Non-Perturbative Definitions of QFT

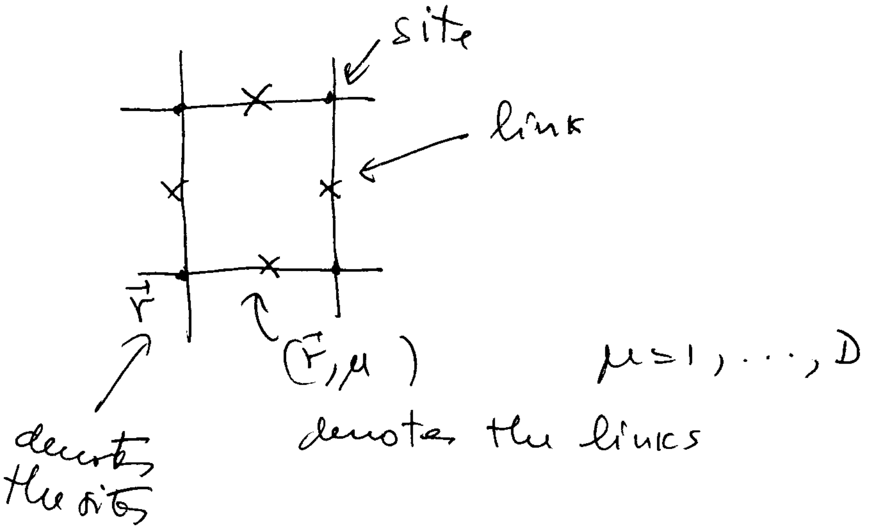
So far we have discussed the behavior of QFT within perturbation theory in powers of a coupling constant g. However we have seen that in a number of cases perturbation theory fails in the sense that the effective coupling constant runs to strong ~~...~~ coupling at low energies. We should regard this behavior as an indication that the fixed point at $g=0$ represents an unstable ground state and that the true ground state have very different physics.

For scalar field theories we were able to use the $\frac{1}{N}$ expansion to investigate the strong coupling phase. In the HW set you looked at a fermionic theory with similar behavior. We need to investigate gauge theories under a similar light. Unfortunately the large N_c limit of gauge theories (where N_c is the number of colors) is not as easy to solve. It has been believed since the late seventies that their ground state represents a confined ground state (or phase). The $N_c \to \infty$ of ~~some~~ a (supersymmetric) version of Yang-Mills has been "solved" recently by Maldacena using String Theory.

The strong coupling behavior of any QFT can be studied by defining the theory with a lattice cutoff. In this representation the gauge theory can be studied using methods ~~statistics~~ borrowed from Statistical Mechanics. Let us define the theory in some detail. For simplicity we will discuss first an analog of Maxwell's ~~E~~ electrodynamics. ~~First~~ In the continuum theory the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 \qquad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

A way to put this theory on a lattice is to replace the continuum by a lattice of points, a hypercubic lattice, and to define a vector field on the lattice.



(see J. Kogut RMP 1979)

Gauge fields are vectors $\Rightarrow$ defined on the links

Matter fields are defined on the sites.

Let $\phi(\vec{r})$ be a matter field, transforming under some (irreducible) representation of a group G.

Global transformation : $\phi'(\vec{r}) = V\,\phi(r)$

$V \in G$

e.g. if ϕ is a group element (principal chiral field)

$\Rightarrow \ \mathrm{tr}\left(\phi^{+}(\vec{r})\,\phi(\vec{r}+\hat{e}_\mu)\right) \to \mathrm{tr}\,\phi^{+\prime}(\vec{r})\,\phi'(\vec{r}+\hat{e}_\mu) =$

$$= tr\left(\phi^+(\vec{r})\; V^{-1}\; V\; \phi(\vec{r}+e_\mu)\right) = tr\left(\phi^+(\vec{r})\; \phi(\vec{r}+e_\mu)\right)$$

since $V^{-1}V = I$

For a <u>local</u> invariance we need a <u>covariant</u> <u>derivative</u>

$$tr\left[\phi^+(\vec{r})\; \underset{\substack{\uparrow \\ \text{defined} \\ \text{on the link}}}{\mathcal{U}_\mu(\vec{r})}\; \phi(\vec{r}+e_\mu)\right]$$

$$\Rightarrow tr\left(\phi'^+(\vec{r})\; \mathcal{U}'_\mu(\vec{r})\; \phi'(\vec{r}+e_\mu)\right) =$$

$$= tr\left[\phi^+(\vec{r})\; V^{-1}(\vec{r})\; \mathcal{U}'_\mu(\vec{r})\; V(\vec{r}+e_\mu)\, \phi(\vec{r}+e_\mu)\right]$$

$$\Rightarrow \mathcal{U}'_\mu(\vec{r}) = V(\vec{r})\; \mathcal{U}_\mu(\vec{r})\; V^{-1}(\vec{r}+e_\mu)$$

is the transformation law.

