F \equiv e^{i\pi\hat{N}}$ is conserved since it commutes with the Hamiltonian, $[(-1)^F, \hat{H}] = 0$. Thus, we can only specify if the fermion number is even or odd. This is natural in a system of Majorana fermions which, as such, do not have a conserved charge (since they are charge neutral) but, instead, the fermion parity is conserved.

If the spin operators obey periodic (or anti-periodic) boundary conditions,

$$\hat{\sigma}_3\left(\frac{L}{2}+1\right) = \eta\hat{\sigma}_3\left(-\frac{L}{2}+1\right) \tag{14.70}$$

with $\eta = \pm 1$, then the fermions obey the boundary conditions

$$\hat{\psi}\left(\frac{L}{2}+1\right) = \hat{Q}\;\eta\;\hat{\psi}\;\left(-\frac{L}{2}+1\right) \tag{14.71}$$

In particular, for periodic boundary conditions (PBCs) for the spins ($\eta = +1$), the fermions obey the boundary conditions

$$\hat{\psi}\left(\frac{L}{2}+1\right) = \hat{Q}\hat{\psi}\left(-\frac{L}{2}+1\right) \tag{14.72}$$

Hence, for an even number of fermions, $\hat{N}$ even ($Q = +1$), the fermions obey PBC's, but for $\hat{N}$ odd ($Q = -1$), they obey APBC's. Nevertheless, since the ground state energy for N odd is greater than the ground state energy for N even, $E_0^- > E_0^+$, we can work in the even sector. In this sense, the fermion-boson mapping is two-to-one.

14.6.2 Diagonalization

The fermion Hamiltonian is a bilinear form in Fermi fields. Thus, it should be diagonalizable by a suitable canonical transformation. Since fermion number is not conserved, this transformation cannot be just a Fourier transform. In the even sector

$$\hat{Q}|\Psi\rangle = |\Psi\rangle \tag{14.73}$$

the Fourier transform, for L even, is

$$\hat{\psi}(j) = \frac{1}{L}\sum_{k=-\frac{L}{2}+1}^{\frac{L}{2}} e^{i2\pi\frac{kj}{L}}\tilde{a}(k), \qquad \tilde{a}(k) = \sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} e^{-i2\pi\frac{kj}{L}}\hat{\psi}(j) \tag{14.74}$$

such that

$$\{\tilde{a}(k),\tilde{a}^\dagger(k')\} = L\delta_{k,k'}, \qquad \{\tilde{a}(k),\tilde{a}(k')\} = \{\tilde{a}^\dagger(k),\tilde{a}^\dagger(k')\} = 0 \tag{14.75}$$

In the thermodynamic limit, $L \to \infty$, the momenta $\frac{2\pi}{L}k \equiv k$ fill up densely the interval $(-\pi,\pi]$ (the first Brillouin zone). In this limit we find a representation of the Dirac delta function,

$$\delta(k) \equiv \lim_{L\to\infty}\frac{1}{2\pi}\sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} e^{ikj} = \lim_{L\to\infty}\frac{L}{2\pi}\delta_{k,0} \tag{14.76}$$

and $\lim_{k\to 0}\delta(k) = \frac{L}{2\pi}$. Also, in the thermodynamic limit we identify the operators $\tilde{a}_k \equiv \hat{a}(k)$, which obey the anticommutation relations

$$\{\hat{a}(k), \hat{a}^\dagger(k')\} = 2\pi\delta(k-k') \tag{14.77}$$

These definitions can be used to derive the identities

$$\begin{aligned}\lim_{L\to\infty}\sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \hat{\psi}^\dagger(j)\hat{\psi}(j) &= \int_{-\pi}^{\pi}\frac{dk}{2\pi}\hat{a}^\dagger(k)\hat{a}(k)\\ \lim_{L\to\infty}\sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \hat{\psi}^\dagger(j)\hat{\psi}(j\pm 1) &= \int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{\pm ik}\hat{a}^\dagger(k)\hat{a}(k)\end{aligned} \tag{14.78}$$

and

$$\begin{aligned}\lim_{L\to\infty}\sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \hat{\psi}^\dagger(j)\hat{\psi}^\dagger(j+1) &= \int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{ik}\hat{a}^\dagger(k)\hat{a}^\dagger(-k)\\ \lim_{L\to\infty}\sum_{j=-\frac{L}{2}+1}^{\frac{L}{2}} \hat{\psi}(j)\hat{\psi}(j+1) &= \int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{-ik}\hat{a}(k)\hat{a}(-k)\end{aligned} \tag{14.79}$$

