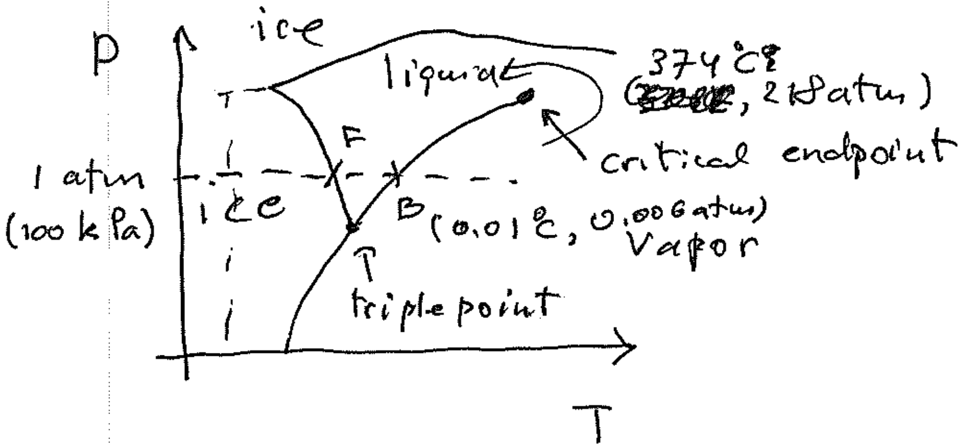
The Concept of Order in CMP

During the history of Humanity many things have been called phases of matter (air, fire, water...)

Water presents itself in many "phases":

liquid water, steam (vapor), ice (20 phases!)



* ice are solid phases of water with ≠ crystal structure
* liquid water and water vapor <u>are not</u> distinct phases
* "Line" of 1st order transitions (where the density jumps) ending at the <u>critical point</u>

Symmetry plays a key role in physics and in the phases of matter. In this class we will see that topology plays a role which is just as important.

Symmetry and phases of matter

Consider a simple model of a ferromagnet: The Ising model.

Square lattice with a spin at each site

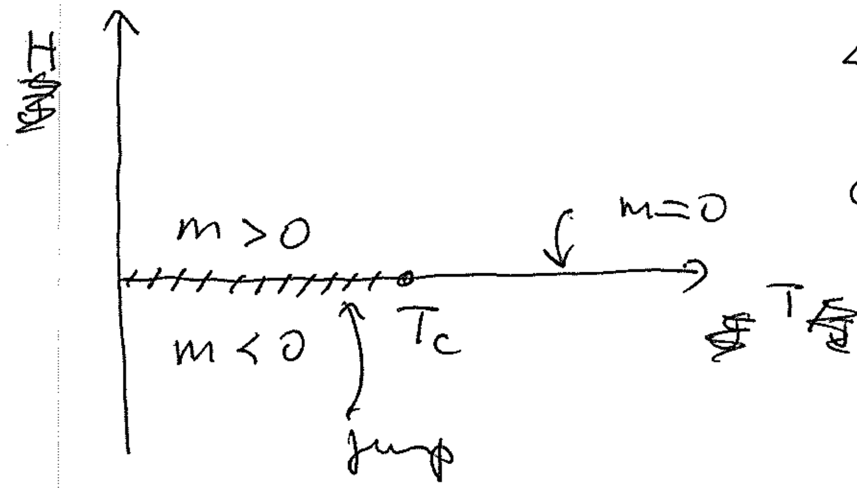
$\sigma = +1\ (\uparrow),\ -1\ (\downarrow)$ (uniaxial)

$$E[\sigma] = -J \sum_{\underbrace{\langle \vec{r}, \vec{r}\,' \rangle}_{\text{n.n.}}} \sigma(\vec{r})\, \sigma(\vec{r}\,') - \underbrace{H}_{\substack{\text{magnetic}\\ \text{field}}} \sum_{\vec{r}} \sigma(\vec{r})$$

$$Z = \sum_{[\sigma]} e^{-E[\sigma]/T} \qquad k_B = 1$$

N sites $\Rightarrow$ 2^N configs.

