Lecture 32 - (4/2): Composite Operators

We have discussed the properties of the massless (critical) theory. In practice one is interested also in the massive theory although interesting phenomena usually take place as the mass vanishes (i.e. T→Te)

For example in s.m. one is interested not just in correlation functions and susceptibilities but also in internal energies, specific heats, etc. Since the temperature like variable is precisely the mass it is of interest to calculate

\[ \frac{\delta F}{\delta m^2} = \text{entropy} \]

\[ \frac{\delta^2 F}{\delta m^2} = \text{specific heat} \]

\[ \frac{\delta F}{\delta T} = -S \]

\[ \frac{\delta F}{\delta m^2} = -\int \langle \phi^2(x) \rangle \; dx \equiv -V \langle \phi^2 \rangle \]

and

\[ \frac{\delta^2 F}{\delta (m^2)^2} = \int dx \int dy \; \langle \phi^2(x) \phi^2(y) \rangle \equiv V \int dr \langle \phi^2(r) \phi^2(r) \rangle \]

This motivates the study of correlations between composite operators like \( \phi^2(x) \). The first problem that arises is that \( \phi(x) \) is not just a function of \( x \) since it may or may not be smooth. Average over configuration \( \phi(x) \) which vary rapidly over short distances are likely to be singular. In fact \( \langle \phi^2 \rangle \) diverges even for a free theory

\[ \langle \phi^2 \rangle = \lim_{x \to y} \langle \phi(x) \phi(y) \rangle \]

Another problem is the meaning of such
a limit. Obviously we do not necessarily mean that \( |x-y| < a \) where 
\( a \) is the lattice spacing. Rather we want to work in the regime
\[ a \ll |x-y| \ll \xi \]
and still be able to remove (or try to remove) the cutoff \( a \).

The first task is then to define Green's functions including \( \phi^2 \)
operators. Formally we define
\[ G^{(N,L)}(x_1,\ldots,x_N; y_1,\ldots,y_L) = \langle \phi(x_1) \cdots \phi(x_N) \phi(y_1) \cdots \phi^2(y_L) \rangle \]

Adding a term to \( \mathcal{S} \) of the form of a spatially varying mass
\[ \delta \mathcal{S} = - \int \frac{t(y)}{2!} \phi^2(y) \, dy \]
we can write
\[ G^{(N,L)}(x_1,\ldots,x_N; y_1,\ldots,y_L) = \frac{\delta^{N+L}}{\delta t(x_1)\cdots\delta t(x_N) \delta t(y_1)\cdots\delta t(y_L)} \bigg|_{t \to 0} \]

And
\[ \frac{\delta^{N+L} \{ F \{ J+t \} \}}{\delta t(x_1)\cdots\delta t(x_N) \delta t(y_1)\cdots\delta t(y_L)} \bigg|_{t=0} = G_c^{(N,L)}(x_1,\ldots,x_N; y_1,\ldots,y_L; \{ t \}) \]

Obviously
\[ G_c^{(N,L+1)}(x_1,\ldots,x_N; y_1,\ldots,y_L, \{ t \}) = \frac{\delta}{\delta t(y_{L+1})} G_c^{(N,L)}(x_1,\ldots,x_N; y_1,\ldots,y_L, \{ t \}) \]

\[ \Rightarrow G_c^{(N,L)}(x_1,\ldots,x_N; y_1,\ldots,y_N, \{ it \}) = \sum_{K=0}^{\infty} \frac{1}{K!} \int \cdots \int dy_{L+1} \cdots dy_{L+K} \cdot \frac{t(y_{L+1}) \cdots t(y_{L+K})}{G_c^{(N+K,L)}(x_1,\ldots,x_N; y_1,\ldots,y_N, \{ it \})} \]
In terms of graphs

\[ I \sim \chi^2 \int dx'_1 \, dx'_2 \, dy'_1 \, dy'_2 \, G_0(x'_1-x'_2) \, G_0(x_1-x'_1) \, G_0(x_1-x'_2) \, G_0(x_2-x'_1) \, G_0(x_2-x'_2) \, G_0(x'_1-x'_2) \, G_0(x_1-y'_1) \, G_0(y'_1-y'_2) \, G_0(y'_2-x'_2) \cdot \]
\[ \times \tau(y'_1) \, \tau(y'_2) \]

\[ \left. \frac{\delta^2 I}{\delta G_{y_1} \, \delta G_{y_2}} \right|_{t \to 0} \sim \delta G_c \]

Next we define 1PI graphs again as those graphs which cannot be separated in 2 parts by cutting just one line, without the external legs coming to the N \( \phi \) fields.

\[ \Rightarrow \Gamma^{(N+L)}(x_1, \ldots, x_m; y_1, \ldots, y_L; t) = \frac{\delta}{\delta \Phi(x'_1)} \cdots \frac{\delta}{\delta \Phi(x'_m)} \Gamma^{(N, L)}(x_1, \ldots, x_m; y_1, \ldots, y_L) \cdot \tau(y_1) \cdots \tau(y_L) \]

Thus a knowledge of all the vertices (including composite \( \Phi \)s) at \( t=0 \) yields the correlation function for \( t \to 0 \).

Obviously

\[
\Gamma^{(N, L)}(x'_1, \ldots, x'_m) = \sum_{L=0}^{\infty} \frac{1}{L!} \int \, dy'_1 \cdots \int \, dy'_L \Gamma^{(N, L)}(x_1, \ldots, x_m; y_1, \ldots, y_L) \cdot \tau(y_1) \cdots \tau(y_L)
\]

and

\[
\Gamma^{(2,1)}(x_1, x'_2, y) = -\delta \Gamma^{(2)}(x_1, x'_2) \, G_c^{(2)}(x_1, x'_2, y) \, \Gamma^{(2)}(x_2, y)
\]
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Momentum space rep:

\[ G^{(N,1)}(x_1, \ldots, x_N, y) = \int dk_1 \ldots dk_N \ d\phi \ G^{(N,1)}(k_1, \ldots, k_N; p) \ e^{-i(k_1 x_1 + \cdots + k_N x_N)} \]

with \( G^{(N,1)}(k_1, \ldots, k_N; p) = \frac{1}{2} \langle \phi(k_1) \cdots \phi(k_N) \ \phi^2(p) \rangle \)

where \( \phi^2(p) \) is the F.T. of \( \phi^2(y) \)

\[ \phi^2(p) = \int dx \ e^{-i \ p \ x} \ \phi^2(x) = \int dq \ \phi(q) \ \phi(p-q) \]

(product \( \Leftrightarrow \) convolution)

\[ \Rightarrow \ G^{(N,1)}(k_1, \ldots, k_N, p) = \frac{1}{2} \int dq \ G^{(N+2)}(k_1, \ldots, k_N; q, p-q) \]

quite generally

\[ G^{(N,1)}(x_1; k_1, \ldots, x_N; k_N) = \frac{1}{2} \int dq_1 \ldots dq_N \ G^{(N+2)}(x_1; k_1, \ldots, q; x_N; q, p_1-\cdots-q_N) \]

example

\[ G^{(2,1)}(k_1, k_2; p) \]

As it is apparent, \( G^{(N,1)} \) are going to be more u.v. divergent (and less i.r. divergent) than the \( G^{\mu} \)’s

Compton-field renormalization:

Let’s calculate \( G^{(2,1)} \) to one loop. Diagrammatically we have

\[ G^{(2,1)} \sim \frac{1}{k_1} \ p + \frac{1}{k_1+k_2} \]

\[ \pi^{(2,1)}(k_1, k_2; p) = 1 - \frac{A}{2} \int \frac{1}{(q^2 + m^2)(q + \mu)^2 + m^2} \]

Note that as \( \lambda \to 0 \), \( \pi^{(2,1)} \to 1 \).

Upon mass and coupling constant renormalization, we have

\[ \pi^{(2,1)}(k_1, k_2; p) = 1 - \frac{g}{2} \int \frac{1}{(q^2 + m^2)((q + p)^2 + m^2)} \]

However, this still diverges logarithmically at \( d = 4 \). Since \( \pi^{(2,1)} \to 1 \), there is no coupling constant renormalization that can absorb this divergence. Thus, we are led to a field renormalization of the composite \( \phi^2 \).

Def. \[ \pi^{(2,1)}_R(k_1, k_2; p) = Z_{\phi^2}^{1/2} \pi^{(2,1)}(k_1, k_2; p) \]

To this order, \[ Z_{\phi^2} = \frac{1}{1 - \frac{g}{2} \int \frac{1}{(q^2 + m^2)^2}} \]

\[ Z_{\phi^2} = 1 + \frac{g}{2} \int \frac{1}{(q^2 + m^2)} \] (to one loop)

However, we also know that \( \phi \) itself gets renormalized (2 loops).

\[ Z_{\phi^2} = 1 + \frac{g}{2} \int \frac{1}{(q^2 + m^2)} + \frac{g^2}{8} \int \frac{1}{(q^2 + m^2)^2} \]
But \( \Gamma^{(2,1)} = \Gamma^{(0)} G^c (2,1) \Gamma^{(2)} \)

and \( \Gamma^{(2)} = \mathcal{Z}_\Phi \Gamma^{(2)}_N \)
\( \mathcal{Z}_\Phi^{-1} \Gamma^{(2)} N = \Gamma^{(2)} \frac{1}{R} \)

we can define
\[
\frac{G^{(2,1)}_{CR}(k_i, m_k^2, \phi_c)}{G^{(2)}_{CR}(k_i) G^{(2)}_{CR}(k_i)} = \Gamma^{(2,1)}_R(k_i, m_k^2, \phi_c) = \mathcal{Z}_\Phi^{-1} \Gamma^{(2,1)}_N(k_i, m_k^2, \phi_c)
\]

Def. \( \mathcal{Z}_\Phi^{-1} = \mathcal{Z}_\Phi^{-1} \mathcal{Z}_\Phi^{-1} = Z^4_{\Phi} \Gamma^{(2,1)}_N(k_i, m_k^2, \phi_c) \)

