Lecture 41 (4/27)

The non-linear σ-model

We have discussed before that a theory with an internal $O(N)$ symmetry could also be realized ~~with a constr~~ non-linearly, i.e. by means of a constraint.

A prime example is the lattice Heisenberg Model

$$H = -\frac{1}{T} \sum_{\langle \vec{r}, \vec{r}' \rangle} \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}') \qquad \text{with} \quad \|\vec{S}\|^2 = 1$$

In the continuum limit we may write

$$\mathcal{H} = \frac{1}{2g} \int d^d x \, (\nabla \vec{S})^2$$

with $\|\vec{S}\|^2 = 1$

Dimensionally we must have $g = T a^{d-2} = T \Lambda^{2-d}$

Since a constraint is imposed the theory is not free. The parameter g, the "temperature", plays the role of the coupling constant. In two dimensions it is dimensionless. Thus we can expect to see marginal behavior if $d=2$.

This model is of interesting for a variety of reasons. Firstly because of its significance in the statistical mechanics of low dimensional magnets. Unexpectedly this model has a number of interesting analogies with 4 dimensional non-abelian gauge theories, being much simpler.

Refs.: A.M. Polyakov (Phys. Lett. 59 B, 79 (1975))
E. Brezin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976); Phys. Rev. B 14, 3110 (1976),
Brezin, Zinn-Justin, Le Guillou, Phys. Rev. D, 14, 2615 (1976)

The $\mathbb{C}P^{N-1}$ models:

$\mathbb{C}P^n$: complex projective space of n-dimensions.

Let $n^a = z^*_\alpha \, \tau^a_{\alpha\beta} \, z_\beta$

where $\sum_{\alpha=1}^{n} |z_\alpha|^2 = 1$ and

$\tau^a_{\alpha\beta}$ are the generators of $SU(N)$ /
$(a = 1, \ldots, N^2-1)$

$$\tau^a_{\alpha\beta} \, \tau^a_{\gamma\delta} = N \, \delta_{\alpha\delta} \, \delta_{\beta\gamma} - \delta_{\alpha\beta} \, \delta_{\gamma\delta}$$

$$\Rightarrow \quad n^a n^a = z^*_\alpha \, \tau^a_{\alpha\beta} \, z_\beta \; z^*_\gamma \, \tau^a_{\gamma\delta} \, z_\delta$$
$$= n \; z^*_\alpha \, z_\beta \, z^*_\beta \, z_\alpha$$
$$- \; z^*_\alpha \, z_\alpha \; z^*_\gamma \, z_\gamma$$
$$= \cancel{N} - 1$$

$$\Rightarrow \quad \| n^a \| = N - 1$$

For $N = 2$ $(SU(2))$ $\Rightarrow$ $\|\vec{n}\| = 1$ and we get

the mapping of $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ (spinor) $\rightarrow$ $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ (vector)

In general we get a field in the coset space

$$\frac{SU(N)}{SU(N-1) \otimes U(1)}$$

$U(1)$: $\quad z_\alpha \rightarrow z_\alpha \, e^{i\phi}$ are equivalent.

Moreover, consider ~~an~~ local $U(1)$ invariance

$$\mathcal{L} = \frac{1}{g^2} \int d^{D}x \; |(\partial_\mu - iA_\mu) z_\alpha|^2 \qquad |z|^2 = 1$$

where $A_\mu \to A_\mu + \partial_\mu \Phi(x)$ under gauge transformations

~~Stage~~ and $\quad z_\alpha \to z_\alpha e^{i\Phi(x)}$

Since A_μ has no dynamics of its own we can integrate it out:

$$\frac{\delta S}{\delta A_\mu} = 0 = \frac{\delta}{\delta A_\mu}\left(|\partial_\mu z|^2 + i\,(z^* \overleftrightarrow{\partial_\mu} z)\, A_\mu + A_\mu^2 \right) = 0$$

$$\Rightarrow \quad A_\mu = -\frac{i}{2}\left(z_\alpha^* \partial_\mu z_\alpha - (\partial_\mu z_\alpha^*)\, z_\alpha \right)$$

If we substitute A_μ back into $\mathcal{L}$ we get a non-linear theory called the CP^{N-1} model.

For $N = 2$ we find

$$\vec{n} = z^\dagger \vec{\tau} z \quad \Rightarrow \quad \|\vec{n}\|^2 = 1 \qquad (\text{if } \|z\| = 1)$$

$$\text{and} \quad \frac{1}{4}(\partial_\mu \vec{n})^2 = |(\partial_\mu - iA_\mu) z|^2 \quad \Rightarrow \quad O(3)\ !$$

These expressions define the projection of the sphere S_3 in z space onto a sphere S_2 in $\vec{n}$-space. This is called the Hopf bundle. We will come back to this when we talk about instantons.