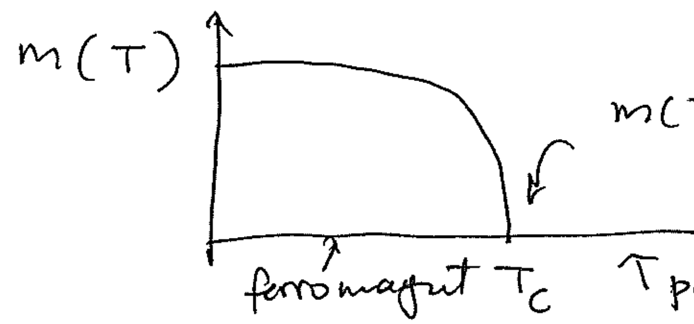
* If N is finite $\Rightarrow$ Z is an analytic function of $\frac{J}{T}$ and $\frac{H}{T}$.
* If $N \to \infty$ $\Rightarrow$ the physics depends on whether $H = 0$ or $H \neq 0$



$\langle \sigma \rangle = m$ (per site)

↑

Order parameter



T paramagnet

exponent

$$m(T) \cong A\,(T_c - T)^{\beta}$$

$$\chi = \frac{dM}{dH}$$

This model also describes the liquid-gas transition and T_c is the critical endpoint

n: density $\Rightarrow$ $\Delta n \Leftrightarrow m$

<u>What symmetry?</u>

$\sigma(\vec{r}) \to -\sigma(\vec{r})$ <u>everywhere</u> $(\mathbb{Z}_2)$

H <u>breaks the symmetry</u> (symmetry breaking field)

<u>explicitly</u>

However if one cools below T_c and <u>then</u> $H \to 0$

$\Rightarrow$ $m > 0$ if $H \to 0^+$ and $m < 0$ if $H \to 0^-$

<u>Crucial</u>: <u>the thermodynamic limit first</u> $(N \to \infty)$

why?

If $N \to \infty$ and $H > 0 \Rightarrow$ configs with $m < 0$ are exponentially suppressed (in $\underline{W}$)

(same with $H < 0$)

$\Rightarrow$ $T < T_c \Rightarrow$ the global $\mathbb{Z}_2$ symmetry is <u>spontaneously broken</u>

($\Leftrightarrow$ the state has <u>less</u> symmetry than the Hamiltonian ($H \to 0$)) $\to$ goto 4'

~~Talk about domain wall representation and corr. functions!~~

<u>Other examples</u>

- Spin system with easy plane anisotropy
- Superfluid and superconductor

Order parameter is a complex #

$$\phi(\vec{r}) = \underset{\uparrow \text{ amplitude}}{|\phi(r)|}\, e^{i \underset{\uparrow \text{ phase}}{\theta(r)}}$$

aslo $m_x = \mathrm{Re}\, \phi$

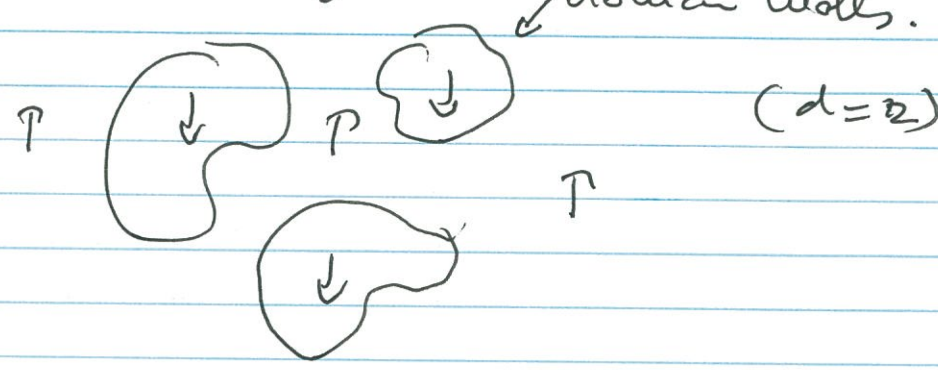
$m_y = \mathrm{Im}\, \phi$

symmetry: $\phi(\vec{r}) \to \phi'(\vec{r}) = \phi(\vec{r})\, e^{i\alpha}$

$\alpha \in [0, 2\pi)$ <u>constant</u>

(Global symmetry) U(1) symmetry

Typical config



For $T < T_c$ the domain walls are suppressed but at T_c they proliferate.

