Why is the large-N limit interesting

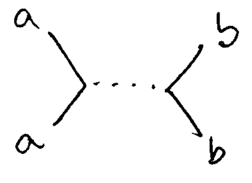
We have been studying the behavior of QFT's at the level of perturbation theory. We have used RG ideas to investigate their behavior non-perturbatively. However, in doing so we need to make some assumptions on how the theory behaves far from the F.P. There are some theories which can be solved exactly and non-perturbatively. The simplest examples appear when the rank of the symmetry group is large (although this does not always work in the sense that the result is not always simple)

Examples:

Consider the $O(N)$ ϕ^4 theory with $(a = 1, \ldots, N)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial_\mu \phi_a + \frac{m_0^2}{2}\phi_a \phi_a + \frac{g}{4!N}(\phi_a \phi_a)^2$$

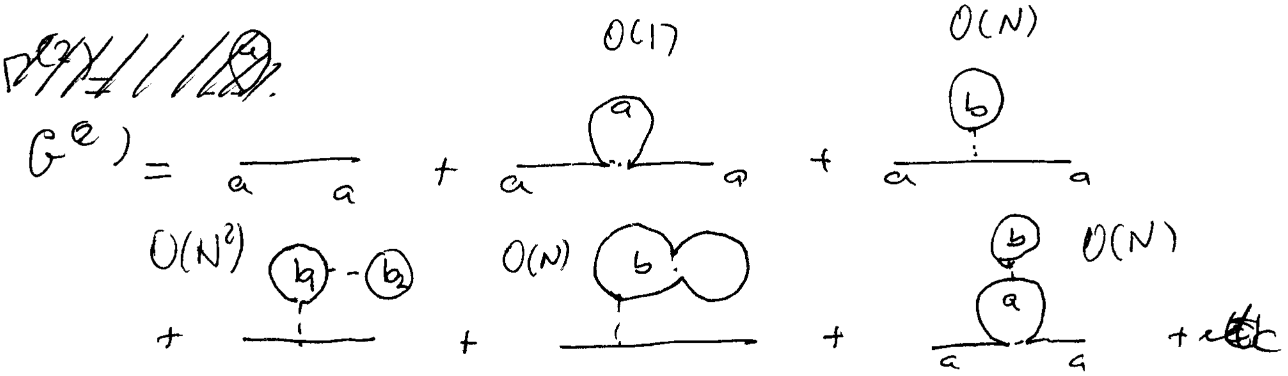
The vertex has the form $\left(\sum_a \phi_a \phi_a\right)\left(\sum_b \phi_b \phi_b\right)$



Q: what is the N-dependence of diagrams?

Consider the 2-point function:

$G^{(2)} =$



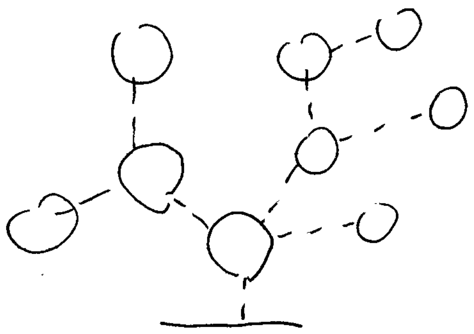
clearly some diagrams $O(N)$, $O(N^2)$ etc.

What diagrams have the largest contribution at a given order in the loop expansion?

clearly  $\sim N \rightarrow$ dominates.

 ~ 1

but



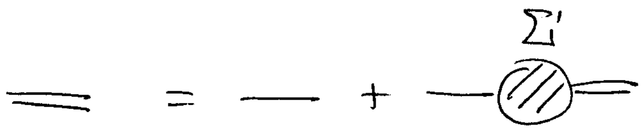
dominates for $\lambda = \frac{g}{N}$, g fixed

Similarly

$\lambda \qquad \lambda^2 N \qquad \lambda^3 N^2$

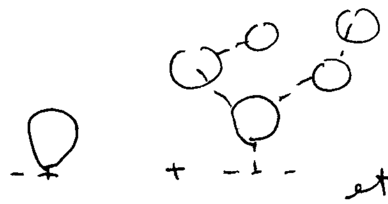
If we redefine $\lambda = \frac{g_0}{N} \Rightarrow \lambda N = g_0$ is fixed

