Classical Dimers and Critical Phenomena

$$S_{2D} = \int d^2x \; \frac{K}{2} (\vec{\nabla} \varphi)^2$$

(see Nienhuis 1987, Henley & Kondev 1996)

& Cauchy-Riemann

$$\partial_\mu \varphi = \varepsilon_{\mu\nu} \partial_\nu \vartheta$$

$$V_n(x) = e^{in\varphi(x)}$$ ("electric") (charge)

$$\tilde{V}_m(x) = e^{im\vartheta(x)}$$ ("magnetic") (vortex)

dimer densities

$$n_x = \frac{1}{4} + \frac{1}{2\pi} (-1)^{x+y} \partial_y \varphi + \frac{1}{2}\left[(-1)^x e^{i\varphi} + c.c.\right]$$

$$n_y = \frac{1}{4} + \frac{1}{2\pi} (-1)^{x+y+1} \partial_x \varphi + \frac{1}{2}\left[(-1)^y e^{i\varphi} + c.c.\right]$$

$\Rightarrow V_1(x) \equiv e^{i\varphi(x)}$ takes the values $1, i, -1, -i$

$\Rightarrow$ columnar order parameter $O_c(x)$

$V_2(x) \equiv e^{2i\varphi(x)}$ takes values $1, -1, 1, -1$

$\Rightarrow$ orientational order $N(x)$ ("nematic")

$\tilde{V}_1(x) = e^{i\vartheta(x)} \Rightarrow$ holes ; $\tilde{V}_2(x) = e^{i2\vartheta(x)}$ pairs of holes (diag. dimer)

$D=2$ ~~[illegible]~~ 9/27/2022

$$\langle \varphi(x)\,\varphi(y)\rangle = -\frac{1}{4\pi K}\ln\left(\frac{|\vec{x}-\vec{y}|}{a}\right)$$

cutoff

$$\langle V_n(x)\,V_{n'}(x')\rangle = \frac{\delta_{n,-n'}}{|\vec{x}-\vec{y}|^{2\Delta_n}}$$

$$\langle \tilde{V}_m(x)\,\tilde{V}_{m'}(x')\rangle = \frac{\delta_{m,-m'}}{|\vec{x}-\vec{y}|^{2\Delta_m}}$$

$$\Delta_n = \frac{n^2}{4\pi K}\,,\quad \Delta_m = \pi K m^2$$

(scaling dimensions)

"e.m." duality $\quad K \to \frac{1}{K} \quad\quad n \leftrightarrow m$

We reproduce the known results for classical dimers on a sq. lattice for $\boxed{K_{free} = \frac{1}{4\pi}}$

Vortex Op. $\tilde{V}_1(x) \Leftrightarrow$ monomer at x

Scaling dim. $\Delta_{vortex} = \frac{1}{4}$

In pple the operator ~~cos~~ $\cos 4\varphi$ must be included (periodicity) but its scaling dimension is 16 >> 2 ⇒ strongly irrelevant.

Quantum ~~Ca~~ Case

$\varphi(\vec{x}, t)$ and $\Pi(\vec{x}, t)$

canonical momentum

$$[\varphi(\vec{x}, t), \Pi(\vec{y}, t)] = i\,\delta^{(2)}(x-y)$$

$$H_0 = \int d^2x \left[\frac{1}{2}\Pi^2(x) + \frac{A}{2}(\vec{\nabla}\varphi)^2 + \frac{\kappa^2}{2}(\nabla^2\varphi)^2 + C\,(\nabla\varphi)^4 + \dots\right]$$

(I) If ~~A<0~~ $A>0$ ⇒ space and time scale the same way $[T]=[L]$ ⇒ $z=1$

⇒ $[\varphi] = \frac{1}{L}$ and $[\Pi] = \frac{1}{L}$

We will see that this is an ordered state

(II) If ~~A>0~~ $A<0$

the G.S. has a modulation

$\varepsilon \sim A\vec{k}^2 + \kappa^2 k^4$ $\quad A<0 \Rightarrow |\vec{k}_0| = \sqrt{\frac{|A|}{\kappa^2}}$ tilt

$\varphi(\vec{x}) \equiv$ ~~$e^{i\vec{k}_0\cdot\vec{x}}\,\tilde{\varphi}(x)$~~ $\vec{k}_0\cdot\vec{x} + \delta\varphi(x)$

III Critical Point at $A=0$

(Lifshitz Transition)

$\Rightarrow \quad [T] = [L]^2 \quad$ and $\quad z=2$

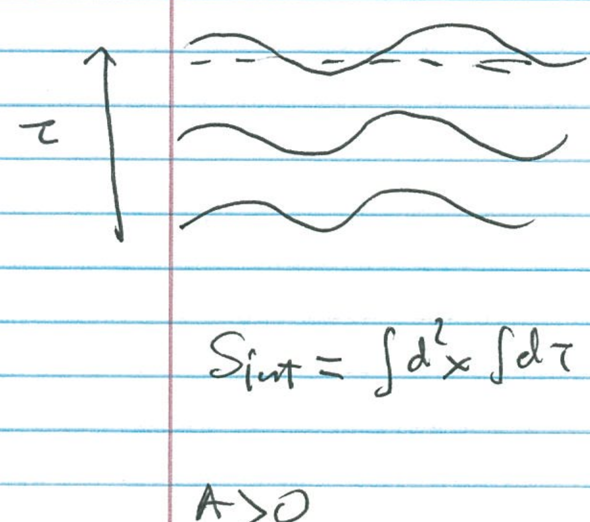
$[\varphi] = [L]^0 = 1 \quad$ dimensionless

