L 15 10/12/2022

$$\Rightarrow \partial_\mu \Pi^{\mu\nu} = 0 \Rightarrow \partial_\mu D^{\mu\nu} = -\partial_\mu \left\langle \frac{\delta J_\mu}{\delta A_\nu} \right\rangle$$

J_μ is conserved $\Rightarrow \partial_\mu J^\mu = 0$

But

$$\cancel{\partial^x_\mu D^{\mu\nu}(x,y) = \frac{i}{\hbar}\partial^x_\mu \langle T J^\mu_0(x) J^\nu_0(y)\rangle}$$

$$\partial^x_\mu D^{\mu\nu}(x,y) = \frac{i}{\hbar}\partial^x_\mu \langle \underset{\text{time-ordering}}{T} J^\mu(x)\, J^\nu(y)\rangle$$

$$= \frac{i}{\hbar}\partial^x_\mu \Big[\Theta(x_0 - y_0)\,\langle J^\mu(x)\, J^\nu(y)\rangle + \Theta(y_0 - x_0)\,\langle J^\nu(y)\, J^\mu(x)\rangle\Big]$$

$$= \frac{i}{\hbar}\,\delta(x_0 - y_0)\,\langle [J_0(x), J_\nu(y)]\rangle$$

$$+ \frac{i}{\hbar}\langle T\, \underbrace{\partial^x_\mu J^\mu(x)}_{=0}\, J_\nu(y)\rangle$$

$$\Rightarrow \partial^x_\mu D^{\mu\nu}(x,y) = \frac{i}{\hbar}\,\delta(x_0 - y_0)\,\langle [J_0(x), J^\nu(y)]\rangle$$

And $\partial^\mu \left\langle \frac{\delta J_\mu(x)}{\delta A_0(y)} \right\rangle = 0$

$\langle J_0(x)\rangle$

$$\partial^\mu \left\langle \frac{\delta J_\mu(x)}{\delta A_\ell(y)} \right\rangle = \partial_k \left\langle \frac{\delta J_k(x)}{\delta A_\ell(y)} \right\rangle = -\frac{e}{Mc^2}\partial^x_k\left[\delta(x-y)\,\langle J_0(x)\rangle\right]$$

$\Rightarrow$ at equal-times

$$\langle [J_0(x_0,\vec{x}), J_k(x_0,\vec{y})] \rangle =$$

$$= \frac{ie}{\hbar Mc^2} \partial_k^x \left(\delta(\vec{x}-\vec{y}) \langle J_0(\vec{x}) \rangle \right)$$

This is a Ward Identity!

(Schwinger term)

$$\Rightarrow \partial^\mu D_{\mu,0}(x,y) = 0$$

$$\partial^\mu D_{\mu,k}(x,y) = \frac{ie}{\hbar Mc^2} \partial_k \left(\delta(x-y) \langle J_0(x) \rangle \right)$$

while $\partial^\mu \Pi_{\mu\nu} = 0$ <u>always</u>

All of this is true <u>in general</u>.

Consider free fermions ~~and the continuum~~

action: $S = \int dx\, \psi^*(x) \left(iD_0 + \mu - \underbrace{h[\langle A_\mu \rangle + A_\mu]}_{\text{one-particle } H} \right) \psi(x)$

$$h[A_\mu] = -\frac{\hbar^2}{2M} \vec{D}_j^2$$

$$Z[A] = \int \mathcal{D}\psi^* \mathcal{D}\psi \, e^{\frac{i}{\hbar} S(\psi,\psi^*,A)}$$

$$\Rightarrow Z[A] = \mathrm{Det}\,(iD_0 + \mu - h[A])$$

$$\Rightarrow S_{eff}(A) = -i\,\mathrm{tr}\ln\left(iD_\mu + \mu - h[A]\right)$$

If A_μ is weak (or $|A_\mu| \ll |\langle A_\mu \rangle|$)

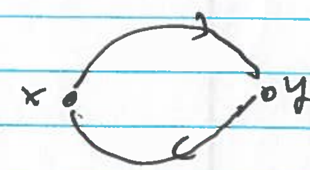
$\Rightarrow$ we can expand in powers of A_μ