$\Rightarrow$ At large N we sum bubble diagrams

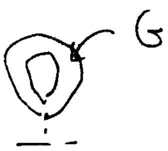
Dyson's Eqn: 

$$G = G_0 + G_0 \Sigma' G$$

$$\Rightarrow G^{-1} = G_0^{-1} - \Sigma$$

For $N \to \infty$ $\quad \Sigma =$  etc

(all trees)

$\Sigma =$  ("Hartree")

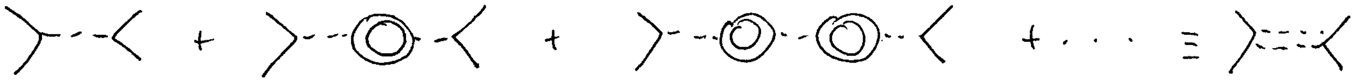
Explicitly:

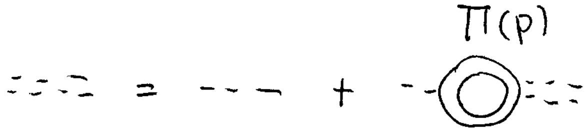
$$\Sigma(p) = -\frac{g}{6} \int \frac{d^d p}{(2\pi)^D} \frac{1}{p^2 + m_0^2 - \Sigma(p)}$$

$$\Rightarrow m^2 = m_0^2 - \Sigma' = m_0^2 + \frac{g}{6} \int_p \frac{1}{p^2 + m^2}$$

$$\Rightarrow m_c^2 = -\frac{g}{6} \int_p \frac{1}{p^2} \qquad \text{as before}$$

and





$$\Pi(p) = \# \int_k G(k)\, G(p+k)$$

$$\Gamma^{(4)} = \Gamma_0^{(4)} + \Gamma_0^{(4)} \Pi\, \Gamma^{(4)}$$

$$\Gamma^{(4)}(p) = \frac{\Gamma_0^{(4)}}{1 - \Gamma_0^{(4)} \Pi(p)}$$

$\Gamma_0^{(4)} \sim$ local coupling $= g/N$

Lecture 44 (5/4) : The O(N) non-linear σ-model in the $N\to\infty$ limit.

L25 In the past lecture we have seen that the exponents of an O(N) non-linear σ-model become very simple as $N\to\infty$. The same is true for the O(N) ϕ^4 theory. This suggests that one looks at this model in the limit in which $N\to\infty$. The result turns out to be equivalent to the spherical model of Berlin and Kac (1952).

The functional integral of the O(N) non-linear σ-model is

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\vec{\pi}\ \delta(\sigma^2+\pi^2-1)\ e^{-\frac{S}{g}}$$

$$S = \int d^dx \left[\frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 - H\sigma - \vec{J}.\vec{\pi}\right]$$

(I will set $\vec{J}=0$ hereafter)

The δ function can be implemented by means of a Lagrange multiplier field $\alpha(x)$.

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\vec{\pi}\, \mathcal{D}\alpha\ e^{-\frac{S}{g} + \int d^dx \frac{\alpha(x)}{2g}(1-\sigma^2(x)-\vec{\pi}^2(x))}$$

Let $\vec{\pi} = \sqrt{g}\,\vec{\varphi}$ and integrate out the $\vec{\varphi}$ fields

$$Z = \int \mathcal{D}\sigma\,\mathcal{D}\alpha\ e^{-\frac{1}{g}\int\left[\frac{1}{2}(\partial_\mu\sigma)^2 - H\sigma - \frac{\alpha}{2}(1-\sigma^2)\right]} \int \mathcal{D}\vec{\pi}\, e^{-\frac{1}{g}\int d^dx\, \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{\alpha\vec{\pi}^2}{2}}$$

$$= \int \mathcal{D}\sigma\,\mathcal{D}\alpha\ e^{-\frac{1}{g}\int\left[\frac{1}{2}(\partial_\mu\sigma)^2 - H\sigma + \frac{\alpha}{2}\sigma^2 - \frac{\alpha}{2}\right]} \left(\det[-\partial^2+\alpha]\right)^{-(N-1)/2}$$