By collecting terms, we find that the Hamiltonian becomes

$$\begin{aligned}H = -L &+ 2\int_{-\pi}^{\pi}\frac{dk}{2\pi}(1+g\cos k)\hat{a}^\dagger(k)\hat{a}(k)\\ &+ g\int_{-\pi}^{\pi}\frac{dk}{2\pi}(e^{ik}\hat{a}^\dagger(k)\hat{a}^\dagger(-k) - e^{-ik}\hat{a}(k)\hat{a}(-k))\end{aligned} \tag{14.80}$$

This Hamiltonian has the same form as a pairing Hamiltonian in the Bardeen-Cooper-Schrieffer theory of superconductivity (Schrieffer, 1964). This is a consequence of the fact that the fact that fermion number is conserved modulo 2.

It is possible to rewrite H of Eq.(14.80) in terms of the spinor field $\hat{\Psi}(k)$

$$\hat{\Psi}(k) = \begin{pmatrix}\hat{a}^\dagger(k)\\ \hat{a}(-k)\end{pmatrix}, \qquad \hat{\Psi}^\dagger(k) = \begin{pmatrix}\hat{a}(k), & \hat{a}^\dagger(-k)\end{pmatrix} \tag{14.81}$$

which, in the theory of superconductivity, is known as the Nambu representation.

Notice that the two components of the spinor $\hat{\Psi}$ are not independent. Indeed if we denote by $\hat{C}$ the matrix

$$\hat{C} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{14.82}$$

we find that the spinor field $\hat{\Psi}(k)$ obeys the Majorana condition (c.f. Eq.(7.107))

$$\hat{\Psi}^\dagger(k) = \left(\hat{C}\ \hat{\Psi}(-k)\right)^T = \hat{\Psi}^T(-k)\ \hat{C} \tag{14.83}$$

This is a real spinor field, a Majorana fermion.

In terms of the spinor fields $\hat{\Psi}(k)$ the Hamiltonian is

$$\begin{aligned} H = +L\ &\left(-1 + \int_0^\pi \frac{dk}{2\pi} 2(1 + g\cos k)\right) \\ - &\int_0^\pi \frac{dk}{2\pi} \hat{\Psi}^\dagger(k) \begin{pmatrix} 2(1+g\cos k) & -2ig\sin k \\ 2ig\sin k & -2(1+g\cos k) \end{pmatrix} \hat{\Psi}(k) \end{aligned} \tag{14.84}$$

We will see shortly that, in the low energy limit, this Hamiltonian reduces to the Dirac Hamiltonian in 1+1 dimensions.

To diagonalize this system we finally perform a Bogoliubov transformation to a new set of fermion operators $\hat{\eta}(k)$,

$$\begin{aligned} \hat{a}(k) &= u(k)\hat{\eta}(k) - iv(k)\hat{\eta}^\dagger(-k) \\ \hat{a}(-k) &= u(k)\hat{\eta}(-k) + iv(k)\hat{\eta}^\dagger(k) \end{aligned} \tag{14.85}$$

where $u(k)$ and $v(k)$ are two real (even) functions of k, that will be determined below. The inverse transformation is

$$\begin{aligned} \hat{\eta}(k) &= u(k)\ \hat{a}(k) + iv(k)\ \hat{a}^\dagger(-k) \\ \hat{\eta}(-k) &= u(k)\ \hat{a}(-k) - iv(k)\ \hat{a}^\dagger(k) \end{aligned} \tag{14.86}$$

