* Let's look at Landau levels in detail

$$H = \frac{1}{2M}\left[\left(-i\hbar\partial_1 - \frac{e}{c}A_1\right)^2 + \left(-i\hbar\partial_2 - \frac{e}{c}A_2\right)^2\right]$$

$$\partial_i \equiv \frac{\partial}{\partial x_i}$$

$$B = \varepsilon_{ij}\partial_i A_j \,, \quad \Phi = BL^2 \quad \text{flux}$$

$$\Phi = N_\phi \phi_0 = N_\phi \frac{hc}{e} = 2\pi N_\phi \frac{\hbar c}{e}$$

if we use units $\hbar = c = e = 1 \Rightarrow \Phi = 2\pi N_\phi$

$(\phi_0 = 2\pi)$

Magnetic length $= l_0 = \left(\frac{\hbar c}{eB}\right)^{1/2} \equiv \frac{1}{\sqrt{B}}$ $(\hbar = c = e = 1)$

<u>Circular gauge</u> $\vec{\nabla}\cdot\vec{A} = 0$

$$A_i = -\frac{1}{2}B\,\varepsilon_{ij}x_j$$

Complex coordinates $z = x_1 + ix_2$

wave functions $\psi(z,\bar{z}) = f(z,\bar{z})\, e^{-|z|^2/4l_0^2}$

works on a disk $\approx S_2$ (except for edge states)

$B>0$ $\Rightarrow$ $$-\frac{2\hbar^2}{M}\partial_z\partial_{\bar{z}} f + \frac{e|B|\hbar}{Mc}\bar{z}\,\partial_{\bar{z}} f + \frac{e|B|\hbar}{2Mc} f = E f$$

Schrödinger Eqn.

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)\,, \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)\,; \quad 4\partial_z\partial_{\bar{z}} = \nabla^2$$

for $B < 0$ $\quad z \leftrightarrow \bar{z}$

Angular momentum:

$$L_z = -i\hbar\,(x_1\partial_2 - x_2\partial_1) = \hbar(z\partial_z - \bar{z}\partial_{\bar{z}})$$

If $f(z,\bar{z}) \equiv f(z)$ (analytic)

is a solution since $\partial_{\bar{z}} f(z) = 0$

$$\Rightarrow E = \frac{1}{2}\frac{e|B|\hbar}{Mc} = \frac{1}{2}\hbar\omega_c$$

Basis of analytic functions: $f_n = z^n$ $\quad (n \geq 0)$

$$L_z f_n = n\hbar f_n$$

$\Rightarrow$ the states with $\psi(z,\bar{z}) = c\, z^n$

have energy $\frac{1}{2}\hbar\omega_c$ and ang. mom. $L_z = \hbar n$

$n = 0, 1, \ldots, N_\phi$

$\bar{z}^m$ is also a solution with

energy $\hbar\omega_c(m + \frac{1}{2})$

and $L_z = -m\hbar$

Degeneracy

Let $\vec{a}$ and $\vec{b}$ be two vectors (non-colinear) on the plane

$B > 0$

Linear Momentum: $\vec{P} = -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}$, $\vec{B} = |B|\,\hat{e}_z$

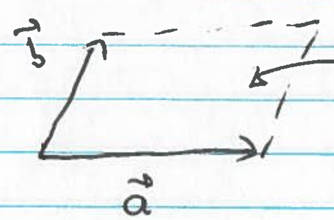
Magnetic Translations

$$k_i = P_i(-B) = P_i - \frac{eB}{c}\,\varepsilon_{ij}\,x_j$$

Finite magnetic translation:

$$\hat{t}(\vec{a}) = e^{\frac{i}{\hbar}\vec{a}\cdot\vec{k}}$$

$$\hat{t}(\vec{a})\,\hat{t}(b) = e^{-i(\vec{a}\times\vec{b})\cdot\hat{e}_z/l_0^2}\;\hat{t}(\vec{b})\,\hat{t}(\vec{a})$$



area $= |\vec{a}\times\vec{b}|$

$$\frac{|\vec{a}\times\vec{b}|}{l_0^2} = |\vec{a}\times\vec{b}|\,\frac{eB}{\hbar c} = \frac{B\,|\vec{a}\times\vec{b}|}{\frac{\hbar c}{e}} = 2\pi\,\frac{\overbrace{B|\vec{a}\times\vec{b}|}^{\text{flux}}}{\phi_0}$$

$$\Rightarrow \frac{|\vec{a}\times\vec{b}|}{l_0^2} = 2\pi \times \frac{\text{Flux}}{\text{flux quantum}}$$

$$\hat t(\vec a)\,\hat t(\vec b) = e^{\frac{i}{2l_0^2}(\vec a\times\vec b)\cdot\hat e_z}\;\hat t(\vec a+\vec b)$$

"cocycle"

Magnetic algebra

$\Rightarrow$ $\hat t(\vec a)$ are ray (projective) representations of the magnetic translations.