We can define an effective action for the σ and α fields.

$$\frac{S_{eff}}{g} = \frac{1}{g}\int d^dx \left[\frac{1}{2}(\partial_\mu \sigma)^2 + \frac{\alpha(x)}{2}\sigma^2(x) - \frac{\alpha(x)}{2} - H\sigma(x)\right] +$$

$$+ \left(\frac{N-1}{2}\right) \mathrm{tr} \ln(-\partial^2 + \alpha) \qquad \alpha = \alpha(x)$$

Define: $g(N-1) = g_0$

$\sigma = \sqrt{g(N-1)}\; m$

$$\Rightarrow \frac{S_{eff}}{g} = (N-1)\int d^dx \left[\frac{1}{2}(\partial_\mu m)^2 + \frac{\alpha(x)\, m(x)^2}{2} - \frac{\alpha}{2g_0} - \frac{Hm}{\sqrt{g_0}}\right] +$$

$$+ \left(\frac{N-1}{2}\right) \mathrm{tr} \ln(-\partial^2 + \alpha)$$

Thus

$$Z = \int \mathcal{D}\alpha \mathcal{D}m\, e^{-(N-1)\bar{S}(m,\alpha)}$$

$$\bar{S} = \int d^dx \left(\frac{1}{2}(\partial_\mu m)^2 + \frac{\alpha m^2}{2} - \frac{\alpha}{2g_0} - \frac{Hm}{\sqrt{g_0}}\right) + \frac{1}{2}\mathrm{tr}\ln(-\partial^2 + \alpha)$$

Since $N \to \infty$, the form of Z suggests a saddle point expansion, i.e. that the functional integral will be dominated by the solutions to the S.P. equation $\delta S = 0$ as $N \to \infty$

$\delta \bar{S} = 0$ is equivalent to two conditions.

$$\frac{\delta \bar{S}}{\delta \alpha(x)} = 0 \qquad \text{and} \qquad \frac{\delta \bar{S}}{\delta m(x)} = 0$$

$$\frac{\delta \bar{S}}{\delta m(x)} = -\partial^2 m(x) + \alpha(x) m(x) - \frac{H}{\sqrt{g_0}} = 0 \qquad \text{is one equation.}$$

$$\frac{\delta \bar{S}}{\delta \alpha(x)} = -\frac{1}{2g_0} + \frac{m^2(x)}{2} + \frac{1}{2}\frac{\delta}{\delta\alpha(x)} \operatorname{tr} \ln(-\partial^2+\alpha)$$

$$\frac{\delta}{\delta\alpha(x)} \operatorname{tr}\ln(-\partial^2+\alpha) = \int dy \frac{\delta}{\delta\alpha(x)} \langle y| \ln(-\partial^2+\alpha)|y\rangle$$

$$= \int dy \langle y| \frac{\delta}{\delta\alpha(x)} \ln(-\partial^2+\alpha)|y\rangle =$$

$$= \int dy \int dz \langle y| \frac{1}{-\partial^2+\alpha}|z\rangle \frac{\delta}{\delta\alpha(x)} \langle z|-\partial^2+\alpha|y\rangle$$

$$= \int dy \int dz \langle y| \frac{1}{-\partial^2+\alpha}|z\rangle \delta(x-z)\,\delta(x-y) =$$

$$= \langle x| \frac{1}{-\partial^2+\alpha}|x\rangle = \frac{\delta}{\delta\alpha(x)} \operatorname{tr}\ln(-\partial^2+\alpha)$$

$$\Rightarrow \frac{\delta\bar{S}}{\delta\alpha(x)} = 0 \Rightarrow \frac{1}{\cancel{2}} \langle x| \frac{1}{-\partial^2+\alpha}|x\rangle + \frac{m^2(x)}{\cancel{2}} - \frac{1}{\cancel{2}g_0} = 0$$

Let $\sigma = \text{const} = M$, $H = \text{const.}$ and look for solutions with $\alpha = \bar{\alpha} = \text{const.}$ $\Rightarrow m^2 = \frac{M^2}{g_0}$

$$\frac{\delta\bar{S}}{\delta m} = 0 \Rightarrow \bar{\alpha}\, m = \frac{H}{g_0} \Rightarrow \bar{\alpha} = \frac{H}{M}$$

$$g_0 \langle x| \frac{1}{-\partial^2+\frac{H}{M}}|x\rangle = 1 - M^2$$

or

$$g_0 \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+\frac{H}{M}} = 1 - M^2$$

In order to solve this equation one has to take care of its divergencies.

