L14 10/11/2022

$$H = -t \sum_{j=1,2} \left(e^{i\hat{P}_j} + e^{-i\hat{P}_j}\right)$$

$\Rightarrow$ eigenstates of H are <u>not</u> eigenstates of $e^{i\hat{P}_j}$

<u>Lattice magnetic translations</u>

$$e^{i\hat{k}_j} = \sum_{\vec{x}} |\vec{x}\rangle\, e^{i A'_j(\vec{x})} \langle \vec{x} + \hat{e}_j|$$

A'_j chosen s.t. $e^{i\hat{k}_j}$ commute with $e^{i\hat{P}_j}$

and hence with H.

$j \neq k \Rightarrow \Delta_j A'_k(x) = \Delta_k A'_j(x)$

$\Rightarrow \hat{k}_j = \hat{P}_j(-B)$ (as before)

$\Rightarrow A'_1 = 0$, $A'_2(x) = -2\pi \frac{p}{q} x_1$ (gauge choice)

$$e^{i\hat{k}_1} e^{i\hat{k}_2} = e^{i 2\pi \frac{p}{q}} e^{i\hat{k}_2} e^{i\hat{k}_1}$$

Consider magnetic translation by n_1 along x_1 and n_2 along x_2

$\hat{T}_j^{n_j} = e^{i n_j \hat{k}_j}$
(no sum)

$\hat{T}_j^{n_j}$ commute with each other if $n_1 n_2 \frac{p}{q} \in \mathbb{Z}$

$\Rightarrow$ the eigenstates of $\hat{H}$ are eigenstates of $\hat{T}_1^{n_1}$ and $\hat{T}_2^{n_2}$

$$[T_1, T_2^q] = [T_1, H] = [T_2^q, H] = 0$$

choose the basis $|k_1, k_2\rangle$ /

$$\hat{T}_1 |k_1, k_2\rangle = e^{ik_1} |k_1, k_2\rangle$$
$$\hat{T}_2^q |k_1, k_2\rangle = e^{iqk_2} |k_1, k_2\rangle$$

with PBC's

$$\hat{T}_1^{L_1} |k_1, k_2\rangle = |k_1, k_2\rangle$$
$$(\hat{T}_2)^{L_2/q} |k_1, k_2\rangle = |k_1, k_2\rangle$$

$\Rightarrow$ (L_1 and $L_2 \to \infty$) $|k_1, k_2\rangle$ belong to the Brillouin zone $-\pi \leq k_1 \leq \pi$ and $-\frac{\pi}{q} \leq k_2 \leq \frac{\pi}{q}$

(L_2 must be divisible by q)

$$\Rightarrow N_\phi = \frac{p}{q} L_1 L_2 \in \mathbb{Z}$$

# of states in the magnetic BZ is $L_1 L_2 / q$

<u>Fourier Transform</u>

$$\psi(x_1, x_2) = \frac{1}{q} \sum_{r=1}^{q} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi/q}^{\pi/q} \frac{dk_2}{2\pi/q} e^{i(k_1 x_1 + k_2 x_2)} \psi(k_1, k_2 + 2\pi \frac{p}{q} r)$$

Let $\psi_r(k_1, k_2) \equiv \psi(k_1, k_2 + 2\pi \frac{p}{q} r)$ $\quad r = 1, \dots, q$

(q - component amplitude)

Schrödinger Eqn: $\Leftrightarrow$ Harper equation

$$-t\{e^{ik_1}\psi_{r+1}(k_1,k_2)+e^{-ik_1}\psi_{r-1}(k_1,k_2)\}$$

$$-2t\cos\left(k_2+\frac{2\pi p}{q}r\right)\psi_r(k_1,k_2)=E(k_1,k_2)\,\psi_r(k_1,k_2)$$

* $\psi_r(k_1,k_2)$ are periodic functions on the magnetic BZ

$$\psi_r(k_1+2\pi n_1,k_2)=\psi_r(k_1,k_2)$$

$$\psi_r\left(k_1,k_2+\frac{2\pi}{q}n_2\right)=\psi_{r+n_2}(k_1,k_2)$$

$$\psi_{r+q}(k_1,k_2)=\psi_r(k_1,k_2)$$

$\Rightarrow$ Magnetic BZ is a 2-torus, and $\psi_r(k_1,k_2)$ is an q-component complex vector field continuous on the torus.