The transformation is canonical since it preserves the anticommutation relations,

$$\{\hat{a}(k),\ a^\dagger(k')\} = 2\pi\delta(k-k') \implies \{\hat{\eta}(k),\ \eta^\dagger(k')\} = 2\pi\delta(k-k') \tag{14.87}$$

which can be achieved provided the functions $u(k)$ and $v(k)$ satisfy the relation

$$u^2(k) + v^2(k) = 1 \tag{14.88}$$

This condition is solved by the choice

$$u(k) = \cos\theta(k) \qquad v(k) = \sin\theta(k) \tag{14.89}$$

We will determine the function $\theta(k)$ by demanding that the fermion-number non-conserving terms in terms of the fields $\hat{\eta}(k)$ cancel out exactly. Let us now define the functions $\alpha(k)$ and $\beta(k)$

$$\alpha(k) = 2(1 + g\cos k), \qquad \beta(k) = 2g\sin k \tag{14.90}$$

The condition that the terms that do not conserve the fermions $\hat{\eta}(k)$ cancel is

$$-2\alpha(k)u(k)v(k) + \beta(k)\ \left(u^2(k) - v^2(k)\right) = 0 \tag{14.91}$$

The solution for the functions $u(k)$ and $v(k)$ of Eq.(14.89) implies that $\theta(k)$ must satisfy the simple relation

$$\tan(2\theta(k)) = \frac{\beta(k)}{\alpha(k)} = \frac{g\sin k}{1 + g\cos k} \tag{14.92}$$

With this choice, the Hamiltonian H takes the form

$$H = \int_0^{\pi} \frac{dk}{2\pi}\, \omega(k) \left(\hat{\eta}^{\dagger}(k)\hat{\eta}(k) + \hat{\eta}^{\dagger}(-k)\hat{\eta}(-k)\right) + \varepsilon_0(g)L \tag{14.93}$$

where $\varepsilon_0(g)$ is given by

$$\begin{aligned}\varepsilon_0(g) &= -1 + 2\int_0^{\pi} \frac{dk}{2\pi} \left(v^2(k)\ 2g\sin k - u(k)v(k)\ 2(1 + g\cos k)\right) \\ &= -1 + \int_0^{\pi} \frac{dk}{2\pi} \left[4g\sin k\sin^2(\theta(k)) - 2\sin(2\theta(k))(1 + g\cos k)\right]\end{aligned} \tag{14.94}$$

and $\omega(k)$ by

$$\begin{aligned}\omega(k) &= \alpha(k)\cos(2\theta(k)) + \beta(k)\sin(2\theta(k)) \\ &= 2\left[(1 + g\cos k)\cos(2\theta(k)) + g\sin k\sin(2\theta(k))\right]\end{aligned} \tag{14.95}$$

We will choose the range of $\theta(k)$, and the signs of $\cos(2\theta(k))$ and $\sin(2\theta(k))$,

$$\operatorname{sgn}\cos(2\theta(k)) = \operatorname{sgn}\alpha(k), \quad \operatorname{sgn}\sin(2\theta(k)) = \operatorname{sgn}\beta(k) \tag{14.96}$$

which guarantee that $\omega(k) \geq 0$. Using these results we can write $\omega(k)$ in the form

$$\omega(k) = |\alpha(k)||\cos 2\theta(k)| + |\beta(k)||\sin 2\theta(k)| \tag{14.97}$$

Hence, we find

$$\omega(k) = 2\sqrt{1 + g^2 + 2g\cos k} \geq 0 \tag{14.98}$$

Similar algebraic manipulations lead to the simpler expression for $\varepsilon_0(g)$,

$$\varepsilon_0(g) = -\frac{1}{2}\int_{-\pi}^{\pi} \frac{dk}{2\pi}\, \omega(k) < 0 \tag{14.99}$$

which is clearly negative.

14.6.3 Energy Spectrum and Critical Behavior

With these choices it is now elementary to find the spectrum of the Hamiltonian.

A: The ground state

The ground state $|G\rangle$ is simply the state annihilated by all the destruction operators $\hat{\eta}(k)$,

$$\hat{\eta}(k)|0\rangle = 0, \qquad \hat{\eta}(-k)|0\rangle = 0 \tag{14.100}$$

The ground state energy density is equal to $\varepsilon_0(g)$, and it is negative.