$$\Rightarrow \mathcal{U}_\mu(\vec{r}) \in G \qquad \mathcal{U}_\mu(\vec{r}) = e^{iA_\mu(\vec{r})}$$

$$A_\mu(\vec{r}) \sim \int_{\vec{r}}^{\vec{r}+e_\mu} dx_\mu\, A^\nu(x) \sim \underset{\substack{\uparrow \\ \text{lattice spacing}}}{a_0}\; A_\mu$$

$A_\mu \in$ algebra of G

$$A_\mu = A_\mu^a \underset{\substack{\uparrow \\ \text{generators.}}}{t^a}$$

$\Rightarrow$ the action of the matter field becomes ($a_0 \to 0$, <u>formally</u>)

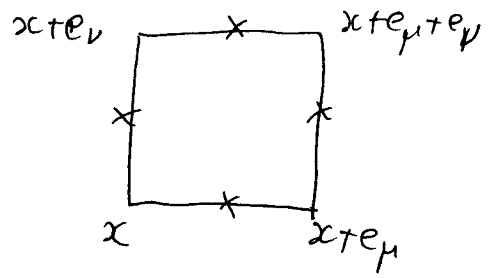
$$-S_{matter} = \frac{K}{2} \sum_{\vec{r},\mu} tr\left(\phi^\dagger(\vec{r})\, U_\mu(\vec{r})\, \phi(r+e_\mu)\right)$$

$$\xrightarrow[a_0 \to 0]{} -\int dx^d \frac{1}{2g^2_{matter}}\, tr\left[(D_\mu \phi)^\dagger D^\mu \phi\right]$$

D_μ : covariant derivative

The action for the gauge fields must also be gauge invariant $\Rightarrow$ we must use Wilson loops:

$$tr \prod_{\substack{(x,\mu) \in \Gamma \\ \uparrow \\ \text{path}}} U_\mu(x)$$



$$\Rightarrow \quad -S_{gauge} = \frac{1}{2g^2} Tr\left(U_\mu(x)\, U_\nu(x+e_\mu)\, U_\mu^{-1}(x+e_\nu)\, \underbrace{}_{U_\nu^{-1}(x)}\right)$$

$$\Rightarrow \quad S_{gauge} \xrightarrow[a_0 \to 0]{} -\frac{1}{4g^2} \int d^D x \; tr\, F_{\mu\nu}^2$$

Wilson Loop: $W[\Gamma] = \left\langle tr \prod_{(x,\mu) \in \Gamma} U_\mu(\vec{x}) \right\rangle$

$$\underset{a_0 \to 0}{\sim} \left\langle tr\, P\, e^{i \oint_\Gamma dx_\mu A^\mu} \right\rangle$$

Pure Gauge Theory

$$Z = \int \mathcal{D}U \; e^{-S[U]}$$

$$S[U] = -\frac{1}{4g^2} \sum_{(\vec{x},\mu,\nu)} \mathrm{tr}\left(U_\mu U_\nu U_\mu^{-1} U_\nu^{-1} \right)$$

Simple cases

ⓐ $U \in \mathbb{Z}_2 \Rightarrow$ Ising Gauge Theory $\qquad U_\mu = \pm 1$

ⓑ $U \in U(1) \Rightarrow$ <u>compact</u> QED $\qquad U_\mu = e^{iA_\mu}$

$0 \leq A_\mu < 2\pi$
(compact)

ⓒ $U \in \mathbb{R} \Rightarrow$ non-compact QED (Maxwell)

$$S = +\frac{1}{4g^2} \sum_{(x,\mu\nu)} \left(\Delta_\mu A_\nu - \Delta_\nu A_\mu \right)^2$$

finite difference
(it does not work for compact groups)

ⓓ YM $\longrightarrow$ $U \in SU(N_c)$

Phases of Gauge Theories

(A) Weak coupling: $g \to 0$ (perturbation theory!)