\( \Rightarrow \Gamma^{(2,1)}_R = Z^4_{\Phi} \mathcal{Z}_\Phi^{-1} \Gamma^{(2,1)}_N \)

in general:
\[\Gamma^{(N,1)}_R = Z^{N/2}_{\Phi} \mathcal{Z}_\Phi^{-1} \Gamma^{(N,1)}_N\]

And we have four additional renormal conditions:
\[\Gamma^{(2,1)}_R(k_i, k_k; p_i, \phi_c) \bigg|_{SP} = 1\]
\[SP \Rightarrow k_i^2 = \frac{3}{4} k_k^2\]
\[k_i, k_k = -\frac{k_k^2}{4}\]

This vertex (\(\Gamma^{(2,1)}_R\)) is also \(\mathcal{Z}_\Phi\) free, but this divergence is not taken care by all we've done so far. Thus we simply subtract it.
Renormalization away from $T_c$ (massive theory)

We start with $m^2_0 = T - T_0$ and write

$$m_0^2 \phi^2 = m_c^2 \phi^2 + (m_0^2 - m_c^2) \phi^2 \equiv m_c^2 \phi^2 + \delta m_0^2 \phi^2$$

where $m_c^2$ s.t. $m_R^2 = 0$

where $\delta m_0^2$ can be considered as the limit of a spatially varying $\phi^2$ term, or $\delta m_0^2(q_i) \to \delta m_0^2 \delta(q)$

$$= \Gamma^{(N,L)}(k_i, p_i, m_0^2, \lambda, \Lambda) = \sum_{M=0}^{\infty} \frac{1}{M!} \int dq_1 \ldots dq_M \delta m_0^2(q_1) \ldots \delta m_0^2(q_M) \Gamma^{(N,L+M)}(k_i, p_i, q_i, q_i = 0; m_c^2, \lambda, \Lambda)$$

as $\delta m_0^2 \to \delta m_0^2 \delta(q)$

$$\Gamma^{(N,L)}(k_i, p_i, m_0^2, \lambda, \Lambda) = \sum_{M=0}^{\infty} \frac{1}{M!} (\delta m_0^2)^M \Gamma^{(N,L+M)}(k_i, p_i, q_i, q_i = 0; m_c^2, \lambda, \Lambda)$$

as $\Lambda \to \infty$ the left has divergences, but

$$\Lambda \to \infty \quad \Gamma^{(N,M)}(k_i, p_i, m_0^2 = 0; q_R, R) = Z_{\phi^2}^{N/2} Z_{\phi^2}^{-2} \Gamma^{(N,M)}(k_i, p_i, m_c^2, \lambda, \Lambda)$$

except for $\Gamma^{(0,2)}$, that cannot be renormalized mult.

$$\Rightarrow \quad Z_{\phi^2}^{N/2} Z_{\phi^2}^{-2} \Gamma^{(N,L)}(k_i, p_i, m_0^2, \lambda, \Lambda) = \sum_{M=0}^{\infty} \frac{1}{M!} (\delta t)^M \Gamma^{(N,M)}(k_i, p_i, q_i, q_i = 0; m_0^2 = 0, \lambda, \Lambda)$$

with $N \neq 0$

$$\delta m_0^2 = T - T_c = Z_{\phi^2} \delta t$$

rem. temp. diff.

$\delta t \to 0 \Rightarrow$ Heaviside is finite a $\Lambda \to \infty$. Thus we can still use the
Param. constants of the massless theory. This is so because the UV divergences are controlled by the region \( \Lambda \gg |k| \gg M_R \) and don't play a significant role in these UV chiral.

\[
\Pi_R^{(\Lambda_0)}(k_i, p_i; \delta t, \delta t', \theta_R, k) = \sum_{M=0}^{\infty} \frac{1}{M!} \Pi_R^{(N+M)}(k_i, p_i; \delta t, \delta t', \theta_R, k)
\]

Similarly one can study the broken phase. To find

\[
\lim_{\Lambda \to 0} \frac{Z_{\phi^2} Z_{\omega_0}^L}{Z_{\gamma_0}} \Pi_R^{(\Lambda_0)}(k_i, p_i; \delta t, \delta t', \theta_R, \lambda, \Lambda) = \Pi_R^{(\Lambda_0)}(k_i, p_i; \delta t, M, \delta k)
\]

\[
= \sum_{M, P=0}^{\infty} \frac{1}{M! P!} \Pi_R^{(N+M+P)}(k_i, p_i; \delta t, \delta t', \theta_R, \gamma_0, \omega_0, \phi, \lambda, \Lambda)
\]

with \( \phi = Z_{\phi^2}^{1/2} M \)

\[
\Rightarrow \text{Eqs of state}
\]

\[
H(\delta t, M, \delta k, k) = \sum_{M, P=0}^{\infty} \frac{1}{M! P!} \Pi_R^{(N+M+P)}(k_i, p_i; \delta t, \delta t', \theta_R, \gamma_0, \omega_0, \phi, \lambda, \Lambda)
\]

\[
H = \sum_{M, P=0}^{\infty} \frac{1}{M! P!} \Pi_R^{(N+M+P)}(k_i, p_i; \delta t, \delta t', \theta_R, \gamma_0, \omega_0, \phi, \lambda, \Lambda)
\]

is the equation of state.
Lecture 34 (4/11)

Comments on renormalization of theories with continuous symmetries.

If the theory has continuous symmetry then the renormalization procedure has to be consistent with it. We saw that there are Ward Identities that guarantee, for example, that in the symmetric theory, the 2-point function remains diagonal to all orders in Perturbation Theory. Likewise the vertices have to be compatible with the symmetry. Thus renormalization must be consistent with the Ward Identities and all vertices must have the same group (or tensorial) structure as the bare ones.

E.g. since \( \Pi^{(2)}_{ij}(k) = \delta_{ij} \cdot \Pi^{(0)}_{kk}(k) \)

so we must only impose conditions on \( \Pi^{(0)} \) itself (identical to those we've discussed above) etc.

We have profound consequences on the renormalization procedure.

(a) They relate renormalized constants
(b) They relate renormalized (finite) quantities in the broken symmetry state

Example: Consider the O(2) theory. We know that

\[ \Pi_{\sigma\sigma}(\mathbf{s}) = \Pi_{\sigma\sigma}(\mathbf{s}) \] 

\( \Rightarrow \) the renormalized constants must be equal if we want to maintain this relationship. It's not hard to show that no parts are compatible with this requirement.

Also

\[ \Pi_{\sigma}(p) - \Pi_{\pi\pi}(p) = \nu \Pi_{\chi\sigma}(0, p, -p) \]
\[ \Gamma^{(n)}_{\lambda \bar{\lambda}}(k_1, \ldots, k_r, m^2, \bar{\phi}, \lambda, \Lambda) = \sum_{i,j=0}^{\infty} \frac{1}{i! j!} \int \frac{d^d k}{(2\pi)^d} \frac{d^d \bar{\phi}(x)}{(2\pi)^d} \frac{d^d \phi(y)}{(2\pi)^d} \delta^n(k_1, \ldots, k_r) \delta^0 (k_i - k_j) \]

We know that \( \Phi_0 \) and \( \Phi_{\mu} \) contain a quadratically divergent piece, while it is easy to show that \( \Phi_{\mu\nu} \) does not. Then the quadratically divergent piece in \( \Phi_{\mu} \) and \( \Phi_0 \) must be equal and cancel exactly. Moreover, the linear piece must equal the linear piece in \( \Phi_{\mu\nu} \).

An easy way to see that is to expand \( \Phi_{\mu\nu} \) around the symmetric theory. Since the \( 3 \)-point function vanishes in the symmetric theory, the lowest contribution comes from the \( 4 \)-point function. \( \Rightarrow \frac{\partial^2}{\partial \rho^2} \Phi_{\mu\nu} \) has no primitive divergences, since \( \Phi_{\mu\nu} \) has \( \mu \neq \nu \) in the \( \rho \) part.

\[ \Rightarrow \frac{\partial}{\partial \rho} \Phi_{\mu\nu} - \frac{\partial}{\partial \rho} \Phi_0 = \nu \frac{\partial}{\partial \rho} \Phi_{\mu\nu}(\rho, \beta, \gamma) \]

Since the LHS has no primitive \( \rho \)-div, \( \Rightarrow \) l.h.s. has none either.

\[ \Rightarrow \] the same subtraction that make \( \frac{\partial}{\partial \rho} \Phi_{\mu\nu} \) finite (by removing the \( \rho \)-div) make \( \frac{\partial}{\partial \rho} \Phi_0 \) finite too and the divergent parts must be equal.

The theory at \( d \neq d_c \)

Up to now I've only discussed the renormalizable case, i.e. \( d = d_c = 4 (\phi^4) \).

