After some manipulation $\Longrightarrow$

$$(\sigma_{xy})_\alpha = \frac{ie^2}{\hbar}\left[\partial_1 \langle\alpha|\partial_2|\alpha\rangle - \partial_2 \langle\alpha|\partial_1|\alpha\rangle\right]$$

(Niu - Thouless - Wu 1985)

where $\partial_j = \frac{\partial}{\partial\theta_j}$

If we average over the BC's

$$\Rightarrow \langle(\sigma_{xy})_\alpha\rangle = \int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}\,(\sigma_{xy})_\alpha$$

$$= -i\frac{e^2}{\hbar}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}\left(\partial_2\langle\alpha|\partial_1|\alpha\rangle - \partial_1\langle\alpha|\partial_2|\alpha\rangle\right)$$

$$-i\langle\alpha|\partial_j|\alpha\rangle \equiv A_j \quad \text{Berry connection}$$

$$\Rightarrow \langle(\sigma_{xy})_\alpha\rangle = \frac{e^2}{\hbar}\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}\,\varepsilon_{jk}\,\partial_j A_k$$

Berry flux through the torus of BC's!

The quantity

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta_1\int_0^{2\pi}d\theta_2\,\varepsilon_{jk}\,\partial_j A_k \equiv \frac{1}{2\pi}\oint_C d\theta_j\,A_j \equiv C_1$$

first Chern number

(C: boundary of T^2)

Fiber Bundles and the Hall conductance

Why is the Hall conductance precisely quantized?

We will show that it is a topological invariant.

The BC's angles, θ_1 and θ_2, are phases and are defined mod $2\pi \Rightarrow$ Each choice of BC is a point $\vec{\theta}$ on a 2-torus $S_1 \times S_2$ of BC's

For each BC, i.e. for each $\vec{\theta}$, we have a unique eigenstate $\Psi_\alpha([\vec{x}]; \vec{\theta})$ of the full many-body Hamiltonian. In math this ~~means that~~ is called a fiber bundle.

The wave function has an amplitude and a phase. The phase of the w.f. is not a physical observable but changes of the phase are.

Let us pick some BC $\vec{\theta}_0$ and let the phase of the w.f. be $\arg[\Psi_\alpha(\vec{\theta}_0)]$. When we turn on a weak e.m. field the phase ~~changes~~ $\vec{\theta}(t)$ (since the ~~vecto~~ vector potential does)

$\Rightarrow \vec{\theta}_0 = \vec{\theta}(t_0) \longrightarrow \vec{\theta}(t)$ (adiabatically) over a long time T

~~Suppose~~ Suppose that after a long time T the BC returns to $\vec{\theta}_0 \Rightarrow$ as a function of time the BC's traced a closed curve Γ on the 2-torus of BC's.

Change in the phase of the w.f. δ_Γ

$$\delta_\Gamma = \Delta \arg \Psi = \Delta \, \mathrm{Im} \ln[\Psi] = \arg\left[\Psi(\vec{\theta}(t_0+T))\right] - \arg\left[\Psi(\vec{\theta}(t_0))\right]$$

If Ψ is an analytic non-vanishing function of $\vec{\theta} \Rightarrow \delta_\Gamma = 0$ (since we can deform Γ to zero without ~~...~~ finding a singularity. In that case the only analytic function on a torus is a constant ($\delta_\Gamma = 0$)

$\Rightarrow$ A non-vanishing adiabatic change in phase of requires that $\ln \Psi$ should have singularities enclosed by the closed loop Γ.

Non-vanishing changes of phases of w.f.'s ~~...~~ in Q.M. are known as Berry Phases (Berry 1984, Simon

$\Rightarrow$ Since Ψ is locally an analytic of $\vec{\theta} \Rightarrow$ ~~...~~ $\ln \Psi$ must have isolated singularities $\Rightarrow$ $\Psi_\alpha(\vec{\theta})$ must

have isolated zeros on the 2-torus of BC's.

$\Rightarrow$ δ_Γ counts the # of zeros enclosed by Γ.