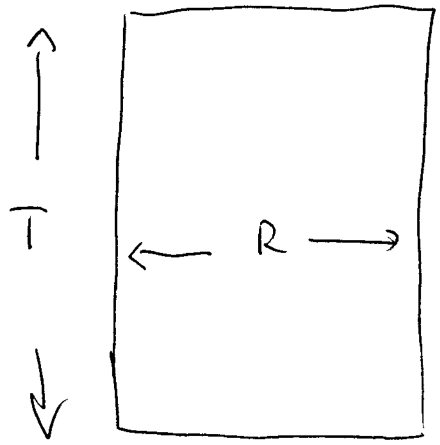
$\Rightarrow$ $e^{+\frac{1}{4g^2}\,\mathrm{tr}\,(\mathcal{U}\mathcal{U}\mathcal{U}^{-1}\mathcal{U}^{-1})}$ is dominated by

flat configurations, i.e. $F_{\mu\nu} = 0$

$\Rightarrow$ $\mathcal{U}_\mu = I$ up to gauge transformation

(i.e. A_μ = gauge transf.)

This is the phase we studied. For a group $U(1)$ it is a Maxwell theory with a photon. We computed the Wilson loop (in Phys. 483) and we saw that for a loop with $a \ll R \ll T$



$$W_\Gamma \sim e^{-T\,V(R)}$$

$$V(R) = \frac{e^2}{R} \quad \text{Coulomb!}$$

(Note: $\frac{T}{R}$ is scale invariant)

We call this a Coulomb phase

In the YM case this phase, described by massless gluons and free quarks, is <u>unstable</u>

since $\beta(g) \sim g^2 \to g$ flows to strong coupling in the IR.

How does it behave for g large?

If g is large $\Rightarrow$

$$e^{+\frac{1}{4g^2}\sum' \mathrm{tr}(UUU^{-1}U^{-1})} \sim 1$$

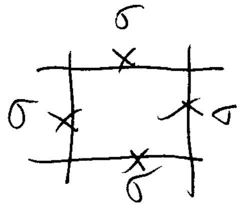
$\Rightarrow$ we must expand in powers of $\frac{1}{g^2}$

This is similar to the <u>high temperature expansion</u> of Classical Statistical Mechanics.

I will do a simple (and generic) case, the Ising Gauge theory. $U \equiv \sigma$

$$Z = \sum_{(\sigma)} e^{+K\sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma} \qquad \sigma = \pm 1$$

$K \sim \frac{1}{g^2}$



$$W[\Gamma] = \langle \prod_{\Gamma} \sigma \rangle$$

We can use the identity

$$e^{K\sigma\sigma\sigma\sigma} \equiv (\cosh K) + (\sinh K)\,\sigma\sigma\sigma\sigma$$

to construct an expansion. Since $\operatorname{tr}\sigma = 0$

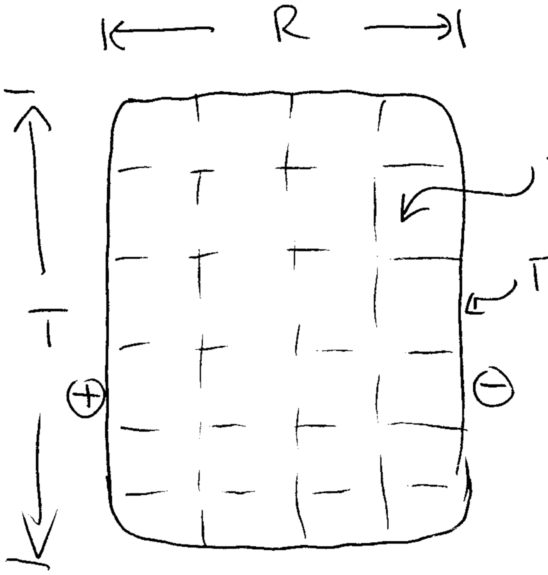
$\Rightarrow$ the only non-zero terms are those in which either there are no σ's or that they come in pairs, since $\sigma^2 = 1$. (This result generalizes to any group. See Kogut's review or the book by ~~Iz~~ Drouffe and Itzykson).

Thus the non-zero terms consists of closed surfaces with a weight $\sim (\tanh K)^{\text{Area (Surface)}} \times$ multiplicity (or entropy)

This is a convergent series.

