Non-Perturbative Definitions of QFT

So far we have discussed the behavior of QFT within perturbation theory in powers of a coupling constant $g$. However, we have seen that in a number of cases, perturbation theory fails, as the effective coupling constant runs to strong coupling at low energies. We should regard this behavior as an indication that the fixed point at $g = 0$ represents an unstable ground state, and that the true ground state has a very different physics.

In scalar theories, we were able to use the $\frac{1}{N}$ expansion to investigate the strong coupling phase. In the HW set you looked at a fermionic theory with similar behavior. We need to investigate gauge theories under a similar light. Unfortunately, the large $N_c$ limit of gauge theories (where $N_c$ is
the number of colors) is not as easy to solve. It has been believed since the late seventies that their ground state represents a confined ground state (or phase). The $N_c \to \infty$ of a (supersymmetric) version of Yang-Mills has been "solved" recently by Maldacena using String Theory.

The strong coupling behavior of any QFT can be studied by defining the theory with a lattice cutoff. In this representation the gauge theory can be studied using methods borrowed from Statistical Mechanics. Let us define the theory in some detail. For simplicity we will discuss first an analog of Maxwell's & electrode vacuum. In the continuum theory the Lagrangian is

$$L = -\frac{1}{4} F_{\mu\nu}^2$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. 
A way to put this thing on a lattice is to replace the continuum by a lattice of points, a hypercubic lattice, and to define a vector field on the lattice. (see J. Kogut RMP 1979)

\[
\begin{array}{c}
\text{site} \\
\text{link}
\end{array}
\]

\[
(F, \mu) \quad \mu = 1, \ldots, D
\]

denote the lines
dejects the sites

gauge fields are vectors \Rightarrow defined on the links
matter fields are defined on the sites.

Let \( \phi(F) \) be a matter field, transforming irreducibly under some representation of a group \( G \).

Global transformation: \( \phi'(r) = V \phi(r) \forall \in G \)

e.g. if \( \phi \) is a group element (principal chiral field)
\[
\Rightarrow \text{tr} (\phi^*(F) \phi(F + \epsilon\mu)) \rightarrow \text{tr} \phi^*(r) \phi'(r + \epsilon\mu) =
\]
\[
\begin{align*}
&= \text{tr} \left( \phi^+(\vec{r}) \ V^{-1} \ V \ \phi(\vec{r}+e_\mu) \right) = \text{tr} \left( \phi^+(\vec{r}) \ \phi(\vec{r}+e_\mu) \right) \\
&\text{since } V^{-1} V = I
\end{align*}
\]

For a local invariance we need a covariant derivative

\[
\text{tr} \left[ \phi^+(\vec{r}) \ U_\mu(\vec{r}) \ \phi(\vec{r}+e_\mu) \right]
\]

\[
\text{defined on the link}
\]

\[
\Rightarrow \ \text{tr} \left( \phi^+_{\mu}(\vec{r}) \ U_{\mu}(\vec{r}) \ \phi(\vec{r}+e_\mu) \right) =
\]

\[
= \text{tr} \left[ \phi^+(\vec{r}) \ V^{-1}(\vec{r}) \ U^\dagger_\mu(\vec{r}) \ V(\vec{r}+e_\mu) \ \phi(\vec{r}+e_\mu) \right]
\]

\[
\Rightarrow U_\mu(\vec{r}) = V(\vec{r}) \quad U_\mu(\vec{r}) = V^{-1}(\vec{r}+e_\mu)
\]

is the transformation law.

\[
\Rightarrow U_\mu(\vec{r}) \in G, \quad U_\mu(\vec{r}) = e^{\imath A_\mu(\vec{r})}
\]

\[
A_\mu(\vec{r}) \sim \int_{\vec{r}} d\vec{x}_\nu \ A^\nu(\vec{x}) \sim \alpha_0 \quad A_\mu
\]

\[
\text{where } \alpha_0 \text{ is lattice spacing}
\]

\[
A_\mu \in \text{algebra of } G
\]

\[
A_\mu = A_\mu^a \ t^a \ \quad t^a \text{ generators}
\]
\[ \Rightarrow \text{the action of the matter field becomes} \]
\[ (a_0 \to 0, \text{ formally}) \]
\[ - S_{\text{matter}} = \frac{K}{2} \sum_i \overrightarrow{\text{tr}} \left( \phi^+ (\alpha) U_{\mu}(x) \phi(\alpha + e_{\mu}) \right) \]
\[ \longrightarrow \to 0 \int dx^d \frac{i}{2 g^2_{\text{matter}}} \text{tr} \left[ (D_{\mu} \phi)^+ D^\mu \phi \right] \]