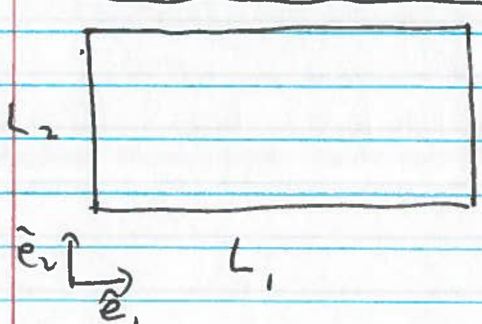
Also $[k_i, P_j] = 0$

$\Rightarrow$ $[k_i, H] = 0$ since $H = \frac{1}{2M}\vec P^2$

but $[k_i, k_j] = -[P_i, P_j] = i\frac{e\hbar}{c}B\,\epsilon_{ij}$

$\Rightarrow$ degenerate states are labelled by either k_1 or k_2 (but not both), or some linear combination

Rectangular system



$\Phi = BL_1L_2$ flux

$N_\phi = \Phi/\phi_0$

$$\frac{L_1L_2}{l_0^2} = 2\pi N_\phi$$

Let $\hat{T}_1 \equiv \hat{t}\left(\frac{L_1}{N_\phi}\hat{e}_1\right)$

$$\hat{T}_2 \equiv \hat{t}\left(\frac{L_2}{N_\phi}\hat{e}_2\right)$$

$\Rightarrow \hat{T}_1 \hat{T}_2 = e^{-i 2\pi/N_\phi}\, \hat{T}_2 \hat{T}_1$

Let $\psi_{n,0}$ be an eigenstate of $\hat{T}_1$

$$H\psi_{n,0} = E_n \psi_{n,0} \;,\quad \hat{T}_1 \psi_{n,0} = e^{i\underset{\substack{\uparrow \\ \text{e.v.}}}{\lambda_0}}\, \psi_{n,0}$$

Let $\psi_{n,m} = \hat{T}_2^{\,m}\, \psi_{n,0}$

Since $[\hat{T}_2, H] = 0 \Rightarrow \psi_{n,m}$ has energy E_n

but $\hat{T}_1 \cancel{\hat{T}_2} \psi_{n,m} = \hat{T}_1 \hat{T}_2^{\,m}\, \psi_{n,0} = e^{-i\frac{2\pi m}{N_\phi}}\, e^{i\lambda_0}\, \psi_{n,m}$

$\Rightarrow$ there are N_ϕ degenerate states for each $\underline{n}$

For wave functions $\rightarrow 0$ as $|x| \rightarrow \infty$ ("disk")

$$k = \frac{i}{2\hbar}(k_1 - ik_2) = \partial_z - \frac{\bar{z}}{4l_0^2}$$

$$\bar{k} = \frac{i}{2\hbar}(k_1 + ik_2) = \partial_{\bar{z}} + \frac{z}{4l_0^2}$$

commute

$$P = \frac{i}{2\hbar}(P_1 + iP_2) = \partial_{\bar{z}} - \frac{z}{4l_0^2}$$

$$\bar{P} = \frac{i}{2\hbar}(P_1 - iP_2) = \partial_z + \frac{\bar{z}}{4l_0^2}$$

$$T = e^{2Lk/N_\phi} \qquad L_1 = L_2 \equiv L$$

$$\bar{T} = e^{iL\bar{k}/N_\phi}$$

$$T\bar{T} = \bar{T}T\, e^{-i2\pi/N_\phi}$$

$\bar{k}\,\psi_{n} = 0$ with $\psi_n = c_n z^n e^{-|z|^2/4\ell_0^2}$

$\Rightarrow \psi_n$ is an eigenstate of $\bar{T}\psi_n = \psi_n$

Complete set

$$\psi_{n,m}(z,\bar{z}) \equiv T^m \psi_n(z,\bar{z}) \equiv C_{n,m}\, e^{2L\frac{m}{N_\phi}k}\, \psi_n(z,\bar{z})$$

$$H\psi_{n,m} = E_n \psi_{n,m}$$

$$\bar{T}\psi_{n,m} = e^{2\pi i m/N_\phi}\,\psi_{n,m}$$

$$H = \frac{2\hbar^2}{M}\left[-P\bar{P} + \frac{eB}{4\hbar c}\right]$$

Toroidal BC's :

can we impose? : $\psi(x_1, x_2) = \psi(x_1 + L_1, x_2) = \psi(x_1, x_2 + L_2)$

No since $\vec{A}$ violates translation invariance

Impose $\qquad$ gauge transf.