For \( d > 4 \) (i.e. \( d < d_c \)) the degree of u.v. divergence grows with the order of p.t. Thus the short distance behavior is ill defined and p.t. is not reliable in this domain. On the other hand, the theory is trivial in the infrared. We saw already that for \( d > d_c \) the graphs are I.R. finite. Even this means that at long distance fluctuations become
unimportant and the classical (i.e. Landau) theory is qualitatively correct. 
(The exponents, etc., are the same) and $\xi \sim |T-T_c|^{-\frac{1}{2}}$ $(d>4)$

But if $d<4$ the situation is inverted. The u.v. sector is trivial. 
The theory is said to be super-renormalizable. Thus, except for the 
mass renormalization, the theory is finite. Moreover, the only divergent (u.v.) 
graph is the tadpole and the theory can be made finite essentially by 
normal ordering the mass term. However, at long distance the 
situation is quite different. As a matter of fact, the degree of I.R. 
divergence grows with the order in p.r. and the massless theory 
becomes very different from the classical theory. Thus approaching the 
critical point becomes problematic. How can we solve this problem? 
Assume that physical quantities change smoothly with the dimension.
For instance, we know that classically, the 2-point function behaves 
like
$\langle p^{(2)}(p) \rangle \sim p^{2}$

In position space we have
$\langle \phi(x) \phi(y) \rangle \sim \frac{1}{|x-y|^{d-2}}$

This is not always true since we know that there are I.R. div. contrib. 
Let's assume that
$\langle \phi(x) \phi(y) \rangle \sim \frac{1}{|x-y|^{d-2+\eta}}$

i.e.
$\langle p^{(2)}(p) \rangle \sim p^{2-\eta}$

If $\eta$ is small we can expand in powers of $\eta$ for fixed $p$
\[ p^2 - \gamma \ln p = 1 - \gamma \ln p + \gamma^2 \ln^2 p + \cdots \]

\[ \Rightarrow \Gamma(p) = p^2 \left[ 1 - \gamma \ln p + \gamma^2 \ln^2 p + \cdots \right] \]

Thus if a small parameter could be found we could devise an expansion and somehow by resumming the p.t. series we could make sense of the divergent contributions of the type shown above. In fact there is such a parameter: it is \( \epsilon = 4 - d \) (in our theory). The strategy that we will use will be to solve the theory at the critical dimension first and to expand in powers of \( \epsilon \) later to explore the theory at \( d < d_c \). Naturally we will not go very far unless we can compute to very high orders. In practice this is three loops (or maybe one more). Recently Zinn Justin et al. have cleverly used these results together with an understanding of the behavior of large orders of p.t. to make predictions at \( d = 3 \).

The regime \( d < d_c \) will be explored by expanding the diagrams in powers of \( \epsilon \).

Lecture 35 (4/13)

The Renormalization Group.

In previous lectures we have discussed that it was possible to render the theory finite by defining a set of renormalization constants, \( \tilde{\Gamma}^{(N)}(k; i, p; j, m_k, g_R, \lambda) = Z_{\phi^L} Z_{\phi^R} \tilde{\Gamma}^{(N)}(k; i, p; j, m_k, g_R, \lambda) \rightarrow \lambda \rightarrow 0 \)
If we consider the massless theory we have \( \mu_c^2 = m_c^2 \) and \( m_c = 0 \).

The constants are then functions of the renormalization scale since

is the only dimensional quantity left.

\[
Z \phi = Z \phi (g_R (k), K, \Lambda)
\]

\[
= \lambda (g_R (k), K, \Lambda)
\]

\[
m_c = m_c (g_R (k), K, \Lambda)
\]

Of course \( K \) is arbitrary. Thus if we compare the same theory

but renormalized at two different scales \( K_1 \) and \( K_2 \), they must

relate to each other since the compared to the same bare theory.

\[
\Gamma_R^{(n)} (p_i ; K_1, g_R (K_1)) = Z_{\phi}^{N/2} (g_R (K_1), K_1, \Lambda) \Gamma^{(n)} (p_i, \lambda, \Lambda)
\]

\[
\Gamma_R^{(n)} (p_i ; K_2, g_R (K_2)) = Z_{\phi}^{N/2} (g_R (K_2), K_2, \Lambda) \Gamma^{(n)} (p_i, \lambda, \Lambda)
\]

\[\Rightarrow \Gamma_R^{(n)} (p_i, g_R (K_1), K_1) = Z (g_R (K_2), K_2, g_R (K_1), K_1) \Gamma_R^{(n)} (p_i, g_R (K_2), K_2)\]

\[Z (g_R (K_2), K_2, g_R (K_1), K_1) = \frac{Z_{\phi} (g_R (K_1), K_1, \Lambda)}{Z_{\phi} (g_R (K_2), K_2, \Lambda)} \quad \text{as} \quad \Lambda \to \infty\]

Since it is the ratio of \( \phi \) renorm. vertex function \( \Rightarrow \) it is finite as \( \Lambda \to \infty \)

\[\Rightarrow \] a change in the momentum scale is equivalent to a rescaling

of the fields by \( Z^{1/2} \) and a change in the renorm. coupling

\[g_1 = g_R (K_1) \Rightarrow g_2 = g_R (K_2)\]

\[Z (K_2, g_2 ; K_1, g_1) = \frac{\partial}{\partial K_2} \Gamma_R^{(2)} (K, g_1, K_1) \bigg|_{K_2 = K_1}^{K_2 = K_2}\]
and \( g_2 = z^{-2} \left. \Gamma_{k_i}^{(y)} (k_i, g_i, k_i) \right|_{k_i = sp(k_i)} = R(k_2, k_1, g_1) \)

and \( R(k, k_1, g) = g \)

\( \Rightarrow \) The renormalization group is the transformations represented above.

Obviously we also have

\[ \Gamma^3 \Phi, \delta_i, k_i = \Gamma^3 \left[ z^{N/2} \Phi \right] R(k_i, k_1, g_1), k_2 \]

Since the bare theory is independent of the choice of renormalization scale we have

\[ \left. \left[ K \frac{\partial}{\partial K} \right] \Gamma^{(w)} (k_1, \lambda, m^2, \lambda) = 0 \right|_{\lambda, \lambda} \]

\( \Rightarrow K \frac{\partial}{\partial K} \left[ z^{N/2} \Gamma^{(w)} (\Phi, g_2(k), k) \right] = 0 \)

\[ \lambda, \lambda \]

Thus it is also true that

\[ \left[ K \frac{\partial}{\partial K} + \frac{\partial}{\partial g} (g, k) \frac{\partial}{\partial g} - \frac{1}{2} N \gamma_\phi (g, k) \right] \Gamma^{(w)} (\Phi, g_2(k), k) = 0 \]

\[ \lambda, \lambda \]

where

\[ \frac{\partial}{\partial g} (g, k) = \left. \frac{\partial}{\partial K} \right|_{\lambda, \lambda} \]

\[ \gamma_\phi (g, k) = \left. \frac{\partial \phi}{\partial K} \right|_{\lambda, \lambda} \]

But the coupling constant is in general dimensionless. Define a dimensionless coupling constant \( u_0 \) s.t.
\[ \lambda = u_0 \ K \varepsilon \quad \text{and} \quad \gamma_k = u \ K \varepsilon \]

In terms of \( u \) we have

\[ \left( K \frac{\partial \gamma}{\partial k} + \beta(u) \frac{\partial \gamma}{\partial u} - \frac{N}{2} \phi(u) \right) \Gamma^{(0)}_R (p_i, u, k) = 0 \quad (\Lambda \to \infty) \]

\[ \beta(u) = K \frac{\partial u}{\partial k} \bigg|_{\lambda, \Lambda} \quad \text{as} \quad \Lambda \to \infty \]

\[ \delta \phi(u) = \frac{\partial \ln Z\phi}{\partial \ln k} \bigg|_{\lambda, \Lambda} \]

These formulas are somewhat awkward since what we have is \( u_0 = f(u) \) and not the other way around.

\[ K \left( \frac{\partial u}{\partial k} \right)_k = -K \left( \frac{\partial \gamma}{\partial k} u \right) \left( \frac{\partial \lambda}{\partial u} \right)_k \]

and \( \lambda = K \varepsilon u_0 (u, K/\Lambda) \) (dimensional and !)

But \( u_0 \) is finite as \( \varepsilon \to 0 \) (\( d \to 4^- \)) for \( \Lambda < \infty \) and it is also \( \infty \) if \( d = 4 \) and \( \Lambda \to \infty \).

It is better to compute thunks at \( d = 4 \) as \( \Lambda \to \infty \).

\[ \lim_{\Lambda \to \infty} K \left( \frac{\partial \gamma}{\partial k} \right)_u = e^{\lambda} \Rightarrow \]

\[ \beta(u) = K \left( \frac{\partial \gamma}{\partial k} \right)_u = -e \left( \frac{\partial \ln u_0}{\partial u} \right)^{-1} \]

Now \( \beta(u) \) is a power series in \( u \) with coeff. which are \( \varepsilon \)-dependent.

Likewise

\[ \delta \phi(u) = \frac{\partial \ln Z\phi}{\partial \ln k} \bigg|_{\lambda} = K \left( \frac{\partial \gamma}{\partial k} \right)_u \frac{\partial \ln Z\phi}{\partial u} = \beta(u) \frac{\partial \ln Z\phi}{\partial u} \]

\[ \delta \phi(u) = \beta(u) \frac{\partial \ln Z\phi}{\partial k} \]
In general for \( p \not= 1 \) we can have
\[
\left[ \frac{k^2 \partial^2}{\partial k^2} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \varphi(u) + L \varphi_2(u) \right] \Phi^{(N)}(k, p_i, u, \kappa) = 0
\]
\[
\varphi_2(u) = - \left( \frac{\partial}{\partial k} \kappa \Phi^2 \right) \lambda = - \beta(u) \frac{\partial}{\partial u} \kappa \Phi^2
\]

Solution of the RG equations

We begin with
\[
\left[ \frac{k^2 \partial^2}{\partial k^2} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \Phi(u) \right] \Phi^{(N)}(p_i, u, \kappa) = 0 \tag{1}
\]

let \( \Phi = \kappa \Phi \) and consider
\[
\Phi^{(N)}(p_i, u, \kappa) = \exp \left\{ \int_{u_2}^{u} \frac{du'}{\beta(u')} \right\} \Phi^{(N)}(p_i, u, \kappa)
\]

\( \Rightarrow \) (1) reduces to
\[
\left[ \frac{\partial^2}{\partial \kappa^2} + \beta(u) \frac{\partial}{\partial u} \right] \Phi^{(N)}(p_i, u, \kappa) = 0
\]

It's easy to solve this again.
\[
\Phi^{(N)}(p_i, u, \kappa) = \Phi^{(N)}(p_i, \kappa - \int_{u_2}^{u} \frac{du'}{\beta(u')}) \quad \text{for arbitrary}
\]

\( u_1, u_2 \) are integration constants.

\( \Rightarrow \)
\[
\Phi^{(N)}(p_i, u, \kappa) = \exp \left\{ N \int_{u_2}^{u} \frac{du'}{\beta(u')} \right\} \Phi^{(N)}(p_i, \kappa - \int_{u_2}^{u} \frac{du'}{\beta(u')})
\]

is thus the solution. Obviously \( \Phi^{(N)} \) cannot be determined in p.t.