\( D_{\mu} \): covariant derivative

The action for the gauge fields must also be gauge invariant \( \Rightarrow \) we must use Wilson loops:

\[ \text{tr} \prod_{(x, \mu) \in \Gamma} U_{\mu}(x) \]

\[ \text{path} \]

\[ \Rightarrow - S_{\text{gauge}} = \frac{1}{2 g^2} \text{tr} \left( U_{\mu}(x) U_{\nu}(x + e_{\mu}) U^{-1}_{\mu}(x + e_{\nu}) \right) \]

\[ \Rightarrow S_{\text{gauge}} \xrightarrow{a_0 \to 0} - \frac{i}{4 g^2} \int dx^d \text{tr} \ F_{\mu\nu} \]

Wilson Loop:

\[ W[\Gamma] = \left\langle \text{tr} \prod_{(x, \mu) \in \Gamma} U_{\mu}(x) \right\rangle \]

\[ \xrightarrow{\text{as} a_0 \to 0} \left\langle \text{tr} \mathcal{P} e^{i \int dx_{\mu} A_{\mu}} \right\rangle \]
Pure Gauge Theory

\[ Z = \int \mathcal{D}u \; e^{-S[u]} \]

\[ S[u] = -\frac{i}{4g^2} \sum_{(x,\mu,\nu)} \text{tr} \left( U_{\mu} U_{\nu} U_{-\mu} U_{-\nu} \right) \]

Simple cases

(a) \( u \in \mathbb{Z}_2 \Rightarrow \text{Ising Gauge Theory} \quad U_\mu = \pm 1 \)

(b) \( u \in U(1) \Rightarrow \text{compact QED} \quad U_\mu = e^{iA_\mu} \quad \text{under} \quad 0 \leq A_\mu < 2\pi \quad (\text{compact}) \)

(c) \( u \in \mathbb{R} \Rightarrow \text{non-compact QED (Maxwell)} \quad S = +\frac{i}{4g^2} \sum_{(x,\mu,\nu)} \left( \Delta_\mu A_\nu - \Delta_\nu A_\mu \right) \quad (\text{finite difference}) \quad \text{it does not work for compact groups} \)

(d) \( \text{YM} \Rightarrow u \in SU(N_c) \)
Phases of Gauge Theories

(A) Weak coupling: \( g \rightarrow 0 \) (perturbative theory)

\[ e^{\frac{i}{\sqrt{2}} g \tau^a (U U') U'^{-1}} \]

is dominated by flat configuration, i.e., \( F_{\mu \nu} = 0 \)

\( \Rightarrow U_{\mu} = I \) up to gauge transformation (i.e., \( A_{\mu} = \) gauge transf.)

This is the phase we studied. For a group \( U(1) \) it is a Maxwell theory with a photon. The computed the Wilson loop (in Phys. 483) and we saw that for a loop with \( g S R \ll T \)

\[ W_{1,2} \sim e^{-T V(R)} \]

\[ V(R) = \frac{e^2}{R} \quad \text{Coulomb!} \]

(Note: \( \frac{I}{R} \) is scale irrelevant.

We call this a Coulomb phase.)
In the YM case this phase, described by massless gluons and free quarks, is unstable since \( \beta(g) \sim g^2 \rightarrow g \) flows to strong coupling.

How does it behave for \( g \) large?

If \( g \) is large

\[
\chi + \frac{1}{4g^2} \sum_{k} \text{tr} \left( U^{-1} U^{-1} \right) \sim 1
\]

\( \Rightarrow \) we must expand in powers of \( \frac{1}{g} \).

This is similar to the high temperature expansion of Classical Statistical Mechanics.

I will do a simple (and generic) case, the Ising Gauge Theory. \( U \in \sigma \)

\[
Z = \sum_{\{\sigma\}} e^{\frac{1}{g_i} \sum_{\text{plaqute},k} \sigma_0 \sigma_\delta} \quad \sigma = \pm 1
\]

\[
W(\Gamma) = \langle \prod_{\Gamma} \sigma \rangle
\]
We can use the identity

$$e^K = \cosh K + \tanh K$$

to construct an expansion. Since $t = 0$, the only non-zero terms are those in which either there are no $s$'s or that they come in pairs, since $s^2 = 1$. (This result generalizes to any group. See Kogut's review on the book by Drouffe and Itzykson.)

Thus the non-zero terms consists of

$$\text{closed area} \times \text{multiplicity}$$

This is a convergent series. This is the analog of the high-T expansion in classical magnets, which is an expansion in closed loops $\sim e^{-\text{length (loop)} \times \text{multiplicity}}$.