$\Rightarrow$ we have to renormalize the theory $\Rightarrow$ $\epsilon = d-2$ (arbitrary !)

$$g_0 = t\, K^{-\epsilon} Z_1$$

$$\left.\begin{aligned} M &= \sqrt{Z}\, M_R \\ H &= \frac{Z_1}{\sqrt{Z}}\, H_R \end{aligned}\right\} \Rightarrow \quad \frac{H}{M} = \frac{Z_1}{Z}\,\frac{H_R}{M_R}$$

$$t\, K^{-\epsilon} Z_1 \int \frac{dp^d}{(2\pi)^d}\, \frac{1}{p^2 + \frac{H_R}{M_R}\frac{Z_1}{Z}} = 1 - Z\, M_R^2$$

$$\Rightarrow \quad t\, K^{-\epsilon} \left(\frac{Z_1}{Z}\right) \int \frac{dp^d}{(2\pi)^d}\, \frac{1}{p^2 + \frac{H_R}{M_R}\frac{Z_1}{Z}} = \frac{1}{Z} - M_R^2$$

use

$$\int \frac{dp^d}{(2\pi)^d}\, \frac{1}{p^2+m^2} = \frac{S_d}{2}\, \Gamma\left(\frac{d}{2}\right) \Gamma\left(1-\frac{d}{2}\right) (m^2)^{\frac{d}{2}-1}$$

$$= -\frac{S_d}{\epsilon}\, \Gamma\left(1+\frac{\epsilon}{2}\right) \Gamma\left(1-\frac{\epsilon}{2}\right) (m^2)^{\epsilon/2}$$

$$-t\, K^{-\epsilon} \left(\frac{Z_1}{Z}\right)^{1+\frac{\epsilon}{2}} \left(\frac{H_R}{M_R}\right)^{\epsilon/2} \frac{S_d}{\epsilon}\, \Gamma\left(1+\frac{\epsilon}{2}\right) \Gamma\left(1-\frac{\epsilon}{2}\right) = \frac{1}{Z} - M_R^2$$

This equation can be made finite with the choice

$$Z_1 = Z \quad \text{and} \quad \frac{1}{Z} = 1 - t\, \frac{S_d}{\epsilon}\, \Gamma\left(1+\frac{\epsilon}{2}\right) \Gamma\left(1-\frac{\epsilon}{2}\right)$$

$$\boxed{1 - M_R^2 = t\, \frac{S_d}{\epsilon}\, \Gamma\left(1+\frac{\epsilon}{2}\right) \Gamma\left(1-\frac{\epsilon}{2}\right) \left(1 - \left(\frac{H_R}{K^2 M_R}\right)^{\epsilon/2}\right)}$$

which is finite as $\epsilon \to 0$. This is the Equation of State

(a) Calculate t_c : set $H_R = 0$ with $M_R \neq 0$ ($t < t_c$)

$$\Rightarrow \quad 1 - M_R^2 = t\kappa^{-\epsilon} \frac{S_d}{\epsilon} \Gamma(1-\tfrac{\epsilon}{2})\Gamma(1+\tfrac{\epsilon}{2})$$

and $M_R \to 0^+$ as $t \to t_c^-$

Thus t_c is s.t. $\boxed{1 = t_c \frac{S_d}{\epsilon} \Gamma(1-\tfrac{\epsilon}{2})\Gamma(1+\tfrac{\epsilon}{2})}$

This equation tells us that $t_c \to 0$ as $\epsilon \to 0$. Hence in 2d there is no ordered phase at $N = \infty$.