By retracing our steps it is easy to see that the free energy of the 2D classical Ising model is related to the ground state energy of the 1D quantum Ising model,

$$\lim_{N_\tau \to \infty} Z = \exp\left(-\frac{N_\tau L f}{T}\right) = \lim_{N_\tau \to \infty} \operatorname{tr} \hat{T}^{N_\tau} = \lim_{\beta \to \infty} \operatorname{tr} e^{-\beta \hat{H}} \tag{14.101}$$

where $\hat{T}$ is the transfer matrix of the 2D classical Ising model at temperature T and $\hat{H}$ is the Hamiltonian of the 1D quantum Ising model with coupling constant g. Hence, the free energy density f of the 2D classical model and the ground state energy density ε_0 of the 1D quantum model are related by the identification

$$f \equiv \varepsilon_0(g) \tag{14.102}$$

From our results for the ground state energy density we find the explicit result

$$\begin{aligned}
\varepsilon_0(g) &= -2\int_0^\pi \frac{dk}{2\pi}\sqrt{(1+g)^2 - 4g\sin^2\left(\frac{k}{2}\right)} \\
&= -\frac{2|1+g|}{\pi}\int_0^{\pi/2} dx\sqrt{1-(1-\gamma^2)\sin^2 x} \\
&= -\frac{2|1+g|}{\pi} E\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right)
\end{aligned} \tag{14.103}$$

where $E(\frac{\pi}{2}, \gamma)$ is the complete elliptic integral of the second kind, and $\gamma = \left|\frac{1-g}{1+g}\right|$ is the modulus of the elliptic integral. Using the expansion of the elliptic integral in the limit $\gamma \to 0$ (Gradshteyn and Ryzhik, 2015),

$$E\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right) \underset{\gamma \to 0}{\approx} 1 + \frac{\gamma^2}{4}\left(\ln\frac{16}{\gamma^2} - 1\right) + O(\gamma^4) \tag{14.104}$$

we can write an expression for $\varepsilon_0(g)$ valid for $g \sim 1$,

$$\varepsilon_0(g) = -2\left(\frac{1-g}{\pi}\right)\left\{1 + \frac{\gamma^4}{4}\left(\ln\frac{16}{\gamma^2} - 1\right) + \dots\right\} \tag{14.105}$$

We can write the ground state energy density $\varepsilon_0(g)$ as a sum of two terms, $\varepsilon_0^{\text{sing}}$ which contains the singular behavior for $g \approx 1$, and $\varepsilon_0^{\text{reg}}$ for the non-singular behavior away from $g = 1$. The singular piece, $\varepsilon_0^{\text{sing}}(g)$ is

$$\varepsilon_0^{\text{sing}} = -\frac{4}{\pi}\left[1 + \frac{t^2}{8}\left(\ln\left(\frac{8}{|t|}\right) - \frac{1}{2}\right) + \dots\right] \tag{14.106}$$

where $t = |1-g|$ plays the role of the "reduced temperature" of the 2D classical model, $t = (T-T_c)/T_c$ (where T is the temperature of the classical problem and T_c is the critical temperature).

We will now use these results to compute the behavior of the specific heat of the classical 2D Ising model close to the phase transition. This follows from the identification of the free energy of the classical model with ground state energy of the quantum model. Since the specific heat is obtained upon differentiating the free energy with respect to temperature twice, it can be obtained equivalently by differentiating twice the ground state energy of the quantum model with respect to the coupling constant g. Hence, if we write the specific heat as a sum of a singular and a regular part, $C = C_{\text{sing}} + C_{\text{reg}}$, the singular part is readily found to be

$$C_{\text{sing}} \approx -\frac{\partial^2 \varepsilon_0^{\text{sing}}}{\partial t^2} \approxeq +\frac{1}{2\pi}\ln\left(\frac{8}{|t|}\right) - \frac{3}{4\pi} + \dots \tag{14.107}$$

This is the famous logarithmic divergence of the specific heat of the 2D classical Ising model, first derived by Onsager (Onsager, 1944). This result also identifies the critical coupling as $g_c = 1$, corresponding to the Onsager critical temperature.

In this section we have worked consistently in the thermodynamic limit, $L \to \infty$, and obtained the expression for the ground state energy density ε_0 shown in Eq.(14.99). We will see in chapter 21 that the Casimir energy, the leading finite size correction to the ground state energy, has a special significance in conformal field theory. There it will be shown that at a fixed point, the Casimir energy, E_{Casimir}, is

$$E_{\text{Casimir}} = -\frac{\pi c v}{6L} \tag{14.108}$$

where L is the linear size of the system, v is the "speed of light", and c is the central charge of the Virasoro algebra of the CFT. We will see below that at $g = 1$ the energy gap of the spectrum of the Ising model vanishes, and the system is at its phase transition point. Using the results of this section, we can compute the Casimir energy by computing the leading finite-size correction to the ground state energy. The result that one readily finds is that in the Ising model the central charge is $c = 1/2$.