$$A_i(x_1 + L_1, x_2) = A_i(x_1, x_2) + \partial_i \beta_1(x_1, x_2)$$

$$A_i(x_1, x_2 + L_2) = A_i(x_1, x_2) + \partial_i \beta_2(x_1, x_2)$$

such that the circulation around the boundary is Φ (the flux)

$$\Rightarrow \left[\beta_2(x_1+L_1,x_2)-\beta_2(x_1,x_2)\right]-\left[\beta_1(x_1,x_2+L_2)-\beta_1(x_1,x_2)\right] = \Phi$$

choose $\beta_i = -\frac{1}{2}\Phi\,\varepsilon_{ij}\,\frac{x_j}{L_j}$ (no sum)

and gauge invariance requires that

$$\Psi(x_1,x_2) \to e^{-\frac{ie}{\hbar c}\Lambda(x_1,x_2)}\,\Psi(x_1,x_2)$$

$\Rightarrow$ under these large gauge transf.

$$\Psi(x_1+L_1,x_2) = e^{\frac{ie}{\hbar c}\beta_1(x_1,x_2)}\,\Psi(x_1,x_2)$$

$$\Psi(x_1,x_2+L_2) = e^{\frac{ie}{\hbar c}\beta_2(x_1,x_2)}\,\Psi(x_1,x_2)$$

These BC's are consistent if under the sequence of translations

$$(x_1,x_2)\to(x_1+L_1,x_2)\to(x_1+L_1,x_2+L_2)$$

$$(x_1,x_2)\to(x_1,x_2+L_2)\to(x_1+L_1,x_2+L_2)$$

we get the same wave function (single valued!)

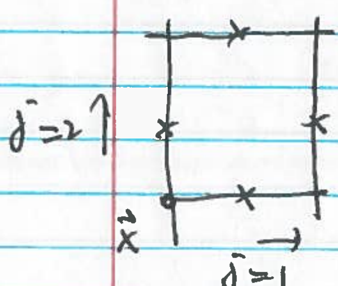
$\Rightarrow \Phi = N_\phi\,\phi_0$ flux quantization.

The Quantum Hall Effect on a Lattice

We will consider now the quantum states of a charged particle on a square lattice in an uniform magnetic field B with flux $\Phi = \frac{p}{q}\phi_0$ per plaquette. This is the famous Hofstadter problem (1976). ~~Ideas~~ We will consider a non-interacting system and, hence, we will focus on the single particle states.

Let $|\vec{x}\rangle$ be a (Wannier) state of the particle localized at a site $\vec{x}$ of the square lattice (with spacing $a = 1$). The Hamiltonian is

$$H = -t \sum_{\vec{x}, j=1,2} |\vec{x}\rangle e^{i\frac{e}{\hbar c}A_j(\vec{x})} \langle \vec{x} + \hat{e}_j| + h.c.$$



$$\sum_{\text{plaquette}} A_j = \Delta_1 A_2 - \Delta_2 A_1 = B$$

lattice difference

Landau gauge: $A_1 = -B x_2, \quad A_2 = 0$

$$0 \le x_i \le L_i \quad (i = 1, 2)$$

<u>Eigenstates:</u> $|\Psi\rangle = \sum_{\vec{x}} \Psi(\vec{x}) \, |\vec{x}\rangle$

$\Rightarrow$ discrete Schrödinger Eqn.

$$-t\left\{ e^{-i 2\pi \frac{p}{q} x_2} \Psi(x_1+1, x_2) + e^{i 2\pi \frac{p}{q} x_2} \Psi(x_1-1, x_2) \right\}$$

$$-t\left\{ \Psi(x_1, x_2+1) + \Psi(x_1, x_2-1) \right] = E \, \Psi(x_1, x_2)$$

H is <u>not</u> invariant under $(x_1, x_2) \to (x_1+1, x_2)$ (same with x_2)

but it is invariant under

$$(x_1, x_2) \to (x_1+q, x_2)$$

$$(x_1, x_2) \to (x_1, x_2+1)$$

$\Rightarrow$ unit cell has q plaquettes $(1 \times q)$

Flux through a cell $\Phi_{cell} = q \, \Phi_{plaquette} = p \, \phi_0$ ✓

Gauge-invariant translation Op.:

$$e^{i\hat{P}_j} = \sum_{\vec{x}} |\vec{x}\rangle \, e^{i A_j(x)} \langle \vec{x} + \hat{e}_j| \quad \text{(unitary!)}$$

$$e^{i\hat{P}_1} e^{i\hat{P}_2} = e^{i 2\pi \frac{p}{q}} e^{i\hat{P}_2} e^{i\hat{P}_1}$$

(discrete magnetic algebra)

$\Rightarrow$ $e^{i n_1 \hat{P}_1}$ and $e^{i n_2 \hat{P}_2}$ do not commute

unless $\frac{p}{q} n_1 n_2 \in \mathbb{Z}$