<u>Correlators</u>

$\langle \sigma(x)\, \sigma(y) \rangle \sim$ prob. to find two parallel spins at distance $R = |x-y|$

$T > T_c$

$$\langle \sigma(x)\, \sigma(y) \rangle \sim e^{-R/\xi}$$

ξ: correlation length

$T < T_c$

$$\langle \sigma(x)\, \sigma(y) \rangle \sim |\langle \sigma(x) \rangle|^2 + \#\, \underbrace{e^{-R/\xi}}_{\text{(connected)}}$$

$\xi \to \infty$ as $T \to T_c$

$$\xi \sim \frac{1}{|T-T_c|^{\nu}}$$

$|\phi(\vec{r})| = \phi_0$ const.

$$\Rightarrow E[\Theta(\vec{r})] = -J\phi_0 \sum_{\langle \vec{r}, \vec{r}' \rangle} \cos(\Theta(r) - \Theta(r'))$$

(X Y model)

$$- H \phi_0 \sum_{\vec{r}} \cos \Theta(\vec{r})$$

not allowed in superfluids and superconductors!

$$Z = \prod_{\vec{r}} \int_0^{2\pi} \frac{d\Theta(r)}{2\pi} \, e^{-E[\Theta(\vec{r})]/T}$$

(A) $\langle e^{i\Theta(\vec{r})} \rangle = 0$ for $T > T_c$

(B) $\langle e^{i\Theta(\vec{r})} \rangle \equiv \phi(\vec{r})$ $T < T_c$

<u>Note</u>: the <u>phase</u> of $\phi(\vec{r})$ is <u>arbitrary</u>!

$\Rightarrow$ SSB of $U(1)$ symmetry

Also $T \ll T_c$, $\Theta(r)$ varies slowly

$$\Rightarrow E[\Theta] \cong + \frac{J\phi_0}{2} \sum_{\langle r, r' \rangle} (\Theta(r) - \Theta(r'))^2$$

"spin-wave approx." (periodicity?)

$$\cong \frac{J\phi_0}{2a^D} \int d^D x \, (\vec{\nabla}\Theta)^2$$

<u>Two symmetries:</u>

<u>Global</u> U(1) symmetry $\quad \Theta(\vec r) \to \Theta(\vec r) + \alpha$

<u>local</u> <u>symmetry</u> (periodicity)

$$\Theta(\vec r) \to \Theta(\vec r) + 2\pi n(r)$$

$$n(r) \in \mathbb{Z}$$

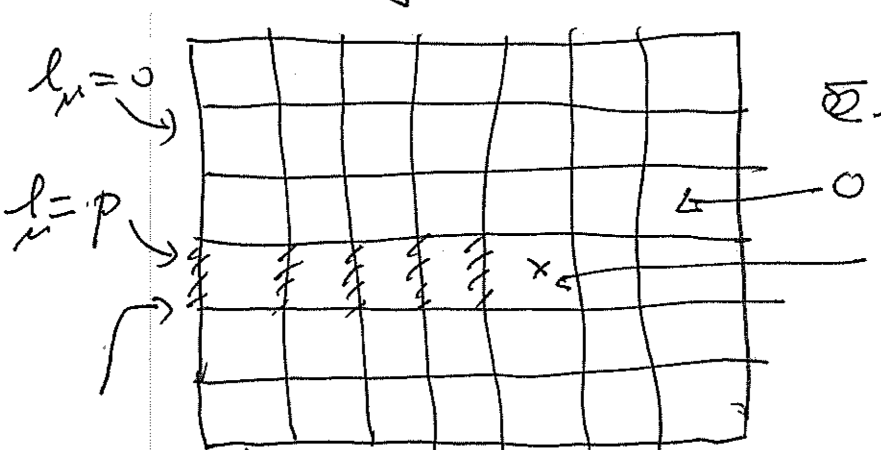
$$Z \cong \prod_{\vec r} \int \frac{d\theta(\vec r)}{2\pi} \sum_{\{l_\mu(\vec r)\}} e^{-\frac{J\ell^2}{T}\sum_{(r,\mu)} \left(\theta(r) - \theta(r+e_\mu) + 2\pi l_\mu\right)^2}$$

links

$e^{i\theta(r)}$ is periodic but breaks U(1)

$\Theta(r)$ breaks <u>both</u>.