This is the analog of the high T expansion in classical magnets, which is an expansion in closed loops $\sim e^{-\text{Length (loop)}} \times$ multiplicity

For the Wilson loop operator it is easy to see that the leading term (of this convergent series) ~~consists~~ is obtained by tiling the minimal surface Σ, whose boundary is Γ.



$$\Rightarrow W_\Gamma \sim (\tanh K)^{\text{Area}(\Sigma)}$$

$$= e^{+(\ln \tanh K) RT}$$

<u>Area Law!</u>

$$\Rightarrow W_\Gamma \simeq e^{-T\, V(R)}$$

$$\Rightarrow \quad V(R) = (-\ln \tanh K)\, R$$

$$\equiv \underset{\uparrow}{\sigma} R$$

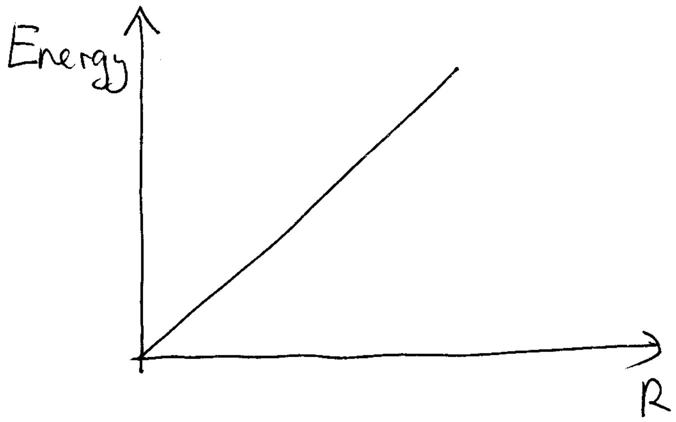
"string tension"

$$\sigma \sim |\ln \tanh K| = |\ln \tanh \tfrac{1}{g^2}|$$

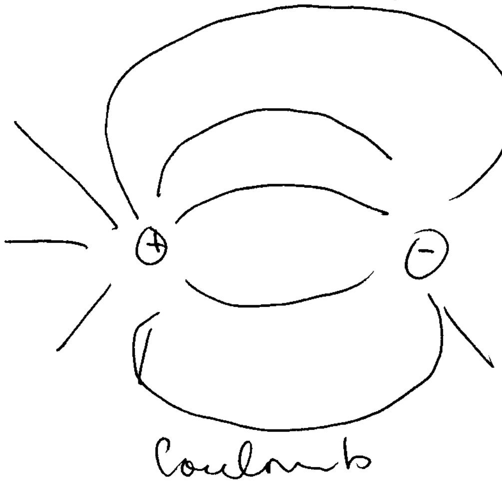
This result holds <u>for all</u> compact gauge groups G.

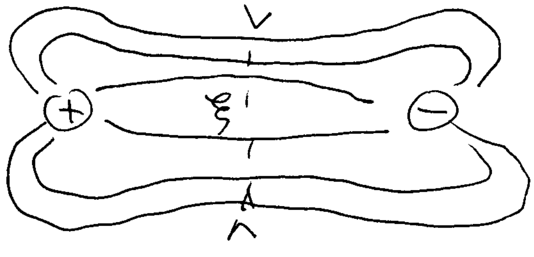
$\Rightarrow$ Since we regard the Wilson loop~~s~~ (or rather $V(R)$) as the ~~amplitudes~~ change of the ground state energy due to the presence of ~~the~~ two sources (~~where~~ (located in space where the Wilson loop intersects a constant time surface) $\Rightarrow$ the effective interaction between the sources is

<u>confining</u> and the same as if there was a string of tension $\sigma \sim |\ln \tanh K|$ stretching ~~t~~ between the sources.



i.e. the Wilson loop represents a source and a sink of "electric field". The fluctuations are so strong that this "electric field" (or chromoelectric field) is squeezed (or "expelled") into a string, instead of being spread out as in a Coulomb field.





ξ: confinement scale

Conversely, for $K \to \infty$ ($g \to 0$) we are in the weak coupling phase and we expect to see a deconfined theory. In this regime, the ground state(s) have "all spins up" (up to gauge transformation) and hence $\prod_P \sigma = 1$ on every elementary plaquette. In this limit the Wilson loop operator is equal to 1. If one link variable is flipped $\Rightarrow$ the ~~plaquette~~ product $\prod_P \sigma = -1$ for all plaquettes P ~~with~~ which share this link. There are $2(D-1)$ such plaquettes for each link. Thus to leading order we get (N = # of sites ; L = perimeter of Γ)

$$\left\langle \prod_\Gamma \sigma \right\rangle = \frac{1 + (N-2L)\, e^{-4(D-1)K} + \cdots}{1 + N e^{-4(D-1)K} + \cdots} \qquad \left(\text{for } \underline{1 \text{ flip}}\right)$$

for n flips: $\frac{1}{n!} (N-2L)^n e^{-4n(D-1)K}$ (ignoring the interaction between flips)