We can put the eqn. of state in the form

$$1 - M_R^2 = \frac{t}{t_c} \Rightarrow M_R = \left(1 - \frac{t}{t_c}\right)^{1/2} \Rightarrow \boxed{\beta = \tfrac{1}{2}}$$

(b) Above t_c

$$1 - M_R^2 = \frac{t}{t_c}\left[1 - \left(\frac{H_R}{M_R \kappa^2}\right)^{\epsilon/2}\right]$$

Remember that $\chi_R = \kappa^2 \frac{M_R}{H_R}$ (by dim. analysis + W-I)

$$1 - M_R^2 = \frac{t}{t_c}\left[1 - [\chi_R(t, H_R)]^{-\epsilon/2}\right]$$

as $H_R \to 0$, $M_R \to 0$ but $\chi_R > 0 \Rightarrow$

$$1 = \frac{t}{t_c}\left(1 - \chi_R^{-\epsilon/2}\right)$$

$$\chi_R = \left(1 - \frac{t_c}{t}\right)^{-2/\epsilon} \Rightarrow \chi_R \sim \left|\frac{t - t_c}{t_c}\right|^{-\gamma} \Rightarrow \boxed{\gamma = 2/\epsilon}$$

(c) Correlation Length: Introduce a source for $\vec{\pi}$ fields, $\vec{J} \Rightarrow$

$$Z[J] = \int \mathcal{D}\alpha\, \mathcal{D}\phi \; e^{\frac{1}{2}\vec{J}\, G_0(\alpha)\, \vec{J}} \; e^{-S/g}$$

at $N=\infty$ $\qquad G_0(\alpha)$ is just the $\langle \vec{\pi}(x)\cdot\vec{\pi}(y)\rangle$ prop.

$$G(x) = \langle \vec{\pi}(x)\cdot\vec{\pi}(y)\rangle = \langle x|\, \frac{1}{-\partial^2+\bar{\alpha}}\, |y\rangle \qquad \langle\vec{\pi}\rangle = 0$$

$$G^{(2)}(\vec{p}) = \frac{1}{p^2+\bar{\alpha}} = \frac{1}{p^2+\frac{H}{M}}$$

$$\Gamma^{(2)}(p) = p^2 + \frac{H}{M} = p^2 + \frac{Z_1}{Z}\,\frac{H_R}{M_R} \equiv p^2 + \frac{H_R}{M_R} \qquad \text{since } Z_1 = Z$$

$$\Rightarrow \quad \xi_{\vec{\pi}} \sim \left(\frac{M_R}{H_R}\right)^{+1/2} \qquad m_\pi^2 = \frac{H_R}{M_R} = \frac{1}{\xi_\pi^2}$$

$$\xi_{\vec{\pi}} \sim \left(\chi_R\, \kappa^{-2}\right)^{+1/2}$$

$$\xi_{\vec{\pi}} \sim \kappa^{-1}\, \chi_R^{1/2}$$

$$\xi_{\vec{\pi}} \sim \kappa^{-1} \left|\frac{t_c - t}{t_c}\right|^{-\gamma/2} \qquad \nu = \frac{\gamma}{2} = \frac{2}{\epsilon}\,\frac{1}{2}$$

$$\boxed{\nu = \frac{1}{\epsilon}}$$

Scaling: $\quad \beta(t) = \kappa \dfrac{\partial t}{\partial \kappa}\Big|_B$

$$g_0 = t\, \kappa^{-\epsilon}\, Z_1$$

$$\beta(t)\left(1 + t\,\frac{\partial \ln Z_1}{\partial t}\right) = \epsilon t$$

$$\beta(t) = \epsilon t - t^2\, S_d\, \Gamma\!\left(1+\tfrac{\epsilon}{2}\right)\Gamma\!\left(1-\tfrac{\epsilon}{2}\right)$$

$$\beta(t) = \epsilon t - \epsilon\frac{t^2}{t_c}$$

$\Rightarrow d>2 \Rightarrow \exists\, t_c$ s.t. $\beta(t_c) = 0$ and $\beta'(t_c) = -\epsilon \Rightarrow \nu = \frac{1}{\epsilon}$

Also $\gamma(t) = \beta(t) \dfrac{\partial \ln Z}{\partial t} = t\, S_d\, \Gamma(1+\frac{\epsilon}{2}) \Gamma(1-\frac{\epsilon}{2}) \equiv \dfrac{t\,\epsilon}{t_c}$