Assume now that all moments are changed by a factor \( p \)
\[
p_i \rightarrow p p_i
\]
Dimensionally it is true that

\[ \Pi_{R}^{(\kappa)}(p, p_i, u, \kappa) = \rho^{N + d - \frac{Nd}{2}} \Pi_{R}^{(u)}(p, u, \kappa) \]

\[ \Rightarrow \Pi_{R}^{(N)}(p, p_i, u, \kappa) = \rho^{N + d - \frac{Nd}{2}} \exp \left\{ \frac{N}{2} \int_{u_1}^{u} \frac{du' \phi(u')}{\beta(u')} \right\} F^{(\kappa)}(p_i, \kappa - \hbar p) \int_{u_2}^{u} \frac{du'}{\beta(u')} \]

since \( k = \hbar \kappa \Rightarrow \kappa \rightarrow \frac{k}{\hbar} \Rightarrow k \rightarrow k - \hbar p \)

Define a scale dependent coupling constant \( u(p) \)

\[ s = \hbar u = \int_{u_1}^{u} \frac{du'}{\beta(u')} \]

\[ \Rightarrow \frac{\partial u(s)}{\partial s} = \beta(u(s)) \quad \text{with} \quad u(s=0) = u \]

And we get

\[ \Pi_{R}^{(\kappa)}(p, p_i, u, \kappa) = \rho^{N + d - \frac{Nd}{2}} \exp \left\{ \frac{N}{2} \left[ \int_{u_1}^{u} \frac{\phi(u')}{\beta(u')} - \int_{u_1}^{u}(u(s)) \phi(u') \right] \right\} x \]

\[ F^{(\kappa)}(p_i, \kappa - \hbar p) \int_{u_2}^{u} \frac{du'}{\beta(u')} \]

\[ \Pi_{R}^{(N)}(p, p_i, u, \kappa) = \rho^{N + d - \frac{Nd}{2}} \exp \left\{ \frac{N}{2} \int_{u_1}^{u} \frac{\phi(u')}{\beta(u')} \right\} \times \]

\[ \Rightarrow \text{a change in the momentum scale is equivalent to} \]

(1) Scaling of \( \Pi^{(\kappa)} \) by its canonical dimension

(2) A running coupling constant

(3) An anomalous factor.
Lecture 36 (4/15) : Fixed points and scaling

In the last lecture we derived a formula connecting the vertex function at two different momentum scales.

\[ \Gamma_R^{(N)}(\mathbf{p}; \mathbf{u}, \mathbf{w}, \mathbf{K}) = \mathcal{S}^{N+d-\frac{Nd}{2}} \exp \left\{ -\frac{N}{2} \int_0^\infty \frac{\gamma \phi(u')}{\beta(u')} \, du' \right\} \times \Gamma_R(\mathbf{p}; \mathbf{u}(s), \mathbf{K}) \]

This is a very complex formula. Let's look at its implications at a fixed point. That is let's assume that there is a value of \( u \), say \( u^* \), such that \( \beta(u^*) = 0 \).

At \( u^* \) we have

\[ \int_0^{s_0} \frac{\gamma \phi(u)}{\beta(u')} \, du' = \int_{s_0}^{s_0+s} \gamma \phi(u^*) \, ds' = \gamma \phi(u^*) \, s = \gamma \phi(u^*) \, \Delta s \]

\[ s = \int \frac{du'}{u \beta(u')} \]

and

\[ \Gamma_R^{(N)}(\mathbf{p}; \mathbf{u}^*, \mathbf{K}) = \mathcal{S}^{N+d-\frac{Nd}{2} - \frac{N}{2} \gamma \phi(u^*)} \Gamma_R(\mathbf{p}; \mathbf{u}^*, \mathbf{K}) \]

For example

\[ \Gamma_R^{(2)}(\mathbf{p}; \mathbf{u}^*, \mathbf{K}) = \mathcal{S}^{2 - \gamma \phi(u^*)} \Gamma_R^{(2)}(\mathbf{p}^*, \mathbf{u}^*, \mathbf{K}) \]

Let \( s = \frac{\mathbf{p}}{\mathbf{p}^*} \rightarrow 1 \) as \( p \rightarrow 0 \)

\[ \Rightarrow \Gamma_R^{(2)}(\mathbf{p}, \mathbf{u}^*, \mathbf{K}) = \left( \frac{\mathbf{p}}{\mathbf{p}^*} \right)^{2 - \gamma \phi(u^*)} \Gamma_R^{(2)}(\mathbf{p}^*, \mathbf{u}^*, \mathbf{K}) \]
For example we can take $p^* = k$

$$\Gamma^{(2)}_K (p, u^*, k) = \left(\frac{p}{k}\right)^2 \frac{\Gamma^{(2)}_K (k, u^*, k)}{k^2 + \ldots}$$

and we know that $\Gamma^{(2)}_K (k, u^*, k) \sim k^2 + \ldots$ ($p \gg k$)

$$\Gamma^{(2)}_K (p, u^*, k) = p^2 \left(\frac{p}{k}\right)^2 \frac{\delta \phi(u^*)}{\delta \phi(u^*)} \text{ at the fixed point}.$$

Thus the 2-point function scales and exhibits an anomalous dimension $\eta = \gamma_{p}(u^*)$. We see therefore that interactions can in fact make a big difference. In particular the dimensions of the various operators, and hence exponents, may not necessarily be those of the classical theory!

Approach to the fixed point: A fixed point could be attractive or repulsive. Let

$$\beta(u) \approx \beta'(u^*) (u-u^*) + O((u-u^*)^2)$$

(i.e. $\beta > 0$ if $u > u^*$)

If $\beta'(u^*) > 0 \Rightarrow$ as $s \to -\infty$ we have $u \to u^*$ (a)

If $\beta'(u^*) < 0 \Rightarrow$ as $s \to +\infty$ we have $u \to u^*$ (b)

(a) $u^*$ is said to be IR stable (UV unstable)

(b) $u^*$ is said to be UV stable (IR unstable)

And let $Y_{\phi(u)} u \left(\begin{array}{c}
\frac{\gamma_{\phi(u)} + \gamma'(u^*) (u-u^*) + O((u-u^*)^2)}{eta'(u^*) (u-u^*) + \ldots}
\end{array}\right) du'$
\[
\sim \frac{\gamma_0^+ (u^+)}{\beta'(u^+)} \ln \left[ \frac{u(s) - u^+}{u - u^+} \right] + \frac{\gamma_0^+ (u^+)}{\beta'(u^+)} (u(s) - u) + \ldots
\]

Thus we find

\[
\exp \left[ - \frac{1}{2} \sum_{u(s)} \frac{\delta \phi(u^+)}{\beta(u^+)} \right] \approx
\]

\[
= \lim_{N \to \infty} \frac{u - u^+}{u(s) - u^+} \left[ \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} \right] e^{- \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} (u(s) - u)}
\]

\[
= e^{- \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} s} - \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} (u(s) - u)
\]

Case (a)

\[
\frac{S \to \infty}{(P \to 0)} \Rightarrow \Gamma_K^0 (\rho k^i, u, k) = \int s - \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} e^{- \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} (u(s) - u)}
\]

\[
\Rightarrow \Gamma_K^0 (\rho k^i, u^*, k) = \frac{\beta^2}{\beta'^2} \left( \frac{\delta}{k} \right)^2 \Gamma_K^0 (\rho k^i, u^*, k)
\]

\[
\Rightarrow \Gamma_K^0 (\rho, u, k) = \left( \frac{\delta}{k} \right)^2 \Gamma_K^0 (\rho k^i, u^*, k)
\]

\[
\frac{\delta = \rho k}{\rho \to 0}
\]

Thus we corrections to F.P. behavior. \(\rho \to 0\) (IR stable F.P.)

Case (b)

\[
\frac{S \to \infty}{(P \to \infty)} \Rightarrow \Gamma_K^0 (\rho, u, k) = \int s - \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} e^{- \frac{N}{2} \frac{\delta \phi(u^+)}{\beta(u^+)} (u^* - u)}
\]

\[
\Rightarrow \Gamma_K^0 (\rho, u, k) = \frac{\beta^2}{\beta'^2} \left( \frac{\delta}{k} \right)^2 \Gamma_K^0 (\rho k^i, u^*, k)
\]

Case (b) Obviously we've got the same situation \(u \to u^*\) at short distance, i.e. for large \( \rho \). The reason is that the long distance behavior is not
controlled by $a^*$ but by some other (IRS) fixed point.

Marginally: If $\beta(u)$ has a double zero we get exceptional situation.