For the Wilson loop operator it is easy to see that the leading term (of this convergent series) is obtained by tiling the minimal surface $\Sigma$, whose boundary is $\Gamma$. 
\[ \Rightarrow W_\mu = e^{-T V(R)} \]

\[ \Rightarrow V(R) = \left( - \ln \tanh k \right) R \]

\[ \Rightarrow \sigma R \]

"string tension"

\[ \sigma \sim |\ln \tanh k| = |\ln \tanh \frac{1}{g_s}| \]

This result holds for all compact gauge groups G.

\[ \Rightarrow \text{Since we regard the Wilson loop as } V(R) \]

\[ \Rightarrow \text{as the expectation change of the ground state energy due to the presence of two sources in space} \]

\[ \Rightarrow \text{the effective interaction between the source is} \]
confining and the same as if there was a string of tension between the sources.

\[ \text{Energy} \]

\[ R \]

\[ \theta \]

\[ \theta \]

i.e. the Wilson loop represents a source and a sink of "electric field". The fluctuation are so strong that this "electric field" (or chromoelectric field) is squeezed (or "expelled") into a string, instead of being spread out as in a Coulomb's field.

\[ \text{Coulomb} \]

\[ \xi: \text{confinement scale} \]
Conversely, for $K \to \infty$ ($g \to 0$) we are in the weak coupling phase and we expect to see a deconfined theory. In this regime, the ground state(s) have "all spins up" (up to gauge transformation) and hence $\prod_P \sigma = 1$ on every elementary plaquette. In this limit the operator is equal to $1$. If one link variable is flipped $\Rightarrow$ the plaquette $\prod_P \sigma = -1$ for all plaquettes $P$ which share this link. There are $2(D-1)$ such plaquettes for each link. Thus, to leading order we get

$$\langle \prod_P \sigma \rangle = \frac{1 + (N-2L) e^{-4(D-1)K}}{1 + N e^{-4(D-1)K} + \ldots} \quad (\text{for 1 flip})$$

for $n$ flips: $\frac{1}{n!} (N-2L)^n e^{-4n(D-1)K}$

and to $Z = \frac{1}{n!} N^n e^{-4N(D-1)K}$
\[ Z \sim 1 + N e^{-4(D-1)K} + \frac{N^2}{2} e^{-8(D-1)K} + \ldots \]

\[ Z = e + N e^{-4(D-1)K} \]

and numerator =

\[ = 1 + (N-2L) e^{-4(D-1)K} + \frac{1}{2} (N-2L)^2 e^{-8(D-1)K} + \ldots \]

\[ = e + (N-2L) e^{-4(D-1)K} \]

\[ = \frac{\langle \prod_{a} \sigma \rangle}{\pi} \]

\[ = e^{(N-2L) e^{-4(D-1)K} - N e^{-4(D-1)K}} \]

\[ = e^{-2 e^{-4(D-1)K} \cdot L} \]

**Perimeter Law**: \[ V(R) = 0 \text{ (or exp. small)} \]

**Note**: This holds for a discrete gauge group (and for a Higgs phase). For a Gauge theory in a Coulomb phase we get a Coulomb interaction.
Elitzur's Theorem:

This is a fundamental result which states that only gauge invariant operators have a non-zero expectation value. This holds for all phases of the gauge theory. Hence, contrary to the case of a global symmetry, a local symmetry cannot be spontaneously broken. Notice that SSB is a theory with a global symmetry requires that we take the thermodynamic limit in the presence of a symmetry breaking field, and that as we remove the field the expectation value of the order parameter $\langle \phi \rangle \neq 0$. However, this works because, in the thermodynamic limit, the degenerate states are infinitely far away from each other (it takes an $\infty$ of order in pert. theory). But of the symmetry is local, boundary conditions don't matter.
Phases of Gauge Theories II

In the absence of matter fields, gauge theories can either have a deconfined phase, in which the Wilson loop has a perimeter law and the interaction between point charges is exponentially decaying in power law $\propto r^\delta$, or in a confined phase, the Wilson loop has an area law and the effective interaction between two point charges grows linearly with distance. Confined and deconfined phases are separated by phase transition.

For gauge theories with discrete symmetry groups, the lowest dimension with a deconfined phase is $1+1=2$ (l.c.d) whereas for a gauge theory with a continuous symmetry group the lower critical dimension (renormalization) is $4=3+1$.