Another example are the Principal Chiral Fields described by $g(x) \in G$ (some Lie group)

$$\mathcal{L} = \cancel{\frac{1}{2g^2}} \frac{1}{2u^2} \operatorname{Tr}\left(\partial_\mu g^{-1} \partial_\mu g\right)$$

Symmetry: $G_R \otimes G_L \qquad g \to h^{-1} g v \quad , h, v \in G$

In general, non-linear sigma models involve fields which are mappings of the space (Euclidean or Minkowski) onto a manifold $\mathcal{M}$ (called the <u>target manifold</u>)

In the previous examples are the cosets:

$$\mathcal{M} = \frac{O(N)}{O(N-1)} = S_{N-1} \qquad O(N)$$

$$\mathcal{M} = \frac{SU(N)}{SU(N-1)\otimes U(1)} \qquad CP^{N-1}$$

In String Theory more general manifolds are considered. In CMP, other examples are

$$\mathcal{M} = \cancel{O(n) \times O(m)} \quad \frac{O(n,m)}{O(n)\otimes O(m)} \qquad \text{Localization}$$

(with $n, m \to 0$!)

In order to discuss the properties of this theory we need to pick a parametrization of the sphere. I pick

$$\vec{S} = \begin{bmatrix} \sigma \\ \vec{\pi} \end{bmatrix} \qquad \text{s.t.} \quad \sigma^2 + \vec{\pi}^2 = 1$$

More explicitly we have to consider a functional integral of the form

$$Z = \int \mathcal{D}\sigma(x)\, \mathcal{D}\vec{\pi}(x)\; \delta(\sigma^2 + \vec{\pi}^2 - 1)\, \exp\Big\{ -\frac{1}{2g} \int d^dx \left[(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2 \right] +$$

$$+ \int d^dx\; J_\sigma\, \sigma + \vec{J}_\pi \cdot \vec{\pi} \Big\}$$

I now proceed to integrate out the σ field.

Remark: $\displaystyle \int d\sigma\; \delta(\sigma^2 + \pi^2 - 1)\; F(\sigma, \pi) = \frac{1}{2} \int \frac{d\sigma^2}{\sigma}\, \delta(\sigma^2 + \pi^2 - 1)\; F(\sigma, \pi)$

$$= \int \frac{1}{2\sqrt{1-\pi^2}}\, F(\sqrt{1-\pi^2}, \vec{\pi})$$

Thus:

$$Z = \int \frac{\mathcal{D}\vec{\pi}}{2\sqrt{1-\vec{\pi}^2}}\, \exp\Big\{ -\frac{1}{2g} \int d^dx \left[\left(\partial_\mu \sqrt{1-\vec{\pi}^2}\right)^2 + (\partial_\mu \vec{\pi})^2 \right] + \int d^dx \left(\sqrt{1-\vec{\pi}^2}\, J_\sigma + \vec{\pi} \cdot \vec{J}_\pi \right) \Big\}$$

Hence it is not sufficient to replace σ by $\sqrt{1-\vec{\pi}^2}$ in the action. There is an additional (Jacobian) factor. Without this factor the integration measure is not invariant under $O(N)$ transformation.

We can write $\displaystyle \left(\delta^d(0) = \lim_{x \to 0} \delta^d(\vec{x}) = \lim_{x \to 0} \int \frac{d^dp}{(2\pi)^d}\, e^{i\vec{p}\cdot\vec{x}} \sim \Lambda^d = \frac{1}{a^d} \right)$

$$S_{eff}[\vec{\pi}] = \frac{1}{2g} \int d^dx \left[\left(\partial_\mu \sqrt{1-\vec{\pi}^2}\right)^2 + (\partial_\mu \vec{\pi})^2 \right] - \frac{1}{2}\, \delta^d(0) \int d^dx\, \ln(1-\vec{\pi}^2) -$$

$$- \sqrt{1-\vec{\pi}^2}\, J_\sigma - \vec{\pi} \cdot \vec{J}_\pi$$

and we find an additional, so called contact term, in the action.