Let $\beta(u) = a (u-u^*)^2$

$\Rightarrow a > 0 \Rightarrow IRS$ (marginally stable)

$\Rightarrow a < 0 \Rightarrow IRU$ (marginally unstable)

Solving the $\beta$-function equation

$$\frac{\partial u}{\partial s} = a (u-u^*)^2 \Rightarrow u(s) = u^* - \frac{u^* - u}{(u^* - u)(a s + 1)}$$

$$u(s) \Rightarrow \int \frac{y_{\phi}(u)}{\beta(u)} \, du' = y_{\phi}(u^*) s + \frac{y_{\phi}'(u^*)}{a} \ln \left( \frac{u(s) - u^*}{u - u^*} \right)$$

and

$$s = -\frac{1}{a} \left[ \frac{1}{u(s) - u^*} - \frac{1}{u - u^*} \right] \xrightarrow{u(s) \to u^*} \frac{1}{a (u(s) - u^*)}$$

$$\exp \left( -\frac{N}{2} \int u(s) \frac{y_{\phi}}{\beta} \right) \xrightarrow{a \to 0} C e^{-\frac{N}{2} y_{\phi}(u^*) s + \frac{N y_{\phi}'(u^*)}{a} \ln s}$$

$$\equiv C \rho - \frac{N y_{\phi}'(u^*)}{a} \left( \ln \rho \right)^{\frac{N y_{\phi}'(u^*)}{a}}$$

Thus

$$\Gamma_C^{(0)}(p, u, k) = (2 - y_{\phi}(u^*))^{\frac{y_{\phi}'(u^*)}{a}} \rho$$

$$\int_0^\infty \frac{y_{\phi}(u^*)}{\rho} d\rho$$

Thus

$$q = \rho \xrightarrow{a \to 0} (u - u^*) \quad (a > 0)$$

$$\Gamma_C^{(0)}(q, u, k) \sim q^{2 - y_{\phi}(u^*)} \left( \ln \frac{q}{k} \right)^{\frac{y_{\phi}'(u^*)}{a}} O(1)$$

Thus in a marginal case there are significant corrections at long distance.
RGalove $T_c$ : the massive theory

We want to study the theory for $dt \neq 0$. As before, we'll expand in powers of $dt$ and use the scaling of composite operators:

$$P^{(N)}_R (k_i, \delta t, u, k) = \lim_{p_i \to 0} \sum_{L=0}^{\infty} \frac{t^L}{L!} \Gamma^{(N)}_R (k_i, p_i, u, k)$$

→ massless theory

Compute

$$\left[ \frac{K^2}{\partial_k} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \phi(u) + \partial \phi(u) \frac{t}{\partial t} \right] \Gamma^{(N)}_R (k_i, t, u, k) =$$

$$= \lim_{p_i \to 0} \sum_{L=0}^{\infty} \frac{t^L}{L!} \left[ \frac{K^2}{\partial_k} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \phi(u) + \partial \phi(u) \frac{t}{\partial t} \right] \Gamma^{(N)}_R (k_i, p_i, u, k)$$

$$\Rightarrow \left[ \frac{K^2}{\partial_k} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \phi(u) + \partial \phi(u) \frac{t}{\partial t} \right] \Gamma^{(N)}_R (k_i, t, u, k) = 0$$

setting $t \to dt$ afterwards. This equation governs the

behavior of the $\Gamma^{(N)}_R$'s away from $T_c$.

* Lecture 37 (9/18)

Fixed Point behavior: $u = u^* \Rightarrow \beta(u^*) = 0$

We've got to solve

$$\left[ \frac{K^2}{\partial_k} - \frac{N}{2} \phi(u^*) + \partial \phi(u^*) \frac{t}{\partial t} \right] \Gamma^{(N)}_R (k_i, t, u^*, k) = 0$$
The solution is:

\[ \mathcal{G} \mathcal{P}_R^n(k_i, t, k) = \frac{N/2}{\mathcal{Q}^{(n)}(\phi_0(u^*), \mathcal{F}^{(n)}(k_i, k t^{\gamma_0}))} \]

where \( \eta = \mathcal{Q}^{(n)}(\phi_0(u^*) \quad \theta = -\mathcal{Q}^{(n)}(\phi_0(u^*)} \)

Using the same arguments as before we get

\[ \mathcal{P}_R^n(k_i, t, k) = \mathcal{Q}^{n+1-d-\frac{N-d}{2}} \mathcal{P}_R^n\left(\frac{k_i}{\mathcal{Q}^{(n)}}, t, k\right) \]

\[ [t] = [\mathcal{Q}^{(n)}] = \mathcal{Q}^{(n+1-d-\frac{N-d}{2})} \]

\[ \Rightarrow \mathcal{P}_R^n(k_i, t, k) = \mathcal{Q}^{n+1-d-\frac{N-d}{2}} \mathcal{Q}^{(n)} \left(\frac{t}{\mathcal{Q}^{(n)}}, \frac{k}{\mathcal{Q}^{(n)}}\right) \]

If we choose \( \frac{k}{\mathcal{Q}^{(n)}} \left(\frac{t}{\mathcal{Q}^{(n)}}\right)^{\gamma_0} = 1 \iff \mathcal{Q}^{(n)} = \mathcal{Q}^{(n+1-d-\frac{N-d}{2})} \left(\frac{k}{\mathcal{Q}^{(n)}}\right) \left(\frac{t}{\mathcal{Q}^{(n)}}\right)^{\gamma_0+1} \)

We find

\[ \mathcal{P}_R^n(k_i, t, k) = k^{n+1-d-\frac{N-d}{2}} \left(\frac{t}{k_i}\right)^{\gamma_0+1} \mathcal{Q}^{(n+1-d-\frac{N-d}{2})} \left(\frac{k}{k_i}\right)^{\gamma_0} \mathcal{Q}^{(n)} \left(\frac{t}{k_i}\right)^{\gamma_0} \mathcal{Q}^{(n)} \left(\frac{k}{k_i}\right) \]

Thus the vertex function depends only on the combination \( k_i \)

\[ \mathcal{Q}^{(n)} \left(\frac{k}{k_i}\right) = \mathcal{Q}^{(n+1-d-\frac{N-d}{2})} \left(\frac{k}{k_i}\right)^{\gamma_0} \mathcal{Q}^{(n)} \left(\frac{t}{k_i}\right)^{\gamma_0+1} \mathcal{Q}^{(n)} \left(\frac{k}{k_i}\right) \]

\[ \Rightarrow \mathcal{Q}^{(n)} \left(\frac{k}{k_i}\right) = \frac{1}{\mathcal{Q}^{(n+1-d-\frac{N-d}{2})} \left(\frac{k}{k_i}\right)^{\gamma_0+1} \mathcal{Q}^{(n)} \left(\frac{t}{k_i}\right)^{\gamma_0+1} \mathcal{Q}^{(n)} \left(\frac{k}{k_i}\right) \}

N=2 \Rightarrow \mathcal{P}_R^2(k, t, k) = k^2 k^{2-2} f(k^2)

and \( f(k^2) \to \text{Const} \)

Also \( \mathcal{P}_R^2(k, t, k) = k^2 k^{2-2} f(k^2) \to k^2 k^{-2+2} g(k^2) \to \text{Const} \)
Thus \( x^{-1} = \Gamma_k^{(2)}(0) \propto \xi^{2-\eta} = t - \gamma \)

\[ \Rightarrow \ \gamma = \nu(2-\eta) \] Scaling relation.

---

**Renormalization Group equations for the bare theory**

I want to discuss now a procedure which, being completely equivalent to what was done so far, is more intuitive and closer to the scheme of Wilson's.

Consider again the critical renormalized theory

\[ \Gamma_k^{(2)}(0, u, \kappa) = 0 \]

\[ \frac{\partial}{\partial k^2} \Gamma_k^{(2)}(k, u, \kappa) \bigg|_{k = k^*} = 1 \]

\[ \rho_k^{(4)}(k, u, \kappa) \bigg|_{\delta \kappa} = u \kappa^\delta \]

Renormalized and bare quantities are related via

\[ \Gamma_k^{(N)}(k, \kappa, \lambda, \Lambda) = Z^{N/2} \Gamma_k^{(N)}(k, \lambda, \Lambda) \quad (\Lambda \to \infty) \]

Thus for \( \Lambda \) large the dependence of \( \Gamma^{(N)} \) on \( \Lambda \) is cancelled by \( Z^{N/2} \).

The renorm. theory is, to leading order, indep. of \( \Lambda \)

\[ \frac{\partial}{\partial \Lambda} \Gamma_k^{(N)}(k, \kappa, \lambda) \bigg|_{u, \kappa} = 0 \]

\[ \Rightarrow \ \left[ \frac{\partial}{\partial \Lambda} + \beta(\overline{\alpha_0}, \epsilon, \kappa) \frac{\partial}{\partial \overline{\alpha_0}} - \frac{N}{2} \gamma_0(\overline{\alpha_0}, \epsilon, \kappa) \right] \Gamma_k^{(N)}(k, \lambda, \Lambda) = 0 \]
with \[ \beta(\bar{u}_0, \epsilon, \lambda) = \lambda \left( \frac{\partial \bar{u}_0}{\partial \Lambda} \right) u, \lambda \]

\[ \gamma(\bar{u}_0, \epsilon, \lambda) = -\lambda \frac{\partial \gamma}{\partial \lambda} \bigg|_{u, \lambda} \]

\[ \bar{u}_0 = \Lambda^{-\epsilon} \lambda \]

Thus we tune the bare parameters as \( \Lambda \) varies while keeping \( \kappa \) fixed. As \( \Lambda \to \infty \) the function reach a finite limit since the \( \phi(\bar{u}_0, \epsilon) \)'s are independent of \( \kappa \)

\[ \Rightarrow \quad \left[ \Lambda \frac{\partial}{\partial \lambda} + \beta(\bar{u}_0, \epsilon) \frac{\partial}{\partial \bar{u}_0} - \frac{N}{2} \gamma(\bar{u}_0, \epsilon) \right] \pi^{(N)}(k, \bar{u}_0, \lambda) = 0 \]

At the r.p. \[ \pi^{(N)}(k, u^*, \lambda) = \Lambda^{N \gamma(\bar{u}_0, \epsilon)} \Phi^{(N)}(k, \epsilon) \]

\[ \Rightarrow \quad \pi^{(2)}(k, u^*, \lambda) = \Lambda^2 \kappa^{2-\epsilon} \]

Likewise one can define a running bare coupling constant \( \bar{u}_0(p) \).

\[ \frac{\partial \bar{u}_0(p)}{\partial \Lambda} = -\beta(\bar{u}_0(p)) \quad \text{with} \quad \bar{u}_0(1) = \bar{u}_0 \]

\[ \Rightarrow \quad \pi^{(N)}(k, \bar{u}_0, \rho \lambda) = e^{\frac{N}{2} \int_{\bar{u}_0}^{\bar{u}_0(p)} \frac{\delta \phi(u)}{\beta(u)} du} \pi^{(N)}(k, \bar{u}_0(p), \lambda) \]
Computation of the renormalization constants and functions:

I will sketch, very briefly, the procedure that one follows in order to compute the $Z$'s, etc.