We will discuss in the next few lectures the physical mechanism of these transitions. However one may
ask how does the presence of dynamical matter fields change this picture. Clearly if the matter field is massive and "uncanceled" (i.e. in an unbroken phase) its effects on the gauge theory should be rather mild and amount to a renormalization of the gauge coupling constant (away from things do). However some observables change. For instance if the matter field carries the fundamental charge, the effective interaction between a pair of static point charges will be linear (in a confining phase) up to some distance at which the energy is equal to the rest energy of a particle-antiparticle pair, "the strong breaks."
However the spectrum does not contain any states with the quantum numbers of the matter field. Instead it is made up of gauge invariant bound states: mesons, hadrons, glueballs.

Another phase is the "Higgs Phase". In this "phase" we imagine that if the coupling \( g \to 0 \) (gauge coupling) \( \Rightarrow \langle \phi \rangle \neq 0 \) and we have SSB \( \Rightarrow \) if the symmetry is continuous \( \Rightarrow \) in pert. theory the gauge field is massless and the symmetry is "broken". However, although the global symmetries can be spontaneously broken, local symmetries cannot be broken.
Higgs Mechanism:

\[ L = -\frac{1}{4} F_{\mu\nu}^2 + \frac{i}{2} |D_\mu \phi|^2 - V(\phi) \]  
(Minkowski)

\[ \phi = \rho e^{i\theta} \]

\[ V(\phi) = V(\phi^* \phi) \equiv V(\rho) \text{ with } V'(\rho)_{\rho_0} = 0 \]
\[ V''(\rho)_{\rho_0} > 0 \]

\[ D_\mu = \partial_\mu + i e A_\mu \]

\[ \phi \rightarrow \rho_0 e^{i\theta} \]

\[ D_\mu \phi = \rho_0 \partial_\mu e^{i\theta} = \rho_0 (i \partial_\mu \theta + e A_\mu) e^{i\theta} \]

\[ |D_\mu \phi|^2 = \rho_0^2 (\partial_\mu \theta + e A_\mu)^2 \]

\[ L = \frac{\rho_0^2}{2} (\partial_\mu \theta + e A_\mu)^2 - \frac{1}{4} F_{\mu\nu}^2 \]

Fix the gauge \( \theta = 0 \) (unitary gauge) (London)

\[ A_\mu \rightarrow A_\mu - \partial_\mu \Phi \]

\[ -e \Phi = \theta \]

\[ \Rightarrow \quad L = \epsilon \rho_0^2 A_\mu^2 - \frac{1}{4} F_{\mu\nu}^2 \quad (\text{i.e. the Goldstone boson is "eaten" by the gauge field!}) \]

\[ m^2 = \epsilon \rho_0^2 \Rightarrow m = \epsilon \rho_0 = \langle \phi \rangle \quad \text{Higgs Mechanism} \]

This is the scene as the expulsion of flux is a superconductor!
The Georgi-Glashow Model

This is a model of Weak Interactions: It involves a three-component real scalar (Higgs) field which couples to an $SU(2)$ gauge field

$$\mathcal{L} = \frac{1}{2} (D\phi)^2 - \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda}{4!} (\phi^2)^2 + \frac{1}{4g^2} F_{\mu\nu}^2 \quad \text{(Euclidean)}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \times A_\nu$$

$$A_\mu = A_\mu^a t^a \quad \text{such generators in the adjoint \& representations}$$

$$D_\mu \phi = \partial_\mu \phi + A_\mu \times \phi \quad \text{covariant \&ation in the adjoint rep.}$$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

Assume the pattern of SSB

$$\phi = \begin{pmatrix} 0 \\ m_0 \\ \frac{m_0}{\sqrt{2}} \end{pmatrix}$$

$\Rightarrow$ the O(3) symmetry is broken down to U(1)

The combination

$$W^\pm = \frac{1}{\sqrt{2}} (A^{(1)}_\mu \mp \xi A^{(2)}_\mu)$$

is a massive gauge field (the "W") with

$$m_W^2 = \frac{g^2 m_0^2}{\lambda}$$

The field $\sigma = \phi_3 - \frac{m_0}{\sqrt{2}}$ is massive (the "Higgs")

and massless "photons" $A^{(3)}_\mu$.

($m_\gamma^2 = 2 m_0^2$)
Q1: Is the Higgs phase stable (non-perturbatively)?
Q2: How does it relate to the other phases?
Q3: What are the observables for this phase and how do they behave non-perturbatively?

One can show that in general $\chi$ is a gauge invariant observable related to $\langle \phi \rangle$.