and to $Z \sim \frac{1}{n!} N^n e^{-4n(D-1)K}$

$$Z \sim 1 + N e^{-4(D-1)K} + \frac{N^2}{2} e^{-8(D-1)K} + \cdots$$

$$\approx e^{+N e^{-4(D-1)K}}$$

and numerator =

$$= 1 + (N-2L)\, e^{-4(D-1)K} +$$

$$+ \frac{1}{2}(N-2L)^2 e^{-8(D-1)K} + \cdots$$

$$= e^{+(N-2L)\, e^{-4(D-1)K}}$$

$$\Rightarrow \left\langle \prod_{\Gamma} \sigma \right\rangle = e^{(\cancel{N}-2L)\, e^{-4(D-1)K} - \cancel{N e^{-4(D-1)K}}}$$

$$= e^{-\left(2 e^{-4(D-1)K}\right) L}$$

<u>Perimeter Law!</u> $\Rightarrow V(R) = 0$ (or exp. small!)

<u>Note:</u> This holds for a discrete gauge group (and for a "Higgs phase"). For a Gauge theory in a Coulomb phase we get a Coulomb interaction.

Elitzur's Theorem:

This is a fundamental result which ~~states~~ that only gauge invariant operators have a non-zero expectation value. This holds for all phases of the gauge theory. Hence contrary to the case of a global symmetry, a local symmetry cannot be spontaneously broken. Notice that SSB in a theory with a global symmetry requires that we take the thermodynamic limit in the presence of a symmetry breaking field, and that as we remove the field the expectation value of the order parameter $\langle \phi \rangle \neq 0$

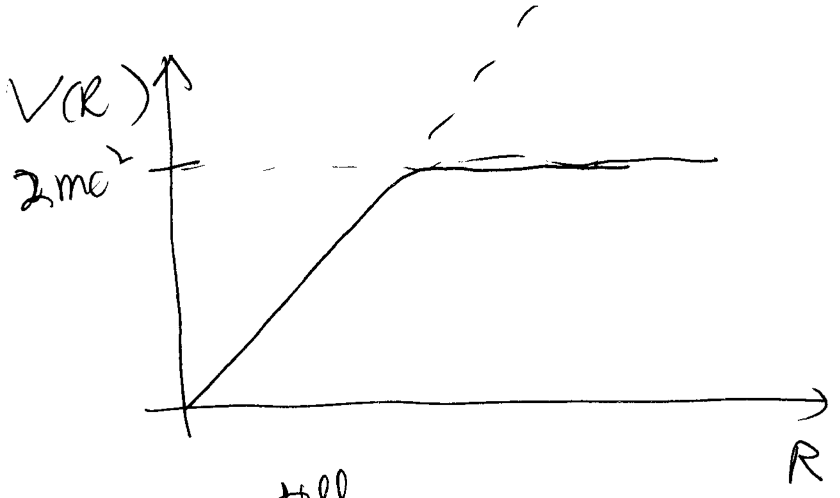
However this works because, in the thermodynamic limit, the degenerate states are infinitely far away from each other (it takes an ∞ # of orders in pert. theory). But if the symmetry is local, boundary conditions don't matter.

Phases of Gauge Theories II

In the absence of matter fields gauge theories can either have a deconfined phase, in which the Wilson Loop has a perimeter law and the interaction between point charges is ~~either~~ either exponentially decaying {or power law}, or in a confined phase, the Wilson loop has an area law and the effective interaction between two point charges grows linearly with distance. Confined and ~~Defin~~ Deconfined phases are separated by phase transitions.

For gauge theories with discrete symmetry groups, the lowest dimension with a deconfined phase is $1+1=2$ (l.c.d) whereas for a gauge theory with a continuous symmetry group the lower critical dimension (renormalizable) is $4 = 3+1$.

We will discuss in the next few lectures the physical mechanism of these transitions. However one may ask how does the presence of dynamical matter fields change this picture. Clearly if the matter field is massive and "uncondensed" (i.e. in an unbroken phase) its effects on the gauge theory should be rather mild and amount to a renormalization of the gauge coupling constant (among other things). However some observables do change. For instance if the matter ~~fiel~~ field carries the fundamental charge, the ~~~~ effective interaction between a pair of static point charges will be linear (in a confining phase) up to some distance at which the energy is equal to the rest energy of a particle-antiparticle pair ⇒ "the string breaks"



However still the <u>spectrum</u> does <u>not</u> contain any states with the quantum #s of the matter field. Instead it's made up of gauge invariant <u>bound states</u>: mesons, hadrons, glueballs

Another phase is the "Higgs Phase". In this "phase" one imagines that if the coupling $g \to 0$ (gauge coupling) $\to$ $\langle \phi \rangle \neq 0$ and we have SSB $\Rightarrow$ if the symmetry is continuous $\Rightarrow$ in pert. theory the gauge field is <u>massive</u> and the symmetry is "broken". However although global symmetries can be spontaneously broken, local symmetries <u>cannot</u> be broken.