Meaning of $\{l_\mu(\vec r)\}$?



plaquette

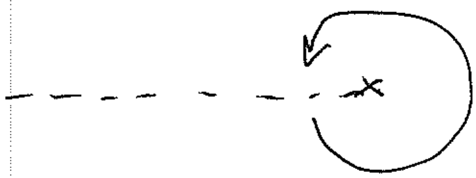
$$\oint l_\mu = \frac{\Phi(\vec r)}{2\pi} \equiv p$$

$\Theta(\vec r)$ jumps by $2\pi p$ on these links

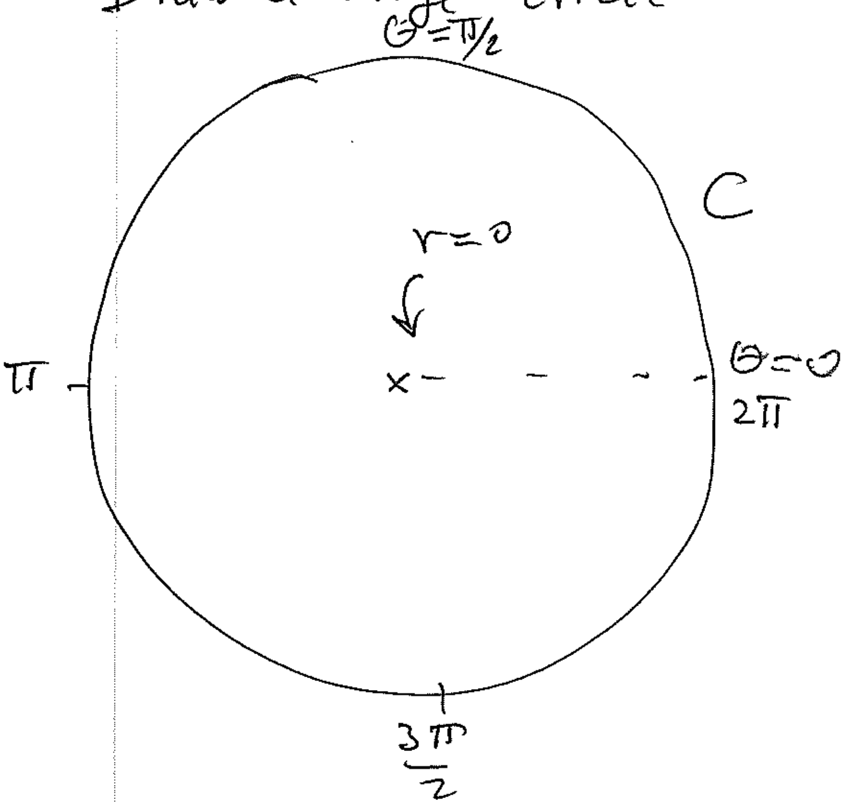
At long distances

Vortex

phase winds by $2\pi p$



Draw a large circle



$$\phi(r) \to 0 \quad @ \; r=0$$

$$\phi(r) \to \phi_0 \, e^{i\theta(r)}$$

on C

On $\underline{C}$ map of $C \to S_1$ circle defined by $\theta(r)$ (mod 2π)

$\Rightarrow$ p is the vorticity or winding #

$$\frac{(\Delta\Theta)_C}{2\pi} = \int_C \frac{d\vec{x}}{2\pi} \cdot \vec{\nabla}\Theta$$

$$\equiv +i \int_C \frac{d\varphi}{2\pi} \, e^{+i\Theta(\varphi)} \, \partial_\varphi \, e^{-i\Theta(\varphi)}$$

$$\equiv p$$

p is a <u>topological invariant</u>

it does not change if C changes smoothly or ~~if~~ the config. changes smoothly.