Moreover, the phase diagram, which embodies the non-perturbative behavior, depends on whether the matter field carries the fundamental charge or not. Let us examine this in the case of a compact $U(1)$ gauge theory coupled to a scalar with charge $q$ (lattice version):

$$S = \sum_{(\mu)} \beta \cos (\Delta \mu \phi - q A_\mu)$$

$$+ \sum_{(\mu, \nu)} K \cos F_{\mu \nu}, \quad K = \frac{1}{\beta q}$$

$$F_{\mu \nu} = \Delta_\mu A_\nu - \Delta_\nu A_\mu$$

$u_\mu = e^{i A_\mu}$, $\phi = e^{i \theta}$
\( g^2 \to 0 \) 
\( \beta \) finite 
\( \) In this limit the gauge fields are flat: \( F_{\mu \nu} = 0 \mod 2\pi \) 

We have the pure matter field with a global symmetry U(1). This theory in general has two phases:

(a) \( \beta < \beta_c \) 
\( \langle e^{i\Theta} \rangle = 0 \) unbroken symmetry 
\( \langle e^{i\Theta(0)} e^{-i\Theta(r)} \rangle \sim e^{-\frac{r}{\beta}} \) 

massive scalar 
\( m^2 \sim \frac{1}{\beta^2} \)

(b) \( \beta > \beta_c \) 
\( \langle e^{i\Theta} \rangle \neq 0 \) SSB 
\( \langle e^{i\Theta(0)} e^{-i\Theta(r)} \rangle \sim |\langle e^{i\Theta} \rangle|^2 + O\left(\frac{1}{\beta^2}\right) \) 

SSB and a Goldstone boson.

For \( g \) small but finite we will expect to be in a Higgs phase for (b) and in a Coulomb phase for (a).
(8) $\beta \to 0$, $g$ fixed.

In this limit the matter fields are massive and their mass $m \sim |\beta| \to \infty$.

$\Rightarrow$ matter fields decouple and we have a pure gauge theory.

For $D > 4$, $\exists \neq g_c /

g < g_c \Rightarrow$ Coulomb phase with massless gluons (photon)

$g > g_c \Rightarrow$ confinement.

(For $G$ non-abelian, the Coulomb phase does not exist for $D = 4$) (For discrete symmetry groups, e.g. $Z_2^k$, this picture extends down $\beta < 1/12$ to $D > 2$.)
(0) For $g \to \infty$ (h $\to 0$) and $\beta$ fixed
the gauge fields fluctuate wildly.
we can go to the unitary gauge $\theta = 0$
when we see that the link gauge fields
decouple from each other $\Rightarrow$ this entire
line (and its neighborhood) are
free of singularities.

(1) What happens as $\beta \to \infty$, $g$ fixed?
It depends on what $g$ is.

(1) $g=1$; the matter field carries the
fundamental charge. This case is
special in two ways. First it is clear
that for $g=1$ there is never an area law
in the Wilson loop except for $\beta \to 0$.

The behavior is always a perimeter law.
This however does not signal deconfinement
but simply that the string breaks
the creation of particle-antiparticle pairs. \Rightarrow saturation of the force but no particles with $q_i = 1$

quantum numbers in the spectrum.

For $\beta \to \infty$, $\cos(\theta - \Delta \mu) = \cos \Delta \mu$

($\theta = 0$)

$\Rightarrow A_{\mu} = 2\pi n (n + Z) \Rightarrow$ gauge fields are fractal. \Rightarrow no singularities.

$Z$ is analytic here!

Higgs and confinement are smoothly connected! They are as different as steam is from liquid water! No global direction \Rightarrow no "true Higgs" phase
\( g \geq 2 \): Now there is a global

distraction in the sense that the Wilson
loop with the fundamental charge has
an area law for \( g \) small and
a perimeter law for \( g \) large and \( g \) small.

To see this we go to \( g \to \infty \Rightarrow \nabla (g \theta - QA) \Rightarrow \theta \leftarrow gA \Rightarrow \)

\[ A_\mu = \frac{2\pi n_\mu}{g} \Rightarrow \text{\( A_\mu \) takes \( g \) values} \]

\[ g = 2 \Rightarrow A_\mu = 0, \pi \Rightarrow \cos A_\mu = \pm 1 \]

Thus,

\( g \) as a function of \( g \) this discrete (\( \mathbb{Z}_2 \))

gauge theory has a phase transition at

\( g = \frac{2\pi n_\mu}{g} \to \text{a confined phase (with an area law) to a deconfined phase (with a perimeter law). This is} \)

\( \text{a Higgs phase where seems that the} \)
Gauge fields are massive but still there is no gauge invariant order parameter.

\[ g > 1 \]