Higgs Mechanism:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}|D_\mu\phi|^2 - V(\phi) \qquad \text{(Minkowski)}$$

$$\phi = \rho\, e^{i\theta}$$

$$V(\phi) = V(\phi^\dagger\phi) \equiv V(\rho) \quad \text{with} \quad V'(\rho)|_{\rho_0} = 0$$
$$V''(\rho)|_{\rho_0} > 0$$

$$D_\mu = \partial_\mu + ieA_\mu$$

$$\phi \to \rho_0\, e^{i\theta}$$

$$D_\mu\phi = \rho_0\, D_\mu e^{i\theta} = \rho_0 (i\partial_\mu\theta + ie A_\mu) e^{i\theta}$$

$$|D_\mu\phi|^2 = \rho_0^2 (\partial_\mu\theta + eA_\mu)^2$$

$$\mathcal{L} = \frac{\rho_0^2}{2}(\partial_\mu\theta + eA_\mu)^2 - \frac{1}{4}F_{\mu\nu}^2$$

Fix the gauge $\theta = 0$ (unitary gauge) (London)

or $A_\mu \to A_\mu - \partial_\mu\Phi$

$$-e\Phi = \theta$$

(i.e. the Goldstone boson is "eaten" by the gauge field!)

$$\Rightarrow \quad \mathcal{L} \equiv \frac{e^2\rho_0^2}{2}A_\mu^2 - \frac{1}{4}F_{\mu\nu}^2$$

A_μ is massive!

$$m^2 = e^2\rho_0^2 \quad \Rightarrow \quad m = e\rho_0 = \langle\phi\rangle$$

Higgs Mechanism

This is the same as the expulsion of flux in a superconductor!

The Georgi-Glashow Model

This is a model of Weak Interactions: It involves a three-component real scalar (Higgs) (O(3)) field which couples to an $\underline{SU(2)}$ gauge field

$$\mathcal{L} = \frac{1}{2}(D\vec{\phi})^2 - \frac{1}{2} m_0^2 \vec{\phi}^2 + \frac{\lambda}{4!}(\vec{\phi}^2)^2 + \frac{1}{4g^2} \mathrm{tr}\, F_{\mu\nu}^2$$

(Euclidean)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \times A_\nu$$

$$A_\mu = A_\mu^a t^a$$ $\longleftrightarrow$ SU(2) generators in the adjoint (vector) representation.

$$D_\mu \phi = \partial_\mu \phi + A_\mu \times \phi$$ covariant derivative in the adjoint rep.

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

Assume the pattern of SSB $\phi = \begin{pmatrix} 0 \\ 0 \\ \frac{m_0}{\sqrt{\lambda}} \end{pmatrix}$

$\Rightarrow$ the O(3) symmetry is broken down to U(1)

The combination $W_\mu^{\pm} = \frac{1}{\sqrt{2}}(A_\mu^{(1)} \mp i A_\mu^{(2)})$ is a massive gauge field (the "W") with $m_W^2 = \frac{g^2 m_0^2}{\lambda}$

The field $\sigma = \phi_3 - \frac{m_0}{\sqrt{\lambda}}$ is massive (the "Higgs") ($m_\sigma^2 = 2m_0^2$)

+ massless "photon" $A_\mu^{(3)}$.

Q_1: Is the Higgs Phase stable (non-perturbatively)

Q_2: How does it relate to the other phases?

Q_3: What are the observables for this phase and how ~~the~~ do they behave non-perturbatively?

One can show that in general $\nexists$ a gauge invariant observable related to $\langle \phi \rangle$.