Let $H(x) = J_\sigma$ $\Rightarrow$

$$Z = \int \prod_{\vec{x}} \frac{d\vec{\pi}(x)}{\sqrt{1-\vec{\pi}^2}} \exp\left\{ \frac{1}{g}\left[-S + \int \vec{J}(x)\cdot\vec{\pi}(x) \right]\right\}$$

with $S = \int d^dx \left[\frac{1}{2}(\partial_\mu\vec{\pi})\cdot(\partial_\mu\vec{\pi}) + \frac{1}{2}\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})(\vec{\pi}\cdot\partial_\mu\vec{\pi})}{(1-\vec{\pi}^2)} - H(x)\sqrt{1-\vec{\pi}^2}\right]$

(where I assumed a lattice regularization)

$\boxed{\text{L23}}$ <u>Ward Identities</u>: The symmetry is really $O(N)/O(N-1)$. Let's perform an infinitesimal rotation mixing σ and $\vec{\pi}$

$$\delta\vec{\pi}(x) = \sqrt{1-\vec{\pi}^2(x)}\;\vec{\omega}$$

$$\delta\cdot\sqrt{1-\vec{\pi}^2(x)} = -\vec{\omega}\cdot\vec{\pi}(x)$$

$\vec{\omega}$ infinitesimal constant vector

Since the action, the measure $\frac{d\pi^2}{\sqrt{1-\pi^2}}$ and the regularization preserve the symmetry, we have

$$\int d^dx \left[J_i(x)\frac{\delta F}{\delta H(x)} - H(x)\frac{\delta F}{\delta J_i(x)}\right] = 0 \qquad i=1,\dots,N-1$$

$F(\vec{J},H) = g\ln Z$ and

$$\Gamma(\vec{\pi},H) = \int d^dx\left[\vec{\pi}\cdot\vec{J} - F(\vec{J},H)\right]$$

$$\frac{\delta\Gamma}{\delta H} = -\frac{\delta F}{\delta H}$$

$$\vec{\pi} = \frac{\delta F}{\delta\vec{J}} \qquad \frac{\delta\Gamma}{\delta H} = H \qquad \frac{\delta\Gamma}{\delta\vec{\pi}} = \vec{J}_\pi$$

Thus we get

$$\int d^dx\left[\frac{\delta\Gamma}{\delta\vec{\pi}(x)}\frac{\delta F}{\delta H(x)} + H(x)\,\vec{\pi}(x)\right] = 0$$

is the W-I for Γ.

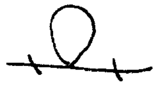
Let us see how does this Ward-Identity provides for an argument which "proves" that the σ-model is just renormalizable in two dimensions.

The first step is to analyze the structure of the divergencies in two dimensions where the coupling constant g is dimensionless. But so is the field $\vec{\pi}$. Thus $\sqrt{1-\vec{\pi}^2}$ is dimensionless and $(\partial_\mu \vec{\pi})^2$ has dimension 2 ~~and~~; also

$$\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})(\vec{\pi}\cdot\partial_\mu\vec{\pi})}{1-\vec{\pi}^2} \quad \text{and} \quad H \quad \text{have dimension 2}$$

Thus even if we expand in powers of $\vec{\pi}$ all operators in such expansion have the same number of derivatives and hence the same dimension. That's to say they are all equally relevant. Indeed this must be so since these op. are related through the symmetry. Dropping any of them, or truncating the expansion, will spoil the symmetry.

Typical graphs $(H, \vec{J}=0)$

 derivatives on the ext. legs $\quad + \quad$ 

this diagram $\sim \int \frac{d^2q}{q^2} \sim \ln\Lambda$ $\qquad \sim \int d^2q \, \frac{1}{q^2} \, q^\mu q^\nu \sim \Lambda^2$

The first graph contributes to $q^2|\vec{\pi}(q)|^2$ while the second to $|\vec{\pi}(q)|^2$ itself (i.e. a mass term). Such a term would be a disaster since it would break the symmetry. However the measure terms in the action

$$\sim -\frac{1}{2}\ln(1-\pi^2) \simeq \frac{\pi^2}{2}$$

cancel such contributions (which are quadratically divergent) (I will not prove this statement here)

In any event dimensional regularization will be used. Within this regularization scheme logarithmic divergencies appear as poles in ϵ $(= d-2)$. Quadratically divergent contributions are regularized to zero and one does not have to worry about these problems. In other schemes, like lattice calculations, the cancellations must be checked explicitely.