For general p and q the spectrum has q bands.

Here we assume that p and q are coprimes (q,p):

$\Rightarrow$ $\mathcal{H}(k_1,k_2)$ has symmetries:

$$\hat{A}_{jk}=\omega^k\delta_{jk}\quad(\text{no sum})\qquad \omega=e^{-i2\pi p/q}$$

$$\hat{B}_{jk}=\delta_{j,k-1}\quad\Rightarrow\quad AB=e^{i2\pi p/q}BA$$

and $\mathcal{H}(k_1,k_2)=e^{-ik_2}\hat{A}+e^{ik_1}\hat{B}+\text{h.c.}$

If $(p,q)=1$, $\Rightarrow \exists\, n, m$ s.t.

$$1 = np + mq$$

$\Rightarrow$ $\tilde{A} = \hat{A}^n$, $\tilde{B} = \hat{B}^m$ satisfy

$$\tilde{A}\, \mathcal{H}(k_1,k_2)\, \tilde{A}^{-1} = \mathcal{H}(k_1 + \tfrac{2\pi}{q} n, k_2)$$

$$\tilde{B}\, \mathcal{H}(k_1,k_2)\, \tilde{B}^{-1} = \mathcal{H}(k_1, k_2 + \tfrac{2\pi}{q}) \quad (*)$$

and $\mathcal{H}(k_1 + \frac{2\pi}{q} - 2\pi m, k_2) = \mathcal{H}(k_1 + \frac{2\pi}{q}, k_2)$

$$\tilde{A}\tilde{B} = e^{-i\, 2\pi \frac{p}{q} n^2}\, \tilde{B}\tilde{A}$$

$\Rightarrow$ If $\psi(k_1,k_2)$ is an eigenstate of $\mathcal{H}(k_1,k_2)$

with $E(k_1,k_2)$ $\Rightarrow$ $\psi'(k_1,k_2) = \tilde{A}\,\psi(k_1,k_2)$

is an eigenstate of $\mathcal{H}'(k_1,k_2) = \mathcal{H}(k_1 + \frac{2\pi}{q}, k_2)$

with the same eigenvalue $E(k_1,k_2)$.

$\Rightarrow$ correspondence $(k_1,k_2) \to (k_1 + \frac{2\pi}{q}, k_2)$

The same happens between $(k_1,k_2) \leftrightarrow (k_1, k_2 + \frac{2\pi}{q})$

Also under translation $(k_1,k_2) \to (k_1+\pi, k_2+\pi)$

$$\mathcal{H}(k_1+\pi, k_2+\pi) = -\mathcal{H}(k_1,k_2)$$

and $E(k_1+\pi, k_2+\pi) = -E(k_1,k_2)$

For q <u>even</u> this is a consequence of $(*)$

$\Rightarrow$ ~~$\bar{g}$~~ if q is even $\Rightarrow$ for each eigenstate with e.v. $E(k_1,k_2)$, $\exists$ an eigenstate with e.v. $-E(k_1,k_2)$

The operator that maps these states is Γ which $\{\Gamma, H\}=0$ and $\Gamma^2 = I$

$\boxed{\Gamma \text{ plays the role of } \gamma_5 \text{ in the Dirac theory}}$

e.g. $\Gamma_{jk} = (-1)^j \, i^{q/2} \, \delta_{k, j+q/2}$

$$\Rightarrow \{\Gamma, \tilde{A}\} = \{\Gamma, \tilde{B}\} = \{\Gamma, \hat{A}\} = \{\Gamma, \hat{B}\} = 0$$

Let us show that for q even there are q states with zero energy. (Generalization of fermion doubling)

Let ~~ω~~ $w = k_1 + ik_2$ complex plane and globally a torus.