As it was stated above, we first use the expressions of the bare coupling constant $Z \phi$ and $Z \phi^2$ in terms of the renormalized coupling constant $\lambda$, the renormalization scale $\Lambda$ and, if a cutoff is used, $\Lambda$. (Otherwise the functions will depend on $\epsilon$.)

The bare dimensionless coupling constant $u_0 = \lambda \Lambda^{-\epsilon}$ has an expansion in powers of the renormalized dimensionless coupling constant $u = g_{\bar{\epsilon}} \Lambda^{-\epsilon}$

$$u_0 = u (1 + a_1 u + a_2 u^2)$$

Also we may ask for $Z \phi$ and $Z \phi^2$

$$Z \phi = 1 + b_1 u^2 + b_2 u^3$$

$$Z \phi^2 = 1 + c_1 u^2 + c_2 u^2$$ (Better def. $\bar{Z} \phi^2 = Z \phi Z \phi^2$)

How do we determine the coeff.? There are two possible alternatives

(i) Use renormalization condition and demand that the new quantities satisfy those conditions.

(ii) Use dimensionless eq. with minimal subtraction and demand that the strong (i.e. poles) dependence on $\epsilon$ be cancelled.

If these coeff. are known, then the RG functions are known since

$$\beta(\lambda) = - \epsilon \left( \frac{\partial}{\partial u} \right) Z \phi - \epsilon u \left( 1 - a_1 u + 2(a_1^2 - a_2) u^2 + \cdots \right)$$
\[ \gamma \phi (u) = \beta (u) \frac{\partial \mu \phi}{\partial u} = - \varepsilon u \left( 2b_2 u + (3b_3 - 2b_2 c_1) u^2 \right) \]

\[ \bar{\gamma} \phi^2 = - \beta (u) \frac{\partial \mu \bar{\phi}^2}{\partial \bar{u}} = \varepsilon u \left( c_1 + (2c_2 - c_1 - a c_1) u \right) \]

(1) Using renormalization conditions:

1. Let \[ \frac{\partial}{\partial k^2} \Gamma^{(2)}(k) \bigg|_{k^2 = k^2} = 1 - B_2 u_0^2 + B_3 u_0^3 \]

   \[ B_2 = B_2 e \frac{\partial}{\partial k^2} D_3 \bigg|_{k^2 = k^2} = B_2 e k^2 D_3^{(')} \]

   \[ B_3 = B_3 e \frac{\partial}{\partial u_0} D_5 \bigg|_{k^2 = k^2} = B_3 e k^3 D_5^{(')} \]

2. \[ \Gamma^{(4)} \bigg|_{\text{sp}} = \kappa^4 \left[ u_0 - A_1 u_0^2 + (A_2^{(1)} + A_2^{(2)}) u_0^3 \right] \]

   \[ A_1 = A_1 \kappa^4 I_{\text{sp}} \]

   \[ A_2^{(1)} = A_2^{(1)} \kappa^4 I_{\text{sp}}^{(2)} \]

   \[ A_2^{(2)} = A_2^{(2)} \kappa^4 I_{\text{sp}}^{(4)} \]

3. \[ \Gamma^{(2)} \bigg|_{\text{sp}} = 1 - \frac{1}{3} C_1 u_0 + (C_2^{(1)} + C_2^{(2)}) u_0^2 \]

   \[ C_1 = C_1 \kappa^4 I_{\text{sp}} \]

   \[ C_2^{(1)} = C_2^{(1)} \kappa^4 I_{\text{sp}}^{(2)} \]

   \[ C_2^{(2)} = C_2^{(2)} \kappa^4 I_{\text{sp}}^{(4)} \]
Renormalization Conditions

(1) \[ \frac{\partial}{\partial k} \left. \Gamma^{(1)}_R \right|_{k^2 = k^2} = 1 = (1 + b_2 u^2)(1 - B_2 u_0^2 + B_3 u_0^3) \]
\[ 1 = 1 + (b_2 - B_2) u^2 + (b_3 + B_3 - 2B_2 a_1) u^3 + \ldots \]
\[ \Rightarrow \]
\[ b_2 = B_2 \quad b_3 = 2B_2 a_1 - B_3 \]

(2) \[ \left. \Gamma^{(1)}_R \right|_{sp} = \kappa^E u = \kappa^E \left( 1 + \frac{2b_2 u^2}{(1 + b_2 u^2)^2} \right) (A_1 u_0^2 + (\alpha_2^{(1)} + \alpha_2^{(2)}) u_0^3) \]
\[ \Rightarrow \]
\[ a_1 = A_1 \]
\[ a_2 = 2(A_1)^2 - (\alpha_2^{(1)} + \alpha_2^{(2)}) - 2b_2 \]

(3) \[ \left. \Gamma^{(2,1)}_R \right|_{sp} = 1 = (1 + c_1 u + c_2 u^2) \left[ 1 - C_1 u_0 + (C_2 + C_2') u_0^2 \right] \]
\[ \Rightarrow \]
\[ c_1 = C_1 \]
\[ c_2 = (a_1 + c_1) C_1 - C_2' - C_2^2 \]

\[ b_2 = \bar{B}_2 \kappa^E D'_3 \]
\[ b_3 = \kappa^{3E} \left[ 2 \bar{B}_2 \bar{A}_1 I_{sp} D'_3 - \bar{B}_3 D'_5 \right] \]
\[ a_1 = \kappa^E \bar{A}_1 I_{sp} \]
\[ a_2 = \kappa^{2E} \left\{ \left[ 2(\bar{A}_1)^2 - \bar{A}_2^{(1)} \right] I_{sp} - \bar{A}_2^{(2)} I_{sp} - 2 \bar{B}_2 D'_3 \right\} \]
\[ E_1 = k \varepsilon \bar{C}_1 I_{s_p} \]

\[ C_2 = k^2 \varepsilon_0 \left\{ \left[ (\bar{A}_1 + \bar{C}_1) \bar{C}_1 - \bar{C}_1^{(1)} \right] I_{s_p}^{-1} - \bar{C}_2^{(2)} I_{s_p}^{-1} \right\} \]

**Coefficients (Aunt)**

\[ \bar{B}_2 = \frac{N+2}{18} \]
\[ \bar{B}_3 = \frac{(N+2)(N+8)}{108} \]

\[ \bar{A}_1 = \frac{N+8}{6} \]
\[ \bar{A}_2^{(1)} = \frac{N^2 + 6N + 20}{36} \]
\[ \bar{A}_2^{(3)} = \frac{5N+22}{9} \]

\[ \bar{C}_1 = \frac{N+2}{6} \]
\[ \bar{C}_2^{(1)} = \left( \frac{N+2}{6} \right)^2 \]
\[ \bar{C}_2^{(2)} = \frac{N+2}{6} \]

**Integrals**

\[ I_{s_p} = \int_{s_p} \cdot \cdot \cdot \]
\[ = \int_{\mathbb{R}} \frac{1}{8^2 (p+q)^2} \left| \begin{array}{c} \frac{\mathbf{D}}{2\pi} \\ p = k^2 \end{array} \right| \]

\[ \frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2} \quad \text{(Feynman)} \]
\[ I_{sp} = \int_0^1 dx \int \frac{d\phi}{(2\pi)^D} \frac{1}{(q^2 + 2x p \cdot \phi + x p^2)^2} \]

\[ = \int_0^1 dx \int \frac{d\phi}{(2\pi)^D} \frac{1}{(q^2 + x(1-x) p^2)^2} \]

\[ \int \frac{\frac{d\phi}{(2\pi)^D}}{(q^2 + m^2)^n} = \frac{1}{\Gamma(n)} \int_0^{\infty} dt \ t^{n-1} \int \frac{d\phi}{(2\pi)^D} e^{-t (q^2 + m^2)} \]

\[ = \frac{1}{\Gamma(n)} \int_0^{\infty} dt \ t^{n-1} \ e^{-t m^2} \]

\[ = \frac{\Gamma(n-D)}{\Gamma(n)} \frac{m^{D-2n}}{(4\pi t)^{D/2}} \]

\[ \Rightarrow I_{sp} = \int_0^1 dx \ \frac{\Gamma(2-D)}{\Gamma(2)} \frac{(x(1-x) p^2)}{(4\pi t)^{D/2}} \]

\[ S_D = \left( 2^{D-1} \pi^{D/2} \Gamma(D/2) \right)^{-1} \]

\[ I_{sp} = k^{-\epsilon} S_D \ \frac{1}{2} \frac{\Gamma(D/2)}{\pi^{D/2}} \Gamma(2-D) \int_0^1 dx \ \left( x(1-x) \right)^{-\epsilon/2} \]

\[ \text{Def:} \quad k^\epsilon I_{sp} = J_{sp} S_D \]
\[ J_{sp} = \frac{1}{2} \Gamma\left(2 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \ (x(1-x))^{-\varepsilon/2} \]

\[ \Gamma\left(\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) = (1 - \frac{\varepsilon}{2}) \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{1+\varepsilon}{2}\right) \rightarrow \]

\[ \rightarrow (1 - \frac{\varepsilon}{2}) \frac{2}{\varepsilon} = \frac{2}{\varepsilon} - 1 \]

where \( x \Gamma(x) = \Gamma(x+1) \)

and \( \Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(\frac{1+\varepsilon}{2}\right) \xrightarrow{\varepsilon \to 0} 4 \)

\[ \Rightarrow J_{sp} = \frac{2}{\varepsilon} (1 - \frac{\varepsilon}{2}) \frac{1}{2} \left[1 - \frac{\varepsilon}{2} \int_0^1 dx \ln(x(1-x)) + \ldots\right] \]

\[ \int_0^1 dx \ln(x(1-x)) = -2 \]

\[ J_{sp} = \frac{1}{\varepsilon} (1 - \frac{\varepsilon}{2}) (1 + \varepsilon) = \frac{1}{\varepsilon} (1 + \frac{\varepsilon}{2} + \ldots) \]

\[ \kappa \varepsilon J_{sp} = S_D \frac{1}{\varepsilon} (1 + \frac{\varepsilon}{2}) \]