$\Rightarrow \gamma(t_c) = \epsilon$

$\eta = \gamma(t_c) - \epsilon = 0 + O(\frac{1}{N})$ at $N = \infty$ $\eta = 0$

Comments on renormalizing Yang-Mills Theory SU(N)
(cf. Itzykson-Zuber, Ramond, Gross (Les Houches), Peskin & Schroeder)

Feynman rules:
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} + \bar\Psi (i\not{D} - M)\Psi$$

$\bar\Psi_i \longrightarrow \Psi_j$ fermion (quark) $\dfrac{i}{\not{p} - M + i\varepsilon}\,\delta_{ij}$ $j, i = 1 \cdots N$

$$D^{ij}_\mu = \delta^{ij}\partial_\mu - ig(t^a)^{ij} A^a_\mu$$

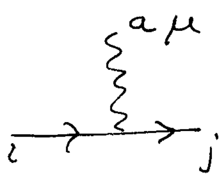
$A^a_\mu \sim\sim\sim A^b_\nu$ gluon (N^2-1) $\nu, \mu = 1 \cdots 4$, $a, b = 1 \cdots N^2-1$ $\dfrac{-i\,\delta_{ab}\, g_{\mu\nu}}{p^2 + i\varepsilon}$ ← (Feynman gauge)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$\bar c^a - - - - - - c^b$ ghost $\dfrac{i\,\delta_{ab}}{p^2 + i\varepsilon}$

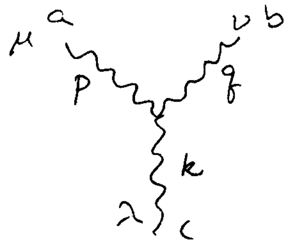
Vertices

quark-gluon



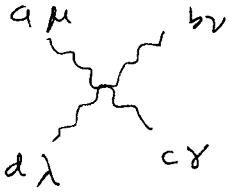
$-ig\gamma^\mu t^a_{ij}$

gluon (trilinear)

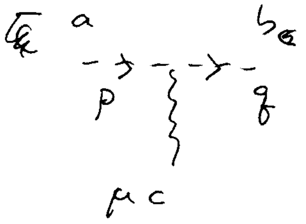


$-g\left((q-k)_\mu g_{\nu\lambda} + 2 \text{ permut}\right) f^{abc}$

quadrilinear

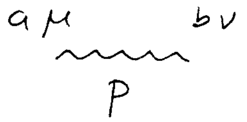


$-ig^2 \left[f^{abe} f^{dce} (g^{\mu\gamma} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\gamma}) + 2 \text{ permut} \right]$



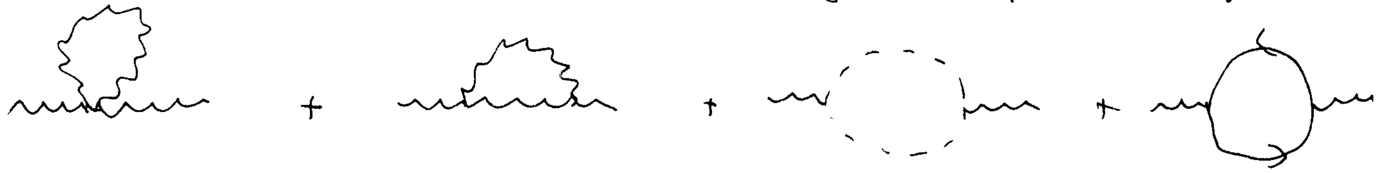
ghost-gluon vertex $-g f_{abc}\, q^\mu$

Gluon propagator $\qquad \Gamma^{(2)\,ab}_{\mu\nu}(p) = -\frac{i\,\delta_{ab}}{\cancel{p^2+i\epsilon}}\, g_{\mu\nu}\, p^2$