B: Excitation spectrum

The first excited state is the fermion state $|k\rangle$ with momentum k,

$$|k\rangle = \hat{\eta}^\dagger(k)|0\rangle, \qquad H|k\rangle = (E_0 + \omega(k))|k\rangle \tag{14.109}$$

Thus, it is an exact eigenstate of the Hamiltonian with excitation energy $\omega(k)$, shown in Fig.14.3 The excitation energy is positive for all momenta k in the first Brillouin zone, $-\pi < k \le \pi$. The excitation energy is smallest at $k = \pi$ where it is equal to $2|1 - g|$.

Therefore, we find that the fermion spectrum has an energy gap $G(g) = \min \omega(k)$. By writing $\omega(k)$ is the form

$$\omega(k) = 2\sqrt{1 + g^2 + 2g\cos k} = 2\sqrt{(1-g)^2 + 4g\cos^2\left(\frac{k}{2}\right)} \tag{14.110}$$

we find that, with our conventions, the gap is reached at $k = \pi$ (for $g > 0$). Thus for all $g \neq 1$ the gap does not vanish, and that only exactly at $g = 1$ the gap vanishes. But this is precisely the phase transition point!

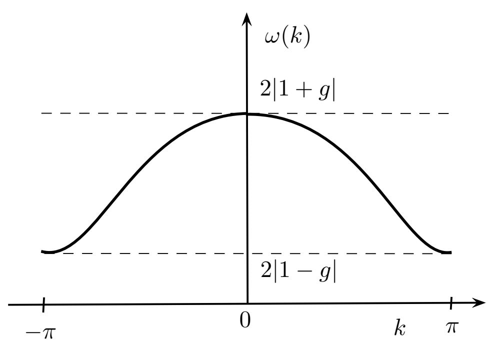

Figure 14.3 The excitation spectrum of the 1D quantum Ising model for $g \neq 1$.

Let's take a closer look at the low energy part of the spectrum. To do that we will consider momenta k close to π by setting $k = \pi - q$, with $|q| \ll \pi$. In this regime, we can approximate the excitation energy to

$$\omega(q) \simeq 2\sqrt{(1-g)^2 + gq^2} \tag{14.111}$$

In the limit $g \to 1$, the spectrum is that of a relativistic fermion with mass $m = |1-g|/2g$ and 'speed of light' $v = 2\sqrt{g}$. We will see shortly that is not an accident. In addition, right at $g = 1$, the spectrum is a linear function of q, i.e. $\omega(q) = v|q|$, and behaves as a relativistic theory of massless fermions.

As $g \to g_c = 1$, the spectral gap $G(g)$ vanishes following a power law

$$G(g) = \mathcal{A}\,|g - g_c|^{\nu} \tag{14.112}$$

where $g_c = 1$, $\mathcal{A} = 2$. We can now identify the critical exponent $\nu = 1$ with the gap exponent. Likewise, the inverse of the mass gap G is the correlation length $\xi = G^{-1}$ of the Ising model (or the Compton wavelength of the relativistic fermion). Hence, the correlation length diverges as the critical point is approached, $g \to 1$, as $\xi \sim |g - g_c|^{-\nu}$, with $\nu = 1$.

What is the physical meaning here of a fermionic excitation? As we saw, a fermion is essentially a kink or domain wall. Hence, the excitation energy of a fermion is essentially the free energy needed to create a domain wall. In the 1D quantum problem, as the phase transition is approached, the domain wall energy vanishes, and it becomes a massless excitation. Thus, the excitations that become massless are effectively topological solitons. On the other hand, the 2D classical picture is equivalent to the Euclidean path-integral picture. In this framework the transition consists of domain walls whose free energy density vanishes. Hence the domain walls proliferate at the phase transition, and span the size of the system.