~~Also~~
<u>current</u> $\vec{J} = |\phi_0| \vec{\nabla}\Theta$

<u>Vorticity</u> $\omega(\vec{x}) = \varepsilon_{\mu\nu} \partial_\mu J_\nu$

$$= \phi_0 \, \varepsilon_{\mu\nu} \, \partial_\mu \partial_\nu \Theta$$

If $\Theta(x)$ is smooth (obviously diff.)

$\Rightarrow \varepsilon_{\mu\nu} \partial_\mu \partial_\nu \Theta = 0 \Rightarrow \omega = 0$

vortex $\omega \neq 0 \Rightarrow$ cross derivatives do not commute at $x = 0$

Energy of a vortex config.

$$\omega(\vec{x}) = \sum_j 2\pi n_j \, \delta^2(\vec{x}-\vec{x}_j)$$

$\uparrow$ vorticity

$$\Rightarrow \Theta(\vec{x}) = \sum_j 2\pi n_j \, \mathrm{Im} \ln(z - z_j)$$

$$z = x_1 + i x_2$$

Cauchy-Riemann

$$\partial_\mu \vartheta = \varepsilon_{\mu\nu} \partial_\nu \Theta$$

$$\Rightarrow \vartheta \text{ obeys } -\nabla^2 \vartheta = \omega(x) \quad \begin{pmatrix}\text{Poisson} \\ \text{Eqn.}\end{pmatrix}$$

$$E = \frac{J\phi_0^2}{2} \int d^2x \, (\partial_\mu \Theta)^2$$

$$= \frac{J\phi_0^2}{2} \int d^2x \, (\partial_\mu \vartheta)^2$$

$$= -\frac{J\phi_0^2}{2} \int d^2x \, \vartheta \nabla^2 \vartheta$$

$$= + \frac{J\phi_0^2}{2} \int d^2x \, \omega(x) \vartheta(x)$$

Solve Poisson

$$\vartheta(x) = \int d^2y \; G(x-y)\,\omega(y)$$

$$-\nabla^2 G(x-y) = \delta^2(x-y)$$

$$G(x-y) = \int \frac{d^Dp}{(2\pi)^D} \frac{e^{ip\cdot(x-y)}}{p^2}$$

$$= \frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D}\,|x-y|^{D-2}}$$

$$G(0) = \lim_{a\to 0} G(a)$$

$$\Rightarrow \quad G(x-y) - G(a) = \frac{1}{2\pi}\ln\left(\frac{a}{|x-y|}\right)$$

$(D\to 2)$

$$E = \frac{J\phi_0^2}{2}\int d^2x \;\omega(x)\,\vartheta(x)$$

$$= \frac{J\phi_0^2}{2}\int d^2x \int d^2y \;\omega(x)\,G(x-y)\,\omega(y)$$

$$\equiv \frac{J\phi_0^2}{2}\sum_{i,j} n_i n_j \; G(x_i-x_j)\cdot 4\pi^2$$

$$= \frac{4\pi^2 J\phi_0^2}{2}\left(\sum_j n_j\right)^2 G(0) + 4\pi^2 J\phi_0^2 \sum_{i>j} n_i n_j \left(G(x_i-x_j) - G(0)\right)$$

$\Rightarrow$ since $G(0) \to \infty$ as $\ln \underset{(\text{size})}{L}$

$\Rightarrow$ only configs with total zero vorticity survive

and

$$\frac{E}{T} = \frac{\cancel{4\pi}\,2\pi J \Phi_0^2}{\cancel{2\pi}\, T} \sum_{i>j} n_i n_j \ln \frac{a}{|x_i - x_j|}$$

2D Coulomb gas!

Energy of a vortex-anti vortex pair at distance R is $\frac{J\Phi_0^2}{2\pi} \ln R/a$ $\quad (n = \pm 1)$

Free energy?

$$F = \underset{\text{energy}}{U} - T\underset{\text{entropy}}{S}$$

$$F_{vortex} = \frac{\cancel{8\pi} J |\Phi_0|^2}{\cancel{2}\,\cancel{2\pi}} \ln\left(\frac{L}{a}\right) - T \ln \underset{\#\text{ of locations}}{\left(\frac{L}{a}\right)^2}$$

$F_{vortex} < 0 \Rightarrow$ vortices proliferate