Vertices:

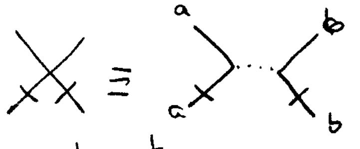
$$\sum_{n=0}^{\infty} (\vec{\pi}^{\,2})^n (\vec{\pi}\cdot\partial_\mu\vec{\pi})(\vec{\pi}\cdot\partial_\mu\vec{\pi})$$

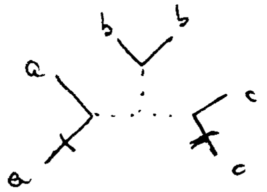
Rescale $\quad \vec{\pi} = \sqrt{g}\,\vec{\varphi}$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\vec{\varphi})^2 + \sum_{n=0}^{\infty} \frac{g^{n+1}}{2} (\vec{\varphi}^{\,2})^n (\vec{\varphi}\cdot\partial_\mu\vec{\varphi})(\vec{\varphi}\cdot\partial_\mu\vec{\varphi})$$

$$- H(x)\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n\,[2(n-1)]!}{2^{2n-1}\,(n-1)!\,n!}\, g^n (\vec{\varphi}^{\,2})^n\right) + \sqrt{g}\,\vec{J}_\pi\cdot\vec{\varphi}$$

+ measure terms.

$n=0$  $\frac{g}{2}\, g^{(1)}_\mu \cdot g^{(2)}_\mu$

$n \geq 1$  etc.

* Lecture 42 (4|29)

I will show now how much do the W-I restrict the possible divergences. By power counting we only expect to worry about operators of dimension two or less. Any op. with dimension higher than two will (a) render the theory non-renormalizable and (b) will be irrelevant at distances much bigger than the cutoff.

Let us expand Γ in powers of g.

$$\Gamma = \sum_{n=0}^{\infty} g^n \, \Gamma^{(n)}$$

This is done by organizing the Feynman graphs in a power series in g.

At lowest order Γ is just the action itself (as usual)

$$\Gamma^{(0)} = S(\vec{\pi}, H) = \int d^2x \left[\tfrac{1}{2}(\partial_\mu \vec{\pi})^2 + \tfrac{1}{2} \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})(\vec{\pi}\cdot\partial_\mu\vec{\pi})}{1-\vec{\pi}^2} - H(x)\sqrt{1-\vec{\pi}^2} \right]$$

At one loop we can use

$$\int d^2x \left[\frac{\delta\Gamma}{\delta\vec{\pi}(x)} \frac{\delta\Gamma}{\delta H(x)} + H(x)\,\vec{\pi}(x) \right] = 0$$

to obtain $\quad \Gamma = \Gamma^{(0)} + g\,\Gamma^{(1)}$, the WI must be satisfied order by order in g.

$$\int d^2x \left(\frac{\delta\Gamma^{(0)}}{\delta\vec{\pi}} \frac{\delta\Gamma^{(1)}}{\delta H(x)} + \frac{\delta\Gamma^{(1)}}{\delta\vec{\pi}} \frac{\delta\Gamma^{(0)}}{\delta H} \right) = 0$$

Define the operator

$$\Gamma^{(0)}{*} \equiv \int d^2x \left[\frac{\delta\Gamma^{(0)}}{\delta\vec{\pi}(x)} \frac{\delta}{\delta H(x)} + \frac{\delta\Gamma^{(0)}}{\delta H(x)} \frac{\delta}{\delta\vec{\pi}(x)} \right]$$

is an element of the algebra of $O(N)/O(N-1)$

$$\Rightarrow \Gamma^{(0)} * \Gamma^{(1)} = 0$$

As the cutoff is removed $\Gamma^{(1)}$ will develop a singular part.

Both divergent and finite parts must satisfy this equation.

$$\Rightarrow \Gamma^{(0)} * \Gamma^{(1)}_{div} = 0$$

We can ~~[illegible]~~ cancel this divergence by adding a counterterm to the action $S(\vec{\pi}, H)$, namely $g S_1(\vec{\pi}, H)$ such that

$$S_1(\vec{\pi}, H) = - \Gamma^{(1)}_{div} + O(g)$$

$\Rightarrow$ The new (renormalized action) $S + g S_1$ satisfies the W-I to all orders in g.

From power counting we now that $\Gamma^{(1)}_{div}$ is a local function with dimension 2 of the field $\vec{\pi}$. Now, H is dimension 2 $\Rightarrow$ $\Gamma^{(1)}_{div}$ is at most $O(H)$. The general form of $\Gamma^{(1)}_{div}$ is

$$\Gamma^{(1)}_{div} = \int d^2x \left[B(\vec{\pi}) + H(x)\, C\{\vec{\pi}\} \right]$$

where B contains <u>at most</u> two derivatives and C is derivative free.

$$\Gamma^{(0)} * \Gamma^{(1)}_{div} = 0$$ yields conditions on B and C.