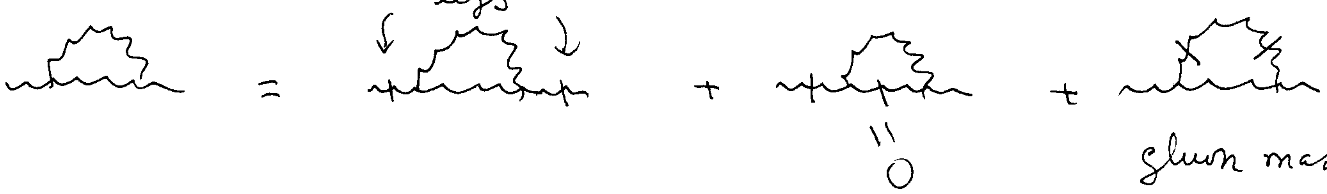
one loop corrections

ghost loop $\qquad$ quark loop.

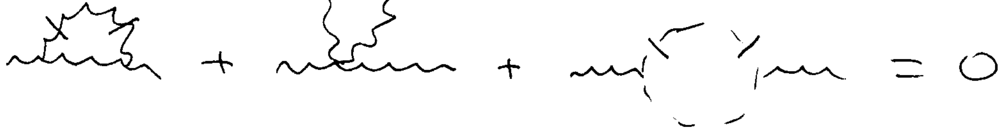


(−1)

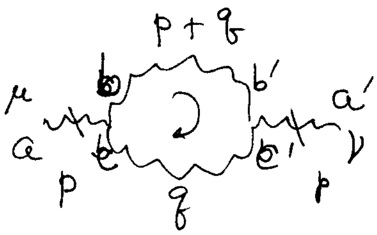
mom. on ext. legs



gluon mass ?



no mass! forbidden by (gauge-invariance)

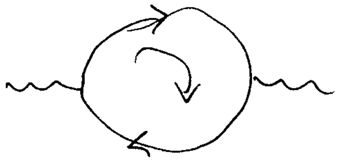


$$\sim \#\, g^2 \left[\int \frac{1}{(p+q)^2 q^2} \frac{d^d q}{(2\pi)^d} \right] p^2 \times i\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \delta_{ab}$$

$d \to 4$ the integral diverges logarithmically

ghosts are fermions

$$- \#' g^2 \left[\int_q \frac{1}{(p+q)^2 q^2} \right] p^2\; i\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) \delta_{ab}$$



(vacuum polarization) $\qquad$ same form

Renormalization (one-loop)

two-point function

$$\cancel{\Gamma(k) \underset{\varepsilon \to 0}{\approx} \frac{g^2 \delta^{ab}}{(4\pi)^2} (k^2 g^{\mu\nu} - k^\mu k^\nu) \Big[\frac{5}{3}}$$

one loop correction : $\Gamma^{(2)\,\mu\nu}_{ab} = - C \delta_{ab} \dfrac{g^2}{16\pi^2} (k^2 g^{\mu\nu} - k^\mu k^\nu) \left[\frac{5}{3} + \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\right] \left[-\frac{2}{\varepsilon} + \ln\left(\frac{k^2}{\mu^2}\right) + \cdots \right]$

(no fermions)

wave-function renorm.

$$Z_3^{-1} \, \Gamma^{(2)\,\mu\nu}_{ab} = \Gamma^{(2)\,\mu\nu}_{R\;ab}$$

$$Z_3 = 1 + \frac{g^2 C}{16\pi^2} \left[\frac{5}{3} + \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\right] \frac{2}{\varepsilon}$$

$$\mathrm{tr}\, T^a T^b = - C \delta^{ab} \qquad \text{(adjoint}$$

$$(T^a)_{cd} = i f^a{}_{cd}$$

$$\mathrm{tr}\, T^a T^b = - f_{cda} f_{cdb} = - C \delta^{ab}$$

$C = N$ for $SU(N)$

$$\mathcal{L} + \delta\mathcal{L} = \mathrm{tr}\, \frac{1}{2} Z_3 (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{\lambda}{2} (\partial \cdot A)^2$$

$$- g Z_1 (\partial_\mu A_\nu - \partial_\nu A_\mu) [A^\mu, A^\nu] + \frac{g^2}{2} Z_4 [A_\mu, A_\nu][A^\mu, A^\nu]$$

$$- \tilde{Z}_3 \, \partial_\mu \bar{\eta} \, \partial^\mu \eta + g \tilde{Z}_1 (\partial_\mu \bar{\eta}_b A^\mu_a \eta_c f_{abc})$$

ghosts

$A = Z_3^{1/2} A_R \quad , \quad \eta = \tilde{Z}_3^{1/2} \eta_R \quad , \quad \bar{\eta} = Z_3^{1/2} \bar{\eta}_R$