* How is this related to σ_{xy} ?

Let $|\Psi_{\vec{\theta}}\rangle$ be a state for some BC $\vec{\theta}$

Define the 1-form: $A_k^{(\alpha)} = i\langle\alpha|\frac{\partial}{\partial\vec{\theta}}|\alpha\rangle$

(Berry connection)

$$\equiv i\langle\Psi_{\vec{\theta}}^{(\alpha)}|\underbrace{\frac{\partial}{\partial\theta_k}}_{\text{infinitesimal change}}|\Psi_{\vec{\theta}}^{(\alpha)}\rangle$$

In this notation

BC average

$$\langle(\sigma_{xy})_\alpha\rangle = \frac{e^2}{h}\underbrace{\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}\left(\partial_1 A_2 - \partial_2 A_1\right)}_{\text{flux of } \vec{A}(\vec{\theta}) \text{ through the 2-torus of BC's (Berry curvature)}}$$

On the other hand, $|\Psi^{(\alpha)}(\vec{\theta})\rangle$ is defined up to an overall phase factor $\Rightarrow$

$|\Psi^{(\alpha)}(\vec{\theta})\rangle$ and $e^{if(\vec{\theta})}|\Psi^{(\alpha)}(\vec{\theta})\rangle$ are equivalent

In other words, we have a "ray" ~~re~~ (projective representation of the state, under this arb. change of phase, the 1-form $\mathcal{A}_k$ changes

$$\mathcal{A}_k(\vec{\theta}) = i\langle\alpha|\partial_k|\alpha\rangle$$

$$\rightarrow i\langle\alpha|\partial_k|\alpha\rangle - \partial_k f(\vec{\theta})$$

(gauge transformation!)

$\Rightarrow$ the arb. phase factors of the w.f. translate into gauge transf. of the 1-form $\mathcal{A}_k(\vec{\theta})$

Let $\Sigma\,/\,\partial\Sigma = \Gamma$ and choose Γ to be a rectangular contour with corners at (θ_1, θ_2), $(\theta_1 + 2\pi, \theta_2)$, $(\theta_1+2\pi, \theta_2+2\pi)$ and $(\theta_1, \theta_2+2\pi)$

$$\Rightarrow \langle(\sigma_{xy})_\alpha\rangle = \frac{e^2}{\hbar}\oint_\Gamma \mathcal{A}_k(\vec{\theta})\, d\theta_k \qquad \text{circulation}$$

A non-zero Hall conductance means that $\vec{\mathcal{A}}$ <u>cannot</u> be a periodic function of $\vec{\theta}$ on the 2-torus

$\Rightarrow$ along the two non-contractible loops of the 2-torus of BC's, A_k must change as

$$A_k(\theta_1+2\pi,\theta_2) = A_k(\theta_1,\theta_2) + \partial_k f_1(\theta_1,\theta_2)$$

$$A_k(\theta_1,\theta_2+2\pi) = A_k(\theta_1,\theta_2) + \partial_k f_2(\theta_1,\theta_2)$$

$$\Psi^{(\alpha)}([\vec{x}];\theta_1,\theta_2+2\pi) = e^{if_2(\theta_1,\theta_2)}\,\Psi^{(\alpha)}([\vec{x}];\theta_1,\theta_2)$$

$$\Psi^{(\alpha)}([\vec{x}];\theta_1+2\pi,\theta_2) = e^{if_1(\theta_1,\theta_2)}\,\Psi^{(\alpha)}([\vec{x}];\theta_1,\theta_2)$$

(This is reminiscent of the Wu-Yang construction of the w.f.'s of charged particles in the presence of a Dirac magnetic monopole.)

Q: Suppose that we have $\Psi^{(\alpha)}(\vec{\theta})$ at some BC $\vec{\theta}$. Can we determine unambiguously $\Psi^{(\alpha)}(\vec{\theta}')$ for some other $\vec{\theta}'$? NO unless $\sigma_{xy}=0$!