$$0 = \int d^2x \left[\frac{\delta \Gamma^{(0)}}{\delta H(x)} \frac{\delta C}{\delta \vec{\pi}} H(x) + \frac{\delta \Gamma^{(0)}}{\delta \vec{\pi}} C(\vec{\pi}) + \frac{\delta \Gamma^{(0)}}{\delta H(x)} \frac{\delta B}{\delta \vec{\pi}} \right]$$

$$\frac{\delta \Gamma^{(0)}}{\delta H(x)} = - \sqrt{1 - \vec{\pi}^2}$$

$$\frac{\delta \Gamma^{(0)}}{\delta \vec{\pi}} = - \partial^2 \vec{\pi}(x) + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})\partial_\mu \vec{\pi}}{1 - \vec{\pi}^2} + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2\, \vec{\pi}}{(1 - \vec{\pi}^2)^2} - $$

$$+ \frac{H(x)}{\sqrt{1 - \vec{\pi}^2}} \vec{\pi}$$

Collecting terms we get

$$\int d^2x \left\{ \left[-\sqrt{1-\vec{\pi}^2}\ \frac{\delta C}{\delta \vec{\pi}} + \frac{\vec{\pi}}{\sqrt{1-\vec{\pi}^2}}\ C(\vec{\pi}) \right] H(x) \right. +$$

$$+ \int d^2x \left\{ \left[-\partial^2\vec{\pi} + \vec{\pi}\ \frac{\partial^2 (1-\pi^2)^{1/2}}{(1-\pi^2)^{1/2}} \right] C - (1-\pi^2)^{1/2} \frac{\delta B}{\delta \vec{\pi}} \right\} = 0$$

$H(x)$ arbitrary ($\vec{\pi}$ arbitrary too?) $\Rightarrow$

$$\sqrt{1-\pi^2}\ \frac{\delta C}{\delta \vec{\pi}} = \frac{\vec{\pi}}{\sqrt{1-\pi^2}}\ C(\vec{\pi}) \quad \Rightarrow \quad \boxed{\frac{\vec{\pi}}{1-\vec{\pi}^2}\ C(\vec{\pi}) = \frac{\delta C}{\delta \vec{\pi}}}$$

$$\int d^2x \left\{ \left[-\partial^2\vec{\pi} + \vec{\pi}\ \frac{\partial^2 (1-\pi^2)^{1/2}}{(1-\pi^2)^{1/2}} \right] C - (1-\pi^2)^{1/2} \frac{\delta B}{\delta \vec{\pi}} \right\} = 0$$

Most General Solution is

$$\Gamma^{(1)}_{div} = \lambda S^{(0)} + \mu \int d^2x \left[\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{(1-\vec{\pi}^2)^2} + \frac{H(x)}{\sqrt{1-\vec{\pi}^2}} \right]$$

$\uparrow$ unperturbed action.

If we set $S_1 = -\Gamma^{(1)}_{div}$ then we write S in terms of a rescaled field $\vec{\pi} \to Z^{1/2}\vec{\pi}$

$$S = \int d^2x \left\{ \frac{Z}{Z_1} \left[\frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{1}{2}\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{\left(\frac{1}{Z} - \vec{\pi}^2\right)} \right] - H\sqrt{\frac{1}{Z}-\vec{\pi}^2} \right\}$$

with

$$\frac{Z}{Z_1} = 1 - \lambda g + O(g^2)$$

$$Z = 1 - 2\mu g + O(g^2)$$

$$g = t Z_1 K^{2-d}$$

$$\frac{H_B}{g} = \frac{H_R K^{d-2}}{t\sqrt{Z}}$$

or $H_B = H_R\, Z_1/\sqrt{Z}$

Thus, to one loop order, it suffices with a renorm. of the wave function and the coupling constant.

However by rescaling the field we modify the transformation laws

$$\delta\vec{\pi} = \sqrt{\frac{1}{Z} - \vec{\pi}^2}\;\vec{\omega} \qquad \text{and} \qquad \frac{d\vec{\pi}}{\sqrt{1-\vec{\pi}^2}} \quad \text{is no longer invariant.}$$

We have to replace it by

$$\frac{d\vec{\pi}}{\sqrt{\frac{1}{Z} - \vec{\pi}^2}}$$ which is an effect only visible to 2 loops. ⇒ the renorm. 1PI Γ

satisfies the W-I.

~~The~~ generalization of this procedure to higher orders can be proved by ~~~~ induction. The key point is always the W-I. To order n we have, assuming we have a renorm. action up to order n-1,

$$\Gamma^{(0)} * \Gamma^{(n)} = -\left(\Gamma^{(1)} * \Gamma^{(n-1)} + \Gamma^{(2)} * \Gamma^{(n-2)} + \dots\right)$$

But on the r.h.s. we have only renormalized terms and thus finite

$$\Rightarrow \quad \Gamma^{(0)} * \Gamma^{(n)}_{div} = 0$$ and hence $\Gamma^{(n)}_{div}$ has the same structure as the lower order divergent terms.