$g = Z_1 Z_3^{-3/2} g_R \qquad \lambda = \lambda_R Z_3^{-1}$

gauge invariance $\Rightarrow \quad \dfrac{Z_4}{Z_1} = \dfrac{Z_1}{Z_3} = \dfrac{\tilde{Z}_1}{\tilde{Z}_3} = \dfrac{Z_{1F}}{Z_2}$

$\uparrow$ fermions.

$$\delta\mathcal{L}_{\bar\psi\psi} = (Z_2 - 1)\, \bar\psi\, i\not\partial\psi - \left(Z_2 \frac{m_0}{m} - 1\right) m \bar\psi\psi$$

$$\delta\mathcal{L}_{\bar\psi A\psi} = (Z_{1F} - 1)\,(-ig\,\bar\psi \not{A}_a T^a \psi)$$

$$\Rightarrow \quad g_0 = Z_g\, g$$

$\uparrow$ total, including fermions

$$Z_g = 1 + \frac{g^2}{16\pi^2}\left(\frac{11}{6} C - \frac{2}{3} T_f\right)\frac{2}{\varepsilon}$$

$$\mathrm{tr}\, T^a T^b = -T_f\, \delta_{ab} \qquad T_f = \tfrac{1}{2}$$

$\uparrow$ depends on the rep.!

$$\beta(g) = -\mu \frac{\partial g}{\partial \mu} = \frac{g^3}{8\pi^2}\left[\frac{11}{6} C - \frac{2}{3} n T_f\right]$$

Asymptotic freedom

$$\beta(g^2) = \frac{g^4}{4\pi^2}\left[\frac{11}{6} N - \frac{2}{3} n T_f\right]$$

several "flavors" of fermions $(n_f) \Rightarrow T_f \to n_f T_f$

$$SU(3) \quad \frac{11}{6}\times 3 - \frac{2}{3} n \frac{1}{2} = \frac{11}{2} - \frac{n}{3} = \frac{33 - 2n_f}{6}$$

$n_f^c \simeq 16$ (!)

$$\kappa \frac{\partial g}{\partial \kappa} = - a\, g^3 \qquad ; \; a = \frac{1}{8\pi^2}\left(\frac{11}{6} N - \frac{2}{3} n\, T_f\right)$$

$$\bar{g} = g^2$$

$$\kappa \frac{\partial \bar{g}}{\partial \kappa} = a\, \bar{g}^2$$

$$\Rightarrow \quad \bar{g}(\kappa) = \frac{1}{\text{Const.} + 2a \ln \kappa}$$

$\Rightarrow$ at large κ (high energies, short distances)

$$\bar{g}(\kappa) \approx \frac{1}{2a \ln \kappa} \xrightarrow[\kappa \to \infty]{} 0 \qquad \text{"asymptotic freedom"}$$

But

$$\frac{1}{\bar{g}(\kappa)} - \frac{1}{\bar{g}(\kappa^*)} = 2a \ln\left(\frac{\kappa}{\kappa^*}\right)$$

Let $\kappa \sim \Lambda$ ("cutoff") and $\bar{g}(\kappa) = g^2$

Q: At what scale $\kappa^* = \frac{1}{\xi}$ does $\bar{g}(\kappa^*) \to \infty$?
(i.e. it is strong)

$$\frac{1}{g^2} - 0 = 2a \ln(\Lambda \xi)$$

$$\Rightarrow \boxed{\xi \sim \frac{1}{\Lambda}\, e^{\frac{1}{2a g^2}}}$$

$\Rightarrow$ perturbation theory works for distances shorter than ξ ("perturbative QCD")

Clearly the physics changes at length scales longer compared with ξ where the effective coupling becomes large.

What is the physics for distances larger than ξ?

This regime is <u>not</u> accessible to perturbation theory!