Moreover the global properties of the phase diagram, which embodies the non-perturbative behavior, depends on whether the matter field carries the fundamental charge or not. Let us examine this in the case of a compact U(1) gauge theory coupled to a scalar with charge q (lattice version)

$$-S = \sum_{(r,\mu)} \beta \cos (\Delta_\mu \theta - q A_\mu)$$

$$+ \sum_{(r,\mu\nu)} K \cos F_{\mu\nu} \quad , \quad K = \frac{1}{g^2}$$

$$F_{\mu\nu} = \Delta_\mu A_\nu - \Delta_\nu A_\mu \quad , \quad u_\mu = e^{iA_\mu} , \quad \phi = e^{i\theta}$$

(A) $g^2 \to 0$, β finite] In this limit the gauge fields are flat: $F_{\mu\nu} = 0 \mod 2\pi$

$\Rightarrow$ we have the pure matter field with a global symmetry $U(1)$. This theory (in general) has two phases:

(a) $\beta < \beta_c$ $\quad \langle e^{i\theta} \rangle = 0$ unbroken symmetry

$$\langle e^{i\theta(0)} e^{-i\theta(r)} \rangle \sim e^{-|r|/\xi}$$

massive scalar
$m^2 \sim \frac{1}{\xi^2}$

(b) $\beta > \beta_c$ $\quad \langle e^{i\theta} \rangle \neq 0$ SSB

$$\langle e^{i\theta(0)} e^{-i\theta(r)} \rangle \sim |\langle e^{i\theta} \rangle|^2 + O\left(\frac{1}{r^{D-2}}\right)$$

SSB and a Goldstone boson.

For g small but finite we will expect to be in a <u>Higgs phase</u> for (B) and in a <u>Coulomb phase</u> for (A).

(B) $\beta \to 0$, g fixed.

In this limit the matter fields are massive and their mass $m^2 \sim |\ln\beta| \to \infty$

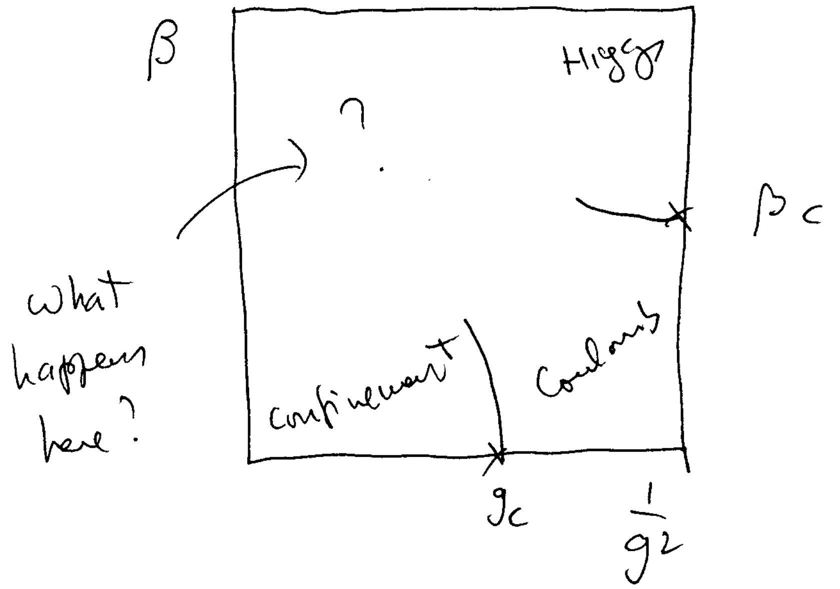
$\Rightarrow$ matter fields decouple and we have a pure gauge theory.

For $D > 4$, $\exists$ g_c /

$g < g_c \Rightarrow$ Coulomb phase with massless ~~[illegible]~~ (photons)

$g > g_c \Rightarrow$ confinement.

(for G non-abelian the Coulomb phase does not exist for $D = 4$) (For discrete symmetry groups, e.g. $\mathbb{Z}_2$, this picture extends down to $D > 2$)



(C) For $g \to \infty$ ($K \to 0$) and β fixed the gauge fields fluctuate wildly. We can go to the unitary gauge $\theta = 0$ where we see that the link gauge fields decouple from each other $\Rightarrow$ this entire line (and its neighborhood) are free of singularities.

(D) What happens as $\beta \to \infty$, g fixed? It depends on what q is.