* Lecture 43 (5/2)

$\boxed{L24}$ In general we get that in two dimensions the renormalized action is

$$\frac{S}{g} = \int d^2x \left\{ \frac{Z}{2Z_1 g}\left[\left(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}\right) + \frac{\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2}{\left(\frac{1}{Z} - \pi^2\right)^{*}}\right] - \frac{H}{g}\sqrt{\frac{1}{Z} - \pi^2} \right\}$$

In general dim.

$$\frac{S}{g} = \frac{\kappa^{d-2}}{2Z_1 t}\int d^dx \left[Z\,\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi} + \partial_\mu\sqrt{1-Z\vec{\pi}^2}\;\partial_\mu\sqrt{1-Z\vec{\pi}^2} - 2\frac{HZ_1}{\sqrt{Z}}\sqrt{1-Z\vec{\pi}^2}\right]$$

$$g = t\,\kappa^{2-d}$$

I will now proceed to sketch the calculation to order one loop.

Let me first observe that the bare propagator is

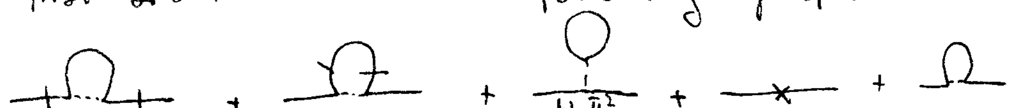
$$G_0^{ij}(p) = \delta^{ij} \, \frac{g}{p^2+H}$$

and that every vertex (except the measure terms) contributes with a weight of $\frac{1}{g}$. The coupling constant g is thus the parameter which organizes the loop expansion.

The zeroth order contribution to the 1PI 2 point vertex is

$$\Gamma^{(2)}(p) = \frac{1}{g}(p^2+H)$$

To first order we have the following graphs



$\frac{H}{g}\frac{\pi^2}{8}$ measure term $= -\frac{\Lambda^d}{(2\pi)^d}\frac{\pi^2}{2}$

$$\cancel{2}\cdot\frac{1}{\cancel{2}}\frac{p^2}{\cancel{g}}\int\frac{d^dq}{(2\pi)^d}\frac{\cancel{g}}{q^2+H} + \frac{\cancel{2}}{\cancel{g}}\cdot\frac{1}{\cancel{2}}\int\frac{d^dq}{(2\pi)^d}\frac{\cancel{g}\,q^2}{q^2+H} + \frac{H}{\cancel{g}\,8}\left(4(N-1)+4\times 2\right)\int_q\frac{\cancel{g}}{q^2+H} - \frac{\Lambda^d}{(2\pi)^d}$$

$$\Gamma^{(2)} = \frac{1}{g}(p^2+H) + p^2\int_q\frac{1}{q^2+H} + \int_q\frac{q^2}{q^2+H} - \frac{\Lambda^d}{(2\pi)^d} + \frac{H(N-1)}{2}\int_q\frac{1}{q^2+H} + H\int_q\frac{1}{q^2+H}$$

$$\int_q\frac{q^2}{q^2+H} - \frac{\Lambda^d}{(2\pi)^d} + H\int_q\frac{1}{q^2+H} = 0$$

(note the cancellation of the quadratic divergence as well as the N indep log. divergence)

Finally $$\Gamma^{(2)}(p) = \frac{1}{g}(p^2+H) + \left(p^2+\left(\frac{N-1}{2}\right)H\right)\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+H}$$

The divergencies in $\Gamma^{(2)}$ can be taken care of by means of coupling constant and wave function renormalization

$$\Gamma_R^{(2)}(p,t,H_R,\kappa) = Z\,\Gamma^{(2)}(p,g,H,\Lambda)$$

with $g = t\kappa^{-\epsilon}Z_1$ and $\frac{H}{g} = \frac{H_R}{t}\frac{\kappa^\epsilon}{\sqrt{Z}}$ or $H = H_R\frac{Z_1}{\sqrt{Z}}$

$$Z = 1 + at + O(t^2)$$

$$Z_1 = 1 + bt + O(t^2)$$

$$\Rightarrow \Gamma_R^{(2)} = \frac{p^2}{t} \kappa^{d-2} \left[1 + t \left(a - b + \kappa^{2-d} \int_q \frac{1}{q^2 + H_R} \right) + \cdots \right]$$

$$+ \frac{H_R}{t} \kappa^{d-2} \left[1 + t \left(\frac{a}{2} + \kappa^{2-d} \left(\frac{N-1}{2}\right) \int_q \frac{1}{q^2 + H_R} \right) + \cdots \right]$$

The coefficients a, b will be chosen s.t. $\Gamma_R^{(2)}$ is finite.