14.7 Continuum Limit and the 2D Ising Universality Class

We will now show that this model defines a non-trivial quantum field theory by taking the continuum limit. We can do this since, as we just saw, as $g \to g_c$ the correlation length diverges, $\xi \to \infty$, and becomes much larger than the lattice spacing. We will construct the continuum limit by looking at the equations of motion of the Majorana fermion fields.

Unlike the Ising spin operators, whose equations of motion are non-linear, the Majorana fermions $\chi_1(j)$ and χ_2 have simple (linear!) equations of motion. Indeed, using the definition of the Majorana fermion operators of Eq.(14.56), one readily finds that the Hamiltonian is

$$H = i\sum_j \chi_1(j)\chi_2(j) - ig\sum_j \chi_2(j)\chi_1(j+1) \tag{14.113}$$

where we assumed periodic boundary conditions (for a chain with an even number of sites). The equations of motion of the Majorana fermions are

$$
\begin{aligned}
i\partial_t\chi_1(j) &= 2i\chi_2(j) + 2ig\chi_2(j-1) \\
i\partial_t\chi_2(j) &= -2i\chi_1(j) - 2ig\chi_1(j+1)
\end{aligned} \tag{14.114}
$$

It is easy to check that the eigenstates, i.e. the Fourier modes of $\chi_1(j)$ and $\chi_2(j)$, have the spectrum of Eq.(14.110).

As shown in Fig.14.3, the low energy states have wave vector $k \simeq \pi$. In order to take the continuum limit we will focus on these low energy states. To this end it is convenient to redefine the Majorana fermion operators as

$$
\chi_1(j) = (-1)^j\tilde{\chi}_1(j), \quad \chi_2(j) = (-1)^j\tilde{\chi}_2(j) \tag{14.115}
$$

whose equations of motion are obtained from those given above by replacing $g \leftrightarrow -g$. In doing so, the low energy Fourier modes of the operators $\tilde{\chi}_i(j)$ are now near $k = 0$. To proceed to the continuum limit, we now restore a lattice constant a_0 and assign to a lattice site labelled by j a coordinate $x_j = ja_0$ with units of length. Since we will be interested only field configurations that vary slowly on the scale of the lattice spacing, we can use the approximations

$$
\begin{aligned}
\tilde{\chi}_2(j-1) &\approx \tilde{\chi}_2(x_j) - a_0\partial_x\tilde{\chi}_2(x_j) + O(a_0^2) \\
\tilde{\chi}_1(j+1) &\approx \tilde{\chi}_1(x_j) + a_0\partial_x\tilde{\chi}_1(x_j) + O(a_0^2)
\end{aligned} \tag{14.116}
$$

Therefore, in this regime we can rewrite the equations of motion in the form of two partial differential equations of the form

$$
\begin{aligned}
\frac{1}{2a_0g}i\partial_t\tilde{\chi}_1 &\simeq i\frac{(1-g)}{a_0g}\tilde{\chi}_2 + i\partial_x\tilde{\chi}_2 \\
\frac{1}{2a_0g}i\partial_t\tilde{\chi}_2 &\simeq -i\frac{(1-g)}{a_0g}\tilde{\chi}_1 + i\partial_x\tilde{\chi}_1
\end{aligned} \tag{14.117}
$$

Upon rescaling and relabeling the time coordinate $t2\sqrt{g}a_0 \mapsto x_0$ (thus setting the "speed of light" to 1), and the space coordinate $x \mapsto x_1$, we can write the equations of motion as

$$
\begin{aligned}
i\partial_0\tilde{\chi}_1 - i\partial_1\tilde{\chi}_2 + i\left(\frac{(1-g)}{a_0g}\right)\tilde{\chi}_2 &= 0 \\
i\partial_0\tilde{\chi}_2 - i\partial_1\tilde{\chi}_1 - i\left(\frac{(1-g)}{a_0g}\right)\tilde{\chi}_1 &= 0
\end{aligned} \tag{14.118}
$$

We will define the *scaling limit* in which $g \to g_c = 1$ and simultaneously $a_0 \to 0$ while keeping finite the following quantity

$$
m = \lim_{\substack{a_0\to 0\\ g\to 1}}\left(\frac{(1-g)}{a_0g}\right) \tag{14.119}
$$

which as we will see it will be identified with a mass scale. In this limit, the equations of motion take the form of a Dirac equation

$$\begin{aligned} i\partial_0\tilde{\chi}_1 - i\partial_1\tilde{\chi}_2 + im\tilde{\chi}_2 = 0 \\ i\partial_0\tilde{\chi}_2 - i\partial_1\tilde{\chi}_1 - im\tilde{\chi}_1 = 0 \end{aligned} \tag{14.120}$$