$[\Pi] = L^{-2}$

Euclidean action

$$S_0 = \int d^2x \int d\tau \left[\frac{1}{2} (\partial_\tau \varphi)^2 + \frac{A}{2} (\vec{\nabla} \varphi)^2 + \frac{K^2}{4} (\nabla^2 \varphi)^2 + \cdots \right]$$

Same as the Gibbs weight of nematic heights $\Rightarrow$ theory of the nematic-smectic



transition in 3D liquid crystals!

$$S_{int} = \int d^2x \int d\tau \; g \cos(4\varphi) \qquad \text{(periodicity)}$$

$A > 0$

$$S_{eff} \simeq \int d^2x \int d\tau \left[\frac{1}{2} (\partial_\tau \varphi)^2 + \frac{A}{2} (\vec{\nabla} \varphi)^2 - g \cos(4\varphi) \right]$$

$\Rightarrow \quad \cos 4\varphi$ is irrelevant (Polyakov) columnar

$\int$

$$\langle e^{i\varphi}\rangle \approx \{1, i, -1, -i\}\, e^{-\frac{1}{2\pi^2}\left(\frac{\Lambda}{\sqrt{A}} - \frac{\pi}{2} m_{eff}\right)}$$

$(\Lambda \sim \frac{1}{a}$: UV cutoff$)$, $m^2_{eff} \simeq \frac{16 g}{\sqrt{A}}$

$\Rightarrow$ ordered state

<u>$A = 0$</u>

$$S_{QLM} = \int d^2x \int d\tau \left[\frac{1}{2}(\partial_\tau \varphi)^2 + \frac{K^2}{2}(\nabla^2 \varphi)^2\right]$$

Quantum Lifshitz

$$H_{QLM} = \int d^2x \left[\frac{1}{2}\Pi^2 + \frac{K^2}{4}(\nabla^2\varphi)^2\right]$$

<u>Dual</u> ? ~~It cannot be Max~~

In 2+1 it has to be a gauge theory but it cannot be Maxwell since $z=2$

<u>Propose</u>

$$H_{QLM\text{-}gauge} = \int d^2x \left(\frac{K^2}{2}(\vec{\nabla}\times\vec{E})^2 + \vec{B}^2\right)$$

electric field

$\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\partial_0 \vec{A}$ / $[E_j(x), A_k(y)] = i\delta_{jk}\delta(x-y)$

Plus a Gauss Law condition ($A_0=0$)

$$\vec{\nabla}\cdot\vec{E}\,|\text{Phys}\rangle=0$$

$\Rightarrow$ Solve the Gauss Law

$$E_j=\varepsilon_{jk}\partial_k\varphi$$

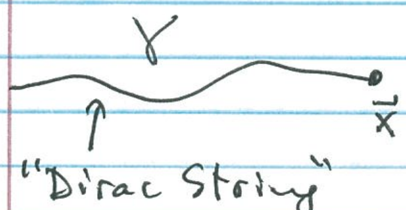
$$\Rightarrow\quad[\varphi(x),B(y)]=i\,\delta(x-y)$$

$$\Rightarrow\quad B\equiv\Pi$$

* Operator that creates a magnetic charge

$$\mathcal{O}_n(x)=e^{in\int_\gamma dy_j\,\varepsilon_{jk}\,\theta(y_j-x_j)\,\delta(y_k-x_k)\,E_k(y)}$$

$$=e^{in\int_{\gamma(x)}dy_j\,\partial_j\varphi(y)\,\theta(y_j-x_j)\,\delta(x_k-y_k)}$$



$$\equiv e^{in\varphi(x)}$$

(compatible with $\varphi\to\varphi+2\pi R$)
$R=1$

Op. that creates an electric charge (gauge theory)

$$\vec{\nabla}\cdot\vec{E}(y)=m\,\delta^2(x-y)$$

$$\Rightarrow\quad E_j(y)=\varepsilon_{jk}\left(\partial_k\varphi(y)+B_k(y)\right)$$

s.t. $\varepsilon_{jk}\,\partial_j B_k(y)=m\,\delta^{(2)}(x-y)$

$A = c(J - V)$ (in terms of the QDM)

Let us derive the wave functional

Configs. $|\{\varphi(\vec{x})\}\rangle$

$$\Psi[\{\varphi(\vec{x})\}] = \langle \Psi | \{\varphi(\vec{x})\} \rangle$$

Schrödinger Rep.

$$\Pi(x) = i \frac{\delta}{\delta \varphi(\vec{x})}$$

$\Rightarrow$ ~~He~~

$$H\,\Psi[\varphi] = \int d^2x \left[-\frac{1}{2} \frac{\delta^2}{\delta \varphi(x)^2} + \frac{\kappa^2}{2} (\nabla^2 \varphi)^2 \right] \Psi[\varphi]$$

$$= E\,\Psi[\varphi]$$

$$= \int d^2x \, \frac{1}{2} \{ Q(\cancel{x}\varphi), Q^{\dagger}(\varphi) \} \Psi(\varphi)$$

$$Q[\varphi] = \frac{1}{\sqrt{2}} \left[-\frac{\delta}{\delta \varphi(x)} + \kappa \nabla^2 \varphi(x) \right]$$

"annihilation op."

$$Q\,\Psi_{GS}[\varphi] = 0$$

$$\frac{1}{\sqrt{2}} \left[-\frac{\delta}{\delta \varphi(x)} + \kappa \nabla^2 \varphi \right] \Psi_0[\varphi] = 0$$