$\Rightarrow$ let $G(x,y)$ be the Green function

$$(iD_0 + \mu - h[A])_x\, G(x,y) = \delta(x-y)$$

$$\Rightarrow G(x,y) = \langle x| \frac{1}{iD_0 + \mu - h[A]} |y\rangle \qquad (\hbar = 1)$$

$$\Rightarrow \tilde{\Pi}_{00}(x,y) = i\, G(x,y)\, G(y,x)$$



$$\tilde{\Pi}_{0j}(x,y) = \frac{1}{2M}\left[G(x,y)\, D^y_j(y,x) - G(y,x)\, D^{y\dagger}_j\, G(x,y)\right]$$

$$\tilde{\Pi}_{j0}(x,y) = \frac{1}{2M}\left[-G(x,y)\, D^{\dagger x}_j(y,x) + G(y,x)\, D^x_j(x,y)\right]$$

$$\tilde{\Pi}_{jk}(x,y) = \frac{i}{M}\delta(x-y)\,\delta_{jk}\, G(x,y) - \frac{i}{4M^2}\{D^x_j G(x,y)\, D^y_k G(y,x)$$

$$- \frac{i}{4M^2} D^{\dagger x}_j G(y,x)\, D^{\dagger y}_k(x,y) + \frac{i}{4M^2} G(x,y)\, D^x_j D^{y\dagger}_k G(x,y)$$

$$+ \frac{i}{4M^2}\left(D^{x\dagger}_j D^y_k G(y,x)\right) G(x,y)$$

In $D=2+1$ with broken time-reversal ($B \neq 0$)
(gapped system)

$$S_{eff}(A) \cong \int d^2x\, dt \left[\frac{\varepsilon}{2} \vec{E}^2 + \frac{\chi}{2} B^2 + \overbrace{\frac{\sigma_{xy}}{4} \epsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda}}^{\text{Hall}} + \dots \right]$$

$$\Rightarrow \langle J_k \rangle_{xy} = \sigma_{xy} \varepsilon_{k\ell} E_\ell$$

$$\sigma_{xy} = \lim_{q \to 0} \left(\frac{i}{2} \frac{\varepsilon^{\mu\nu\lambda} q_\lambda}{q^2} \Pi_{\mu\nu}(q) \right)$$

$$= \lim_{q_0 \to 0} \left(\frac{i}{q_0} \Pi_{xy}(q_0, \vec{q}=0) \right)$$

frequency!

However, in a metal there is no gap in the spectrum and the effective action becomes non-local. A manifestation of this feature is that the uniform limit ($\vec{q} \to 0$) and the static limit ($q_0 \to 0$) <u>do not commute</u>. In a metal (even if there is a magnetic field) ~~the~~ CRR the the symmetric part of $\Pi_{\mu\nu}$ yields the longitudinal conductivity which reflects that there is dissipation.

~~&~~ The conductivity is also found in the limit $\vec{q} \to 0$ first and $q_0 \to 0$ later.

The Kubo formula for σ_{xy}

Consider a system in its gnd. state $|\Psi_0\rangle$, ~~and~~ assume that there is a gap in the spectrum, and that there is a $B \neq 0$. (2D)

Suppose we introduce adiabatically a weak external e.m. perturbation. Adiabatic here means that the rate of change is small in the scale of the gap. Since the pert. is adiabatic we can use the Born-Oppenheimer approx. to evaluate its effects.

Let $|\alpha(t)\rangle$ be an instantaneous eigenstate of the ~~Hamilt~~ Hamiltonian $H(t)$ with energy $E_\alpha(t)$.

Then, to first order in time derivatives (adiabatic!) the perturbed eigenstates are

$$|\Psi_\alpha(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t dt'\, \mathcal{E}_\alpha(t')} \times$$

$$\times \left[|\alpha(t)\rangle + i\hbar \sum_{\beta\neq\alpha} \frac{|\beta(t)\rangle\langle\beta(t)|\partial_t|\alpha(t)\rangle}{\mathcal{E}_\beta(t) - \mathcal{E}_\alpha(t)} \right. + \cdots$$

which is a parametric function of t.

In the gauge $A_0 = 0$, time enters in $H(t)$ through the time-dependence of $\vec{A}(t)$

$$\delta\vec{A}(t) = \vec{E}(t)\, t$$

where $\vec{E}(t)$ is very weak and slowly varying

For an observable $\hat{M}$

$$\langle\Psi_\alpha(t)|\hat{M}|\Psi_\alpha(t)\rangle =$$

$$= i\hbar \sum_{\beta\neq\alpha} \frac{\langle\alpha|\hat{M}|\beta\rangle\langle\beta|\partial_t|\alpha\rangle + \langle\alpha|\partial_t|\beta\rangle\langle\beta|\hat{M}|\alpha\rangle}{\mathcal{E}_\beta(t) - \mathcal{E}_\alpha(t)} + \cdots$$

Here we will be interested in $\hat{M}$ being the current $\vec{J}(\vec{x})$.