Likewise, the same methods can be used to compute \( J_{4sp} \) and \( D_3 \).
Useful Integrals: (Unit 4)

\[ \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + 2p \cdot q + m^2)^\alpha} = \]
\[ = \frac{\text{SD}}{2} \frac{\Gamma\left(\frac{D}{2}\right) \Gamma'\left(\alpha - \frac{D}{2}\right)}{\Gamma(\alpha)} \left( m^2 - p^2 \right)^{\frac{D}{2} - \alpha} \]

\[ \frac{1}{A_1^{\alpha_1} \cdots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \cdots + \alpha_n)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \]

\[ \times \int d\mathbf{x}_1 \cdots d\mathbf{x}_n \quad x_i^{\alpha_1 - 1} \cdots x_{n-1}^{\alpha_{n-1} - 1} \quad (1 - \sum_{j=1}^{n-1} x_j^{\alpha_j} - 1) \]
\[ \sum_{j=1}^{n-1} A_j x_j^{\alpha_j} + (1 - \sum_{j=1}^{n-1} x_j^{\alpha_j}) A_n \]

\( 0 \leq x_i \leq 1 \)
\( \sum_{j=1}^{n-1} x_j \leq 1 \)
The results are

\[ I_{qsp} = (Sd)^2 \kappa^{-2\varepsilon} \frac{1}{2\varepsilon^2} \left( 1 + \frac{3}{4} \varepsilon \right) \]

\[ D_3' = -\frac{1}{8\varepsilon} \left( 1 + \frac{5}{4} \varepsilon \right) \kappa^{-3\varepsilon} (Sd)^2 \]

\[ D_5' = -\frac{1}{8\varepsilon} \left( 1 + \frac{5}{4} \varepsilon \right) \kappa^{-3\varepsilon} \frac{1}{6\varepsilon^2} \]

It is convenient to redefine \( u \to uSd \)

With these results the RG functions can be evaluated readily.

\[ b_2 = -\frac{1}{8\varepsilon} \left( 1 + \frac{5}{4} \varepsilon \right) \frac{1}{48} \]

\[ b_3 = -\frac{1}{8\varepsilon} \left( 1 + \frac{5}{4} \varepsilon \right) \frac{1}{48} \]

\[ q_1 = \frac{3}{2\varepsilon} \left( 1 + \frac{5}{2} \varepsilon \right) \]

\[ q_2 = \frac{a}{4\varepsilon^2} + \frac{a}{48\varepsilon} \frac{37}{24\varepsilon} \]

\[ q_3 = \frac{1}{2\varepsilon} \left( 1 + \frac{5}{2} \varepsilon \right) \]

\[ q_4 = \frac{1}{2\varepsilon^2} + \frac{3}{8\varepsilon} \]

Thus \( (N=1) \)

\[ \beta(u) = -\varepsilon u + \frac{3}{2} \left( 1 + \frac{5}{2} \varepsilon \right) u^2 - \frac{17}{12} u^3 + \ldots \]

\[ \gamma(u) = \frac{1}{24} \left[ (1 + \frac{5}{4} \varepsilon) u^2 - \frac{3}{4} u^3 \right] \]

\[ \delta_1 (u) = \frac{u}{2} \left( 1 + \frac{5}{2} \varepsilon - \frac{u}{2} \right) \]

\[ \delta_2 (u) = \frac{1}{24} \left[ (1 - \frac{5}{4} \varepsilon) u^2 - \frac{3}{4} u^3 \right] \]
Renormalization by Dimensional Regularization
and Minimal Subtraction (à la 't Hooft–Veltman)

We will now do the same but within
dimensional regularization + minimal
subtraction. We will consider the massless
theory in dimension $D$. As we saw
the integrals develop poles in $\varepsilon = 4-D$
(as $D \to 4$). In fact the only
vertex functions which develop poles (primarily)
are $\frac{2}{d!} F^{(2)}$, $F^{(4)}$ and $F^{(2,1)}$,
i.e. those corresponding to a log divergence
at $D=4$.

Once again we write
\[ u_0 = \lambda R^{-\varepsilon} = u \left( 1 + \sum_{n=1}^{\infty} a_n(\varepsilon) u^n \right) \]
\[ z \phi = 1 + \sum_{n=1}^{\infty} b_n(\varepsilon) u^n \]
\[ z \phi^2 = 1 + \sum_{n=1}^{\infty} c_n(\varepsilon) u^n \]
such that
\[ \Xi \Phi \ 
\begin{align*}
\Pi^{(2)}(k; \nu_0, \kappa) &= \Pi^{(2)}_R(k; \nu, \kappa) \\
\Pi^{(4)}(k_i; \nu_0, \kappa) &= \Pi^{(4)}(k_0; \nu, \kappa) \\
\Pi^{(2)}(k_i, k_j; \nu_0, \kappa) &= \Pi^{(2)}_R(k_i, k_j, p; \nu, \kappa)
\end{align*}
\]
are finite as \( D \to 4 \) (i.e. all singular terms are cancelled).

\( a_n(\xi), b_n(\xi) \) and \( c_n(\xi) \) are determined by subtracting the poles via

\[ \underline{\text{minimally}} \]

(under the bare vertex functions (at general momenta))

\[ \Pi^{(2)}(k; \nu_0, \kappa) = k^2 \left( 1 - B_2 \nu_0^2 \right) \]

\[ \Pi^{(4)}(k_i; \nu_0, \kappa) = \nu_0^2 \left( 1 - A_1 \nu_0 + (A_2^{(1)} + A_2^{(2)}) \nu_0^2 \right) \]

\[ \Pi^{(2)}(k_i, k_j; \nu_0, \kappa) = 1 - C_1 \nu_0 + (C_2^{(1)} + C_2^{(2)}) \nu_0^2 \]

where

\[ A_1 = \frac{N+8}{18} \left[ I\left(\frac{k_i+k_2}{k}\right) + I\left(\frac{k_i+k_3}{k}\right) + I\left(\frac{k_1+k_2}{k}\right) \right] \]

\[ A_2 = \frac{5N+22}{54} \left[ I_4\left(\frac{k_1}{k}, \frac{k_2}{k}, \frac{k_3}{k}, \frac{k_4}{k}\right) \right] \text{ \ \permut} \]
Consider \( P^{(2)}_R \to O(u^3) \), \( u_0^2 \to u^2 \)

\[
P^{(2)}_R = k^2 \left( 1 + b_1 u + b_2 u^2 \right) \left( 1 - B_2 u^2 \right) =
\]

\[
= k^2 \left[ 1 + b_1 u + (B_2 - B_2) u^2 + O(u^3) \right]
\]

\( \Rightarrow b_1 = 0 \) and \( b_2 = \left[ B_2 \right]_{\text{singular}} = \left[ B_2 \right]_s \)

\( \Rightarrow b_2 = -\left( \frac{N+2}{144} \right) \) \( \frac{1}{\xi} \) (compare with the earlier result!)

and \( \varphi = 1 - \left( \frac{N+2}{144\xi} \right) u^2 \)

\[
P^{(4)}_R \to O(u^3):
\]

\[
P^{(4)}_R (k_1, \ldots, k_4; u, \xi) = k^4 \left( 1 + 2 b_2 u^2 \right) (u + a_1 u^2 + a_2 u^3)
\]

\[
- \left( u^2 + 2 a_1 u^3 \right) \left( \frac{N+8}{18} \right) \left[ I \left( \frac{k_1 + k_2}{\xi} \right) + 2 \text{ perm.} \right]
\]

\[
+ u^3 \left\{ \left( \frac{N^2 + 6N + 20}{108} \right) \left[ I^2 \left( \frac{k_1 + k_2}{\xi} \right) + 2 \text{ perm.} \right] 
\]

\[
+ \left( \frac{5N + 22}{104} \right) \left[ I_4 \left( \frac{k_1}{\xi}, \ldots, \frac{k_4}{\xi} \right) + 5 \text{ perm.} \right] \right\}
\]

= \]
\[
\begin{aligned}
\kappa^2 \{ & u + u^2 \left[ q_1 - \left( \frac{N+8}{18} \right) \left[ I \left( \frac{k_1 + k_2}{K} \right) + 2 \text{ perm.} \right] \right] \\
+ & u^3 \left[ q_2 + 2b_2 - 2q_1 \left( \frac{N+8}{12} \right) \left[ I \left( \frac{k_1 + k_2}{K} \right) + 2 \text{ perm.} \right] \right] \\
+ & \left( \frac{N^2 + 6N + 20}{108} \right) \left[ I^2 \left( \frac{k_1 + k_2}{K} \right) + 2 \text{ perm.} \right] \right) \\
+ & \left( \frac{5N + 22}{5 - 4} \right) \left[ I_4 \left( \frac{k_1}{K} \right) + 5 \text{ perm.} \right] \right) \\
\Rightarrow & q_1 = \left( \frac{N+8}{18} \right) \left[ I \left( \frac{k_1 + k_2}{K} \right) + 2 \text{ perm.} \right] \right) \\
= & \left( \frac{N+8}{18} \right) \frac{3}{\kappa} = \left( \frac{N+8}{6 \kappa} \right) \\
q_2 = & \left[ -2b_2 + 2q_1 \left( \frac{N+8}{18} \right) \left[ I \left( \frac{k_1 + k_2}{K} \right) + 2 \text{ perm.} \right] \right] \\
- & \left( \frac{N^2 + 6N + 20}{108} \right) \left[ I^2 \left( \frac{k_1 + k_2}{K} \right) + 2 \text{ perm.} \right] \right) \\
- & \left( \frac{5N + 22}{5 - 4} \right) \left[ I_4 \left( \frac{k_1}{K} \right) + 5 \text{ perm.} \right] \right) \\
\end{aligned}
\]
In $I$ there is a term $\sim \ln(k_1 + k_2)^2$
which is finite as $E \to 0$, but since $q_1 \sim \frac{1}{E}

\Rightarrow q_1, I \sim \frac{1}{E} \left[ \ln(k_1 + k_2)^2 + 2 \text{ perturb.} \right]

which must cancel against terms coming from $I^2$. $I_q$ or $q_2$ will depend on $k$
(which it cannot!). All of this
works (as it should) and all momentum dip
singlet terms cancel.