For instance I choose

$$a = - \kappa^{(2-d)} (N-1) \int_q \frac{1}{q^2 + \kappa^2} \Rightarrow Z = 1 - (N-1) t \kappa^{2-d} \int_q \frac{1}{q^2 + \kappa^2}$$

$$b = - (N-2) \kappa^{2-d} \int_q \frac{1}{q^2 + \kappa^2} \qquad Z_1 = 1 - (N-2)\, t\, \kappa^{2-d} \int_q \frac{1}{q^2 + \kappa^2}$$

$$a - b = - \kappa^{2-d} \int_q \frac{1}{q^2 + \kappa^2}$$

Next I ϵ-expand the integrals around $d = 2$

$$\int_q \frac{1}{q^2 + \kappa^2} = \frac{S_d}{2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) (\kappa^2)^{\frac{d}{2} - 1} = - \frac{S_d}{2} \Gamma\left(\frac{d}{2}\right) \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\frac{d}{2} - 1} (\kappa^2)^{\frac{d}{2} - 1}$$

$d = 2 + \varepsilon \qquad \frac{d}{2} = 1 + \frac{\varepsilon}{2} ; \qquad 2 - \frac{d}{2} = 1 - \frac{\varepsilon}{2} \qquad ; \quad \kappa^{d-2} \equiv \kappa^{\epsilon} \simeq 1 + \epsilon \ln \kappa + \cdots$

$$\Gamma\left(1 + \frac{\varepsilon}{2}\right) \Gamma\left(1 - \frac{\varepsilon}{2}\right) \xrightarrow[\varepsilon \to 0]{} 1$$

$$\int_q \frac{1}{q^2 + \kappa^2} \approx - \frac{S_d}{\varepsilon} + O(1)$$

I absorb S_d in the coupling constant.

$$Z \simeq 1 + \frac{(N-1)}{\epsilon} t \quad ; \quad Z_1 = 1 + \left(\frac{N-2}{\epsilon}\right) t$$

(this procedure is equivalent to minimal subtraction)

I can now compute the β-function

$$\beta(t) = \kappa \frac{\partial t}{\partial \kappa}\Big|_B$$

$$g = \kappa^{-\epsilon} Z_1 t$$

$$\kappa \frac{\partial g}{\partial \kappa} = 0 \Rightarrow \quad 0 = -\epsilon t + \beta(t) \left(1 + t \frac{\partial \ln Z_1}{\partial t}\right)$$

$$\frac{\partial \ln Z_1}{\partial t} = \frac{N-2}{\epsilon}$$

$$\boxed{\beta(t) \simeq \epsilon t - (N-2) t^2 + O(t^3)}$$

Analogously we get

$$\gamma(t) = \kappa \frac{\partial \ln Z}{\partial \kappa}\Big|_B \equiv \beta(t) \frac{\partial \ln Z}{\partial t} = (N-1) t + O(t^2)$$

$$\boxed{\gamma(t) = (N-1) t + O(t^2)}$$

~~Fixed Points: t_c s.t. $\beta(t_c) = 0$~~

~~(a) $t_c = 0$ is the only F.P. $d \leq 2$~~

The renorm. $\Gamma^{(2)}$ is $\quad \Gamma_R^{(2)}(\vec{p}, t, H, \kappa=1) = \frac{1}{t}(p^2 + H) - \frac{1}{2}\left(p^2 + \frac{(N-1)}{2} H\right) \ln H + \cdots$

(note that I cannot set $H \to 0$ in $\Gamma^{(2)}$)

RG Equation:

$$\kappa \frac{\partial \Gamma_B^{(N)}}{\partial \kappa}(p, g, H_B, \Lambda) = 0 \Rightarrow$$

$$\left[\kappa \frac{\partial}{\partial \kappa} + \beta(t) \frac{\partial}{\partial t} - \frac{N}{2} \gamma(t) + \left(\frac{\gamma(t)}{2} + \frac{\beta(t)}{t} - (d-2)\right) H \frac{\partial}{\partial H}\right] \Gamma_R^{(N)}(\vec{p}, t, H, \kappa) = 0$$

Fixed Points: $\quad (\beta(t_c) = 0)$

(a) $d \leq 2 \quad t_c = 0$ is the only F.P. and it is IR unstable.
At $d = 2$ it is marginally unstable.