Let us define the 2×2 gamma matrices

$$\gamma_0 = -\sigma_2, \qquad \gamma_1 = i\sigma_3, \qquad \gamma_5 = \sigma_1 \tag{14.121}$$

in terms of which we find that the spinor field $\tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2)$ obeys the $1+1$-dimensional Dirac equation

$$(i\partial\!\!\!/ - m)\tilde{\chi} = 0 \tag{14.122}$$

where the spinor field $\tilde{\chi}$ obeys the reality condition

$$\tilde{\chi}^\dagger \equiv \tilde{\chi}^T \tag{14.123}$$

and, hence, it is a massive relativistic Majorana fermion. From now on, to alleviate the notation we will denote $\tilde{\chi} \equiv \chi$.

The Lagrangian of the theory of free Majorana spinor χ (in the Minkowski signature) is

$$\mathcal{L} = \bar{\chi} i\partial\!\!\!/ \chi - \frac{1}{2} m\bar{\chi}\chi \tag{14.124}$$

Here $\bar{\chi} = \chi^T\gamma_0$, where we used the condition that the Majorana spinor is a real Fermi field. Notice that the (Majorana) mass term is the hermitian operator

$$\frac{1}{2}\bar{\chi}\chi = \frac{1}{2}\chi^T\gamma_0\chi = \frac{i}{2}\epsilon_{\alpha\beta}\chi_\alpha\chi_\beta = i\chi_1\chi_2 \tag{14.125}$$

where $\alpha, \beta = 1, 2$ label the two components of the Majorana spinor (in 1+1 dimensions).

Thus, in the *scaling limit* of $g \to g_c = 1$, i.e. asymptotically close to the phase transition, and at distances large compared to the lattice spacing $a_0 \to 0$ (the "continuum limit") the 2D classical Ising model (and the 1D quantum Ising model) are equivalent to (or rather, define) the continuum field theory of free Majorana fermions. Notice that this works only at distances long compared with the lattice constant (i.e. $a_0 \to 0$) but comparable to the diverging correlation length $\xi \approx \frac{1}{|m|}$.

The results of these past few sections show that the 2D Ising model can be understood in terms of a theory of Majorana fermions which are non-local objects in terms of the spins of the lattice model. It should not be a surprise that the the computation of the correlators of the spin operators themselves may be a non-trivial task in the Majorana fermion basis. In fact, it is possible to compute the correlation function of two spins, although the computation is rather technical and somewhat outside the scope of this book. The results of such a computation are nevertheless instructive. While far from the critical point the results are what is expected, i.e. exponential decay of correlations in the symmetric phase and long range order in the broken symmetry state, it is also found that, at the critical point, the two point function decays as a power law $G(R) \sim R^{-1/4}$. This result is very different from it would have been guessed using dimensional analysis in a free scalar field theory. We will see in a later chapter that the 2D Ising model is in a different universality class characterized by a fixed point of free Majorana fermions which, qualitatively, are the domain walls of the Ising spins. Since domain walls are non-local objects in the language of local spins, it should be no surprise that the results are actually so different. We will return to these questions in chapter 21 where we discuss Conformal Field Theory.

We now summarize what we have done in this Chapter. The 2D classical Ising model is a non-trivial theory which happens to be secretly integrable in terms of a theory of Majorana fermions. While it is great to have an exact solution, we should not loose sight of the important lessons that we can draw from a special case. In fact most quantum field theories of interest are not solvable. The important lesson that we should draw is that interacting quantum field theories have properties that largely cannot be guessed from perturbation theory. In the case of the 2D Ising model, the fluctuations of the individual spin fields are not simple and cannot be accessed perturbatively. In earlier chapters we used symmetry arguments to argue that the Ising model should be equivalent to ϕ^4 field theory. However, while this is true, we will see in later chapters that the perturbative construction of ϕ^4 theory only works near 4 space-time dimensions. What we did in this Chapter shows that below 4 dimensions we cannot rely on perturbation theory.