The states $|\alpha(t)\rangle$ obey the time-dep. Schrödinger Eqn.

$$\Rightarrow \langle\alpha|\partial_t|\beta\rangle = \frac{\langle\alpha|\partial_t\hat{H}|\beta\rangle}{\varepsilon_\alpha(t) - \varepsilon_\beta(t)}$$

but the time dependence of $\hat{H}$ comes from its dependence on $\vec{A}(t)$. Since the current is $\frac{\delta H}{\delta\vec{A}} \Rightarrow$

$$\langle\sigma_{xy}\rangle_\alpha = -i\hbar L_1L_2 \sum_{\beta\neq\alpha} \frac{\langle\alpha|\hat{J}_1|\beta\rangle\langle\beta|\hat{J}_2|\alpha\rangle - \langle\alpha|\hat{J}_2|\beta\rangle\langle\beta|\hat{J}_1|\alpha\rangle}{(\varepsilon_\alpha(t) - \varepsilon_\beta(t))^2}$$

Generalized Toroidal BC's

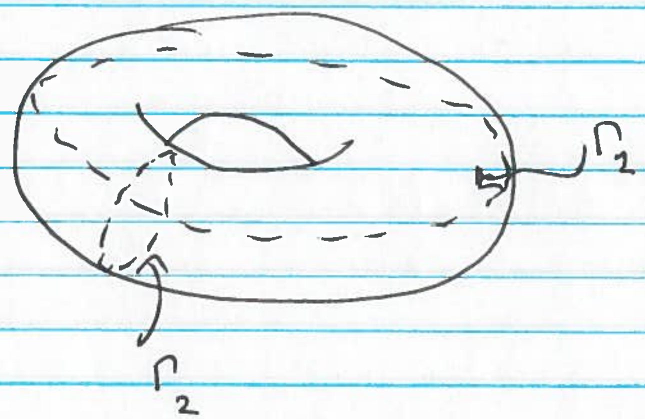
Consider an $L_1 \times L_2$ system with an electrostatic potential $U(\vec{x}) = \vec{E}\cdot\vec{x}$ and $\vec{E} = \vec{\nabla}U$

$$\Rightarrow \delta\vec{A} = \vec{E}\,t = \vec{\nabla}(U(\vec{x})\,t)$$

since $\vec{A}$ is a gradient we can get rid of it by a gauge transf.

$$\Psi(\vec{x}) \to \Psi(\vec{x})\, e^{i \frac{e}{\hbar c} U(\vec{x}) t}$$

This transformation cannot change the value of $\delta\vec{A}(t)$ on a closed <u>non-contractible loop</u> of the torus (Γ_1 and Γ_2)



$$I_j \equiv \oint_{\Gamma_j} \delta\vec{A}\cdot d\vec{\ell} = t \oint_{\Gamma_j} \vec{E}\cdot d\vec{\ell} = t\, E_j L_j$$

(no sum)

holonomies of $\delta\vec{A}$

Let $\Psi(\vec{x}_1, \ldots, \vec{x}_N)$ be the wave function of an N-particle system with the ("twisted") BC's

$$\Psi(\vec{x}_1, \ldots, \vec{x}_N) = e^{i\vec{\theta}\cdot\vec{L}}\, \Psi(\vec{x}_1 + \vec{L}, \ldots, \vec{x}_N + \vec{L})$$

~~where~~

where $\vec{\Theta}$ is an <u>arbitrary</u> two-component vector (of phases) and $\vec{L}$ is a displacement ~~along~~ by L_1 along x_1 and by L_2 along x_2.

In the presence of a magnetic field the BC's are

$$A_1(x_1, x_2+L_2) = A_1(x_1, x_2) + \partial_1 \beta_2(x_1, x_2)$$

$$A_2(x_1+L_1, x_2) = A_2(x_1, x_2) + \partial_2 \beta_1(x_1, x_2)$$

$$\Rightarrow \Psi(\{x_1^{(j)}+L_1\}; \{x_2^{(j)}\}) = $$

$$= e^{-\frac{ie}{\hbar c}\sum_{j=1}^{N} \beta_1(x_1^{(j)}; x_2^{(j)}) + i\theta_1}$$

$$\Psi(\{x_1^{(j)}\}; \{x_2^{(j)}+L_2\}) =$$

$$= e^{-\frac{ie}{\hbar c}\sum_{j=}^{N} \beta_2(x_1^{(j)}, x_2^{(j)}) + i\theta_2}$$

and $\theta_j = \frac{et}{\hbar c} E_j L_j \equiv \frac{e}{\hbar c} I_j$

$\Rightarrow$ the time dependence enters through changes of $\vec{\theta}$