(b) $d>2$ $\quad t_c=0$ is a IR stable F.P.: it defines a stable phase with spontaneously broken symmetry and Goldstone modes

$d=2+\epsilon$

$t_c = \frac{\epsilon}{N-2}+O(\epsilon^2)$ is an IR unstable F.P. which signals a phase transition into a disordered phase ($t>t_c$) and no goldstone behavior.

Dimensionally we have $\quad [\Gamma^{(N)}] = \kappa^d$

$$\Rightarrow \quad \Gamma^{(N)}(p_i, t, H, \kappa) = \rho^d \; \Gamma^{(N)}\left(\frac{p_i}{\rho}, t, \frac{H}{\rho^2}, \frac{\kappa}{\rho}\right)$$

The same arguments exposed above yield a correlation length ξ and a magnetization σ satisfying

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(t)\frac{\partial}{\partial t}\right)\xi(t,\kappa)=0$$

$$\Rightarrow \quad \xi(t,\kappa) = \kappa^{-1} e^{\int_0^t \frac{dt'}{\beta(t')}} \qquad (t<t_c)$$

and $\quad \left(\beta(t)\frac{\partial}{\partial t} + \frac{\gamma(t)}{2}\right)\sigma(t)=0 \qquad$ (follows from $\sigma_R(t) = Z^{-1/2}\sigma_B(t)$)

$$\sigma(t) = e^{-\frac{1}{2}\int_0^t \frac{\gamma(t')}{\beta(t')}dt'}$$

Since $\beta'(t_c) = -\epsilon$ and $\xi \sim \left|\frac{t-t_c}{t_c}\right|^{-\nu} \Rightarrow \nu = -\frac{1}{\beta'(t_c)} = \frac{1}{\epsilon}$

Also $\quad \sigma(t) \sim |t_c - t|^{\beta} \quad$ with $\beta = -\frac{\gamma(t_c)}{2\underbrace{\beta'(t_c)}_{\beta\text{-function}}} = \frac{(N-1)}{2(N-2)} + O(\epsilon)$

Note: $\lim_{N\to\infty} \beta = \frac{1}{2} + O\left(\frac{1}{N}\right)$

(exponent!)

At $H=0$ and $N>2$ we have

$$\Gamma^{(2)}(p,t,K) = \rho^{d}\,\Gamma^{(2)}\left(\frac{p}{\rho}, t, \frac{K}{\rho}\right) = \rho^{d} e^{-\int_t^{t(\rho)} \frac{\gamma(t')}{\beta(t')}}\,\Gamma^{(2)}\left(\frac{p}{\rho}, t(\rho), K\right)$$

$d=2+\epsilon$

$$\Rightarrow \Gamma^{(2)}(p,t,K) = \rho^2\, e^{-\int_t^{t(\rho)} dt' \left(\frac{\gamma(t')-\epsilon}{\beta(t')}\right)}\,\Gamma^{(2)}\left(\frac{p}{\rho}, t(\rho), K\right)$$

$$\rho = e^{\int_t^{t(\rho)} \frac{dt'}{\beta(t')}} \quad \text{as usual} \qquad \left(\rho^{\epsilon} = e^{\int_t^{t(\rho)} \frac{\epsilon\, dt'}{\beta(t')}}\right)$$

At the F.P. $\quad t(\rho) = t_c$

$$\Gamma^{(2)}(p,t_c,K) = \rho^2 e^{-(\gamma(t_c)-\epsilon)\ln\rho}\,\Gamma^{(2)}\left(\frac{p}{\rho}, t_c, K\right)$$

but $\Gamma^{(2)}(p,t_c,K) = \rho^{2-\eta}\,\Gamma^{(2)}\left(\frac{p}{\rho}, t_c, K\right)$

then $\eta = \gamma(t_c) - \epsilon \Rightarrow \quad \eta = (N-1)\left(\frac{\epsilon}{N-2}\right) - \epsilon = \frac{\epsilon}{N-2} + O(\epsilon^2)$

is the exponent η.

$d=2$

$$\gamma = (N-1)\,t$$

$$\beta(t) = -(N-2)\,t^2$$

$$\int_{t_0}^{t} \frac{dt'}{\beta(t')} = \frac{1}{N-2}\left(\frac{1}{t} - \frac{1}{t_0}\right)$$

$$\xi(t,K) = \left[K^{-1} e^{-\frac{1}{(N-2)t_0}}\right] e^{\frac{1}{(N-2)t}}$$

(asymptotic freedom)

Same behavior as QCD in $d=4$