Let's assume that all eigenstates of $\mathcal{H}$ at w are $\neq 0$. $\Rightarrow$ We can write (in the basis with Γ diagonal)

$$\Gamma = \underbrace{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}}_{\frac{q}{2}\times\frac{q}{2}}, \quad \mathcal{H}(\vec{k}) = \begin{pmatrix} 0 & h^\dagger(k) \\ h(k) & 0 \end{pmatrix}$$

If there are no zero energy states $\Rightarrow \mathcal{H}$ locally is invertible $\Rightarrow \det \mathcal{H} = -|\det h|^2 \neq 0$

~~Let $D = \det h$ is locally an analytic function of w.~~

~~Define a vector field $A_i = D^{-1} \frac{\partial}{\partial k_i} D$~~

Let $D(k) = \det h(\vec{k})$ is locally analytic

Define the 1-form $A_i(k) = D^{-1}(k)\frac{\partial}{\partial k_i} D(k)$.

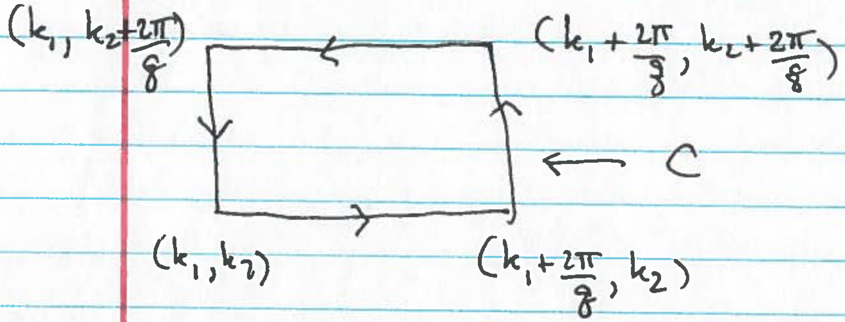
A_i is a 1-form. In any neighborhood of $\vec{k}$ where $D \neq 0$, A_i is closed

i.e. $\varepsilon_{ij}\, \partial_i A_j = \varepsilon_{ij}\, \partial_i \partial_j \ln D \neq 0$

$\left(\partial_i = \frac{\partial}{\partial k_i} \right)$ but it is not exact meaning that

$$\nu = \frac{1}{2\pi} \oint_C d\vec{k} \cdot \vec{A}(\vec{k}) \neq 0 \quad \text{(in general)}$$

If $\nu \neq 0 \Rightarrow D(\vec{k}) = \det h(\vec{k})$ must have a zero at $\vec{k}_0$ inside C



$D(k_1, k_2)$ satisfies

$$D(k_1,k_2) = -D^*(k_1 + \tfrac{2\pi}{g}, k_2) = -D(k_1 + \tfrac{2\pi}{g}, k_2 + \tfrac{2\pi}{g})$$
$$= D^*(k_1, k_2 + \tfrac{2\pi}{g})$$

$\Rightarrow$ the phase of D must wind as C is traversed

Since $D \in \mathbb{C}$ it will trace a path $\mathcal{D}$ on the complex plane as $\vec{k}$ is varied along C. If D does not have a zero inside

$\Rightarrow$ ν will vanish and C (and $\mathcal{D}$) can be deformed to zero. But if there is a zero of D inside C $\Rightarrow$ $\mathcal{D}$ and C cannot be deformed to zero and D will have a singularity.

$\Rightarrow$ $\mathcal{D}$ will wind ν times as $\vec{k}$ traverses C.

Since $D(\vec{k}) \neq$ const. $\Rightarrow$ it has zeros at isolated locations (local analyticity). But if $\vec{k}_0$ is a zero $\Rightarrow$ $\vec{k}_0 + \frac{2\pi}{q}(n_1\hat{e}_1 + n_2\hat{e}_2)$ is also a zero. $\Rightarrow$ $\exists$ a lattice of zeros which is periodic $\Rightarrow$ $\vec{k}_0$ must be $(\frac{\pi}{2}, \frac{\pi}{2})$ and its translations.

$\Rightarrow$ $\exists$ q zeros

$\Rightarrow$ If q is even $\Rightarrow$ $\mathcal{H}(\vec{k})$ has q zeros

We saw an example of this in the flux phase in which $q = 2 \Rightarrow 2$ zeros.

This discussion is the same as the family index theorem of Dirac Operators (related to supersymmetry) (see Wen and Zee, NPB 316, 641 (1989))

Linear Response Theory and Correlation Functions

How does one calculate a Hall conductance?