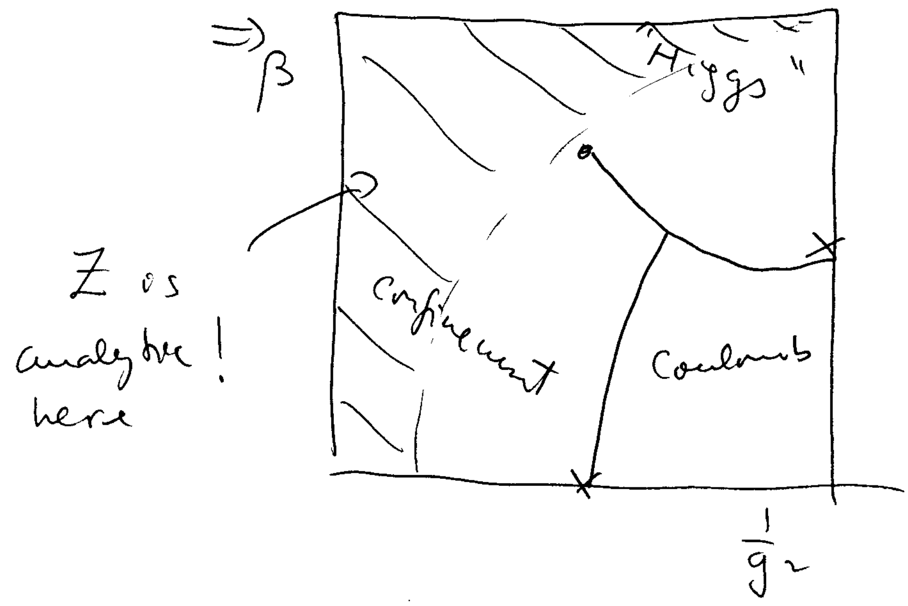
(1) $q = 1$; the matter field carries the fundamental charge. This case is special in two ways. First it is clear that for $q = 1$ there is never an area law for the Wilson loop except for $\beta \to 0$. The behavior is always a perimeter law. This however does not signal deconfinement but simply that the string breaks

~~due~~ to the cration of particle - antiparticle pairs. $\Rightarrow$ saturation of the force. but no particles with the $q=1$ quantum ~~numbers~~ in the spectrum.

For $\beta \to \infty$ $\qquad \cos(\Delta_\mu \theta - A_\mu) \equiv \cos A_\mu$ ($\theta = 0$ gauge)

$\Rightarrow A_\mu = 2\pi n \quad (n \in \mathbb{Z}) \Rightarrow$ gauge fields are frozen. $\Rightarrow$ no ~~single~~ singularities.

$\Rightarrow$



$q=1$

Higgs and confinement are smoothly connected! They are as different as steam is from liquid water! No global distinction $\Rightarrow$ no "true Higgs" phase

(2) $q \geq 2$; Now there is a global destruction in the sense that the Wilson loop with the fundamental charge has an <u>area law</u> for β small and a <u>perimeter law</u> for β large ~~and~~ & g ~~to~~ small

To see this we go to $\beta \to \infty$ $\Rightarrow$

$$\beta \cos(\Delta_\mu \theta - q A_\mu) \equiv \beta \cos q A_\mu \Rightarrow$$

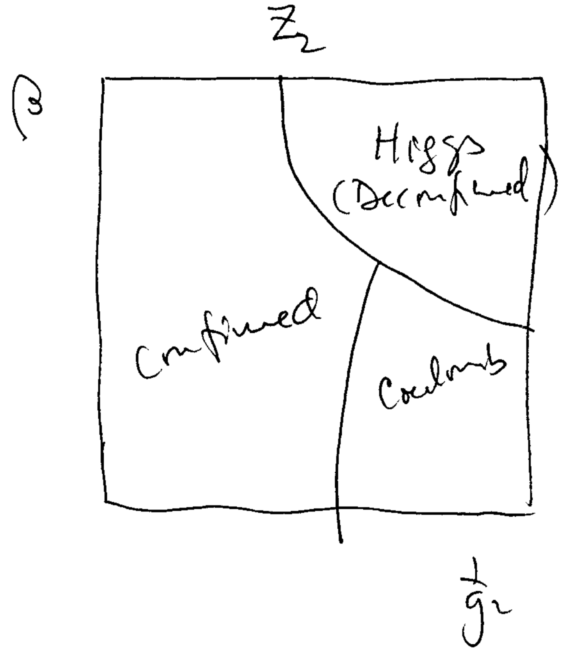
$$A_\mu = \frac{2\pi}{q} n_\mu \quad \Rightarrow \quad A_\mu \text{ takes } \underline{\underline{q}} \ \underline{\underline{\text{values}}}$$

$$q = 2 \quad \Rightarrow \quad A_\mu = 0, \pi \quad \Rightarrow \quad \cos q A_\mu = \pm 1$$

<u>Ising!</u>

$\Rightarrow$ as a function of g this <u>discrete</u> ($\mathbb{Z}_2$) gauge theory has a phase transition at some g_c from a confined phase ~~state~~ (with an area law) to a deconfined phase (with a perimeter law). This is a Higgs phase in the sense that the

Gauge fields are massive but still there is no gauge invariant order parameter!



$q > 1$