\Rightarrow q_2 = \frac{(N+8)^2}{36 \, E^2} - \frac{3N+14}{24 \, E}

\Rightarrow u_0 = u \left\{ 1 + \frac{N+8}{6 \, E} u + \left( \frac{(N+8)^2}{36 \, E^2} - \frac{3N+14}{24 \, E} \right) u^2 \right\}

Likewise $\bar{Z} \phi^2 = 1 + \frac{(N+2)}{6 \, E} u +$

\[ + \left[ \frac{(N+2)(N+5)}{36 \, E^2} - \frac{(N+2)}{24 \, E} \right] u^2 \]
We note that $u_0$, $2\phi$, and $2\phi^2$ are functions of $u$ and $\varepsilon$, and have poles of arbitrary order in $\varepsilon$. However $\beta$, $\gamma\phi$ and $\gamma\phi^2$ are finite as $\varepsilon \to 0 \Rightarrow \left( \frac{\partial u}{\partial u_0} \right)^{-1}$ can have at most a simple pole in $\varepsilon$ at every order in $u \Rightarrow$ the higher order poles must cancel at every order and all coefficients in $\left( \frac{\partial u}{\partial u_0} \right)^{-1} \sim \frac{1}{\varepsilon}$.

In fact we obtain:

$$\beta(u, \varepsilon) = -u \left( \varepsilon - \frac{N+8}{6} u + \frac{3N+14}{12} u^2 \right)$$

$$\gamma\phi(u, \varepsilon) = \frac{N+2}{72} u^2$$

$$\gamma\phi^2 = \frac{N+2}{6} u \left( 1 - \frac{u}{2} \right)$$

Fixed Point: $u^* = \frac{6}{N+8} \varepsilon + \frac{18(3N+14)}{(N+8)^2} \varepsilon^2$

which is somewhat different from the expression we found earlier, but with the same exponents.
Fixed Points:

(a) $d > 4$: $u^* = 0$ is the only ($u > 0$) fixed point.

This is the trivial (Gaussian or free field) fixed point here. We have $\phi(u^*) = 0$ and $\hat{\phi}^2(u^*) = 0$. Then there are no anomalous dimensions, and all exponents are classical.

$\eta = 0$, $\nu = 1$, etc. ($\gamma = 1$, $\beta = 1/2$).

Since $\beta'(0) = - \epsilon > 0$ then it's IR.

(b) $d < 4$ In addition to $u^* = 0$ (which is UV), there is an IR fixed point at $u^* = \frac{2}{3} \epsilon (1 + O(\epsilon^3))$

$\beta'(u^*) = - \epsilon$ ($u^* = 0$) (UV)

$\beta'(u^*) = \epsilon \Theta(\epsilon^{-1})$ ($u^* = \frac{2}{3} \epsilon (1 + O(\epsilon^3))$) (IR)

At the non-trivial f.p.:

$\gamma \phi(u^*) = \frac{\epsilon^2}{54} (1 - \frac{3}{4} \epsilon) (1 + O(\epsilon)) = \eta$

And $\hat{\gamma} \phi^2(u^*) = 2 - \frac{1}{\nu} - \gamma = \frac{\epsilon}{3} (1 + \frac{5}{6} + O(\epsilon^3))$

$\Rightarrow \frac{1}{\nu} = 2 - \frac{6}{3} - \frac{2 \epsilon^2}{27} + O(\epsilon^3)$

$\Rightarrow \nu = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^3)$

$\therefore$
\( d=4 \) this is special. \( \epsilon = 0 \)

\[ \beta(u) = \frac{3}{2} u^2 - \frac{12}{12} u^3 + \cdots \]

The only p.p. is still at \( u^* = 0 \). However, the approach to the p.p. is very slow (no linear term).

Let's look at some consequences of this slow approach to the p.p.

**Corrections to Scaling:**

Let's study the effects of the slow approach to the weak limit of the susceptibility.

\[ \chi^{-1}_R \equiv (b, u, \kappa) = \Pi^{(2)}(0; b, u, \kappa) \]

\( \chi^{-1}_R \) satisfies the RG equation

\[
\left[ \frac{\partial}{\partial t} + \frac{3}{2} u \frac{\partial}{\partial u} - \eta(u) - \Theta(u) \frac{\partial}{\partial t} \right] \chi^{-1}_R = 0
\]

\[ \Theta(u) = -\gamma \phi^2(u) \equiv -\gamma \phi - \bar{\delta} \phi^2 = \frac{1}{\nu(u)} - 2 \]

\[ \eta(u) = \gamma \phi(u) \]

**Solution:**

\[ \chi^{-1}_R (t, u, \kappa) = \exp \left\{ \int_{u_i}^{u} \frac{\eta(u')}{\beta(u')} \, du' \right\} \Phi \left[ t \exp \left\{ \int_{u_i}^{u} \frac{\Theta(u') \, du'}{\beta(u')} \right\} \right] ^{-1} \]

\[ \Phi \left[ t \exp \left\{ \int_{u_i}^{u} \frac{\Theta(u') \, du'}{\beta(u')} \right\} \right] ^{-1} = \Phi \left[ t \exp \left\{ \int_{u_i}^{u} \frac{\Theta(u') \, du'}{\beta(u')} \right\} \right] ^{-1} \]

D-A gives

\[ \chi^{-1}_R (t, u, \kappa) = p^2 \chi^{-1}_R (t p^2, u, \kappa s^{-1}) \]

\[ = s^2 e^{-\int_{u_i}^{u} \frac{\eta(u')}{\beta(u')} \, du'} \chi^{-1}_R (t s, u(s), \kappa) \]
Lecture 40 (4/25)

Define a growing temperature \( t(p) \) as:

\[
p = e^\int_u u(p') \frac{du'}{\beta(u')} \quad \leftrightarrow \quad \beta(u) = \frac{2u}{\beta(u)}
\]

and

\[
t(p) = \frac{t}{\rho^2} e^{-\int_u u(p') \frac{du'}{\beta(u')}}
\]

Then we have

\[
X_R^{-1}(t, u, k) = \left[ \rho^2 \exp\left\{-\int_u u(p') \frac{du'}{\beta(u')}\right\} \right] x
\]

\[
\quad \times \exp\left\{\int_{u_1}^{u(p')} \frac{du'}{\beta(u')}\right\} \times \Phi\left[t(p) \exp\left\{\int_{u_2}^{u(p')} \frac{du'}{\beta(u')}\right\}, k \exp\left\{\int_{u_3}^{u(p')} \frac{du'}{\beta(u')}\right\}\right]
\]

\[
X_R(t, u, k) \equiv \rho^2 \exp\left\{-\int_u u(p') \frac{du'}{\beta(u')}\right\} \times X_R^{-1}(t(p), u(p'), k)
\]

i.e. a change in scale is again equivalent to an adjustment of \( u \) and \( t \) together with an overall factor.

I can also write

\[
t(p) = \frac{t}{\rho^2} \exp\left\{-\int_u u(p') \frac{du'}{\beta(u')}\right\} = t \exp\left\{-\int_u \frac{du'}{\beta(u')} \frac{2 + \theta(u')}{\beta(u')}\right\}
\]

\[
= t \exp\left\{-\int_u \frac{du'}{\nu(u') \beta(u')} \right\}
\]

where \( \frac{1}{\nu(u')} = 2 + \theta(u') \equiv 2 - \Phi^2(u) \equiv 2 - \eta(u) - \overline{\Phi}(u) \)
Let's integrate the flow equation

$$\beta(u) = \frac{\partial u}{\partial s} = \frac{3}{2} u^2 + \ldots$$

$$u(p) = \frac{u(0)}{1 - \frac{3}{2} u(0) p}$$

is the solution s.t. \( \lim_{p \to 1} u(p) = u(0) \equiv u \)

As \( p \to 0 \) (infrared (long distance) limit), we have

$$u(p) \approx \frac{2}{3} \left[ \text{ln} p \right]^{-1} \quad (p \to 0)$$

Let \( p \) be st. \( t(p) \to 1 \) as \( p \to 0 \) (or \( t \to t^* \) well within the p.t. domain)

$$\int_0^{u(p)} \frac{\Theta(u')}{\beta(u')} \, du' = -\frac{1}{2} \int_0^{u(p)} \frac{u' \, du'}{u'^2} = -\frac{1}{3} \int_0^{u(p)} \frac{du'}{u}$$

$$\Theta(u) = -\chi_R^{-1}(u) = -\frac{u}{2} + O(u^2)$$

\[ \Rightarrow \frac{t(p)}{p^2} = \left[ \frac{\frac{3}{2} u}{\text{ln} p} \right]^{-\frac{1}{3}} \]

\[ \Rightarrow \frac{t^*}{t(p)} \approx \left[ \frac{3}{2} u \left( \frac{t}{t(p)} \right) \right]^{-\frac{1}{3}} \quad t(p) \sim t^* \text{ (large)} \]

Also

$$\int_0^{u(p)} \frac{du'}{\beta(u')} \quad \frac{1}{3} (u(p) - u) \quad \to \quad -\frac{u}{3e}$$

Hence

$$\chi_R^{-1}(t, u, x) = \left( \frac{3u}{4} \right) e^{\frac{u}{3e}} \left( \frac{t}{t(p)} \right)^{-\frac{1}{3}} \text{ln} \left( \frac{t}{t(p)} \right) \chi_R^{-1} \quad \text{as} \quad t \to 0$$

( for \( t \to t^* \) )

or

$$\chi_R \sim C \left| \text{ln} t \right|^{-\frac{1}{3}} \quad \text{as} \quad t \to 0$$

(since \( \chi_R(t(0), 0) = \text{finite} \) )

We thus find strong corrections to scaling \((\sim \frac{1}{t})\) behavior (tricritical points, etc.)