Let A_μ be an external (i.e. not quantized) electromagnetic field and let $Z[A_\mu]$ be the partition function (the path integral) in the presence of $A_\mu \neq 0$. We will assume that A_μ is "small" so that we can use it as a perturbation. ⇒ A_μ will be small compared with the uniform magnetic field B.

⇒ $\langle A_\mu \rangle$ is part of the def. of the system

⇒ we set $\langle A_\mu \rangle = 0$ for the perturbation.

$$\Rightarrow Z[A_\mu] = Z[0] \exp\left[\frac{i}{2}\int d^Dx \int d^Dy\, A_\mu(x)\, \Pi^{\mu\nu}(x,y)\, A_\nu(y)\right.$$

$D = d+1$ dimensions

polarization tensor.

Gauge invariance ⇒ $A_\mu = A'_\mu + \partial_\mu \phi(x)$

$$\psi(x) = e^{i\frac{e}{\hbar c}\phi(x)}\, \psi'(x)$$

⇒ $Z[A]$ does <u>not</u> change ⇒ $\partial_\mu \Pi^{\mu\nu}(x,y) = 0$

where we assumed that $\phi(x) \xrightarrow[|x|\to\infty]{} 0$

If the system has boundaries $\Rightarrow$ the values of the gauge field at the boundaries have physical meaning (voltages etc.)

Also a surface without boundaries such as a torus has non-contractible loops Γ and

$$\oint_\Gamma A_\mu dx^\mu \text{ is gauge-invariant (holonomies)}$$

What is the current induced by A_μ?

$$J_\mu(x) = \frac{\delta \overset{\text{action}}{S}}{\delta A_\mu(x)}$$

and $$\langle J_\mu(x)\rangle = \frac{-i}{\hbar Z[A]} \frac{\delta}{\delta A_\mu(x)} Z[A]\Big|_{A=0}$$

In a non-relativistic system (no spin)

$$J_0(x) = -e\,\psi^\dagger(x)\,\psi(x)$$

$$J_j(x) = \frac{e\hbar}{2iMc}\left(\psi^\dagger \partial_j \psi - \partial_j \psi^\dagger \psi\right) - \frac{e^2}{Mc^2} A_j \psi^\dagger \psi$$

$$\equiv \frac{e\hbar}{2iMc}\left[\psi^\dagger(x)\, D_j \psi(x) - \left(D_j \psi(x)\right)^\dagger \psi(x)\right]$$

where $D_j = \partial_j - \frac{ie}{\hbar c} A_j$ is the covariant derivative

On a lattice

$$J_j(x) = \frac{\overset{\text{hopping}}{t}}{2i}\left[\psi^\dagger(x)\, e^{\frac{ie}{\hbar c}\int_x^{x+e_j} \vec{A}\cdot d\vec{z}}\, \psi(x+e_j) - h.c.\right]$$

$$\Rightarrow \Pi_{\mu\nu}(x,y) = -i\hbar \frac{\delta^2}{\delta A_\mu(x)\, \delta A_\nu(y)} \ln Z[A]\Big|_{A=0}$$

$$= -i\hbar \frac{\delta}{\delta A_\mu(x)}\left[\frac{1}{Z[A]} \frac{\delta Z(A)}{\delta A_\nu(y)}\right]$$

$$= i\hbar \left(\frac{1}{Z[A]} \frac{\delta Z(A)}{\delta A_\mu(x)}\right)\left(\frac{1}{Z(A)} \frac{\delta Z(A)}{\delta A_\nu(y)}\right)$$

$$-\frac{i\hbar}{Z(A)} \frac{\delta^2 Z(A)}{\delta A_\mu(x)\, \delta A_\nu(y)}$$

$$\Rightarrow \Pi_{\mu\nu}(x,y) = \frac{i}{\hbar}\langle J_\mu(x)\, J_\nu(y)\rangle_{\text{connected}} + \left\langle \frac{\delta J_\mu(x)}{\delta A_\nu(y)}\right\rangle$$

~~Since gauge invariance requires~~ $\cancel{\partial_\mu \Pi^{\mu\nu} = 0}$

$\Rightarrow$ Def. connected time-ordered correlator

$$D_{\mu\nu}(x,y) \equiv \frac{i}{\hbar}\langle J_\mu(x)\, J_\nu(y)\rangle_{\text{connected}}$$