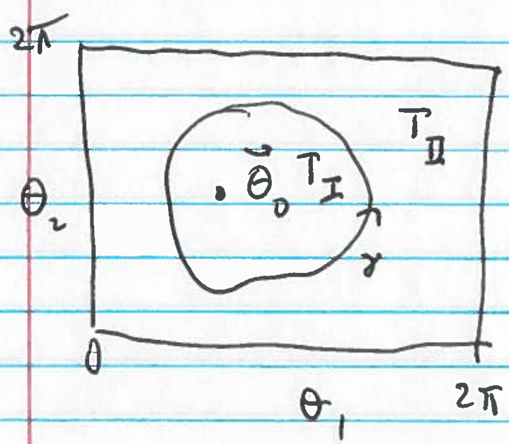
$\Rightarrow$ $\Psi^{(\alpha)}(\vec{\theta})$ must be defined on patches of the 2-torus.

Consider first the case in which $\Psi^{(\alpha)}(\vec{\theta})$ has a single isolated zero at ~~some~~ some point ~~point~~ $\vec{\theta}_0$ of the 2-torus

Let us split $S_1 \times S_1$ into two disjoint regions T_I and T_{II} s.t. $\vec{\theta}_0$ is in T_I

Since T_{II} does not have a zero (i.e. $\vec{\theta}_0$ is not on T_{II}) $\Rightarrow$ we can choose the phase of $\Psi^{(\alpha)}(\vec{\theta})$ on T_{II} to be constant, e.g. zero.
(i.e. we can make $\Psi^{(\alpha)}(\vec{\theta})$ to be real on T_{II})

But on T_I, there is a point where $\Psi^{(\alpha)}(\vec{\theta}_0) = 0$



$\Rightarrow$ we can define the phase of $\Psi^{(\alpha)}(\vec{\theta}_0)$ to be some value and extend analytically its value over some neighborhood of $\vec{\theta}_0$

$\Rightarrow$ we have two definitions of the phase of $\Psi^{(\alpha)}$ on T_I and on T_{II} $\Rightarrow$ they must be related by a gauge transformation

$$\Psi^{I}_{\alpha}([\vec{x}], \vec{\theta}) = e^{if(\vec{\theta})} \Psi^{II}_{\alpha}([\vec{x}], \vec{\theta})$$

where $f(\vec{\theta})$ is a smooth function on the closed contour γ (the common boundary of I and II

Also $\mathcal{A}_k^{I}(\vec{\theta}) - \mathcal{A}_k^{II}(\vec{\theta}) = \partial_k f(\vec{\theta})$

Now

$$\langle (\sigma_{xy})_\alpha \rangle = \frac{e^2}{4\pi^2\hbar} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \left(\partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 \right)$$

$$= \frac{e^2}{4\pi^2\hbar} \left\{ \int_{T_I} d\theta_1 d\theta_2 \left(\partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 \right) + \int_{T_{II}} d\theta_1 d\theta_2 \left(\partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 \right) \right\}$$

$$= \frac{e^2}{4\pi^2\hbar} \left[\oint_\gamma \vec{\mathcal{A}}_I \cdot d\vec{\theta} - \oint_\gamma \vec{\mathcal{A}}_{II} \cdot d\vec{\theta} \right]$$

$$= \frac{e^2}{4\pi^2\hbar} \int_\gamma \vec{\partial} f \cdot d\vec{\theta} \equiv \frac{e^2}{h} \frac{1}{2\pi} \oint_\gamma \vec{\partial} f \cdot d\vec{\theta}$$

$\Rightarrow \langle (\sigma_{xy})_\alpha \rangle$ counts # of ~~zeros~~ times $f(\vec{\theta})$ winds by 2π

$$C_1 = \frac{1}{2\pi} \oint_\gamma \vec{\partial} f \cdot d\vec{\theta} \quad \text{(winding number)}$$

is a topological invariant known as the Chern number

C_1 counts the # of zeros enclosed by γ.

Hall conductance and Fiber Bundles

At every point $\vec{\theta}$ of $S_1 \times S_1$ (i.e. for each BC)

we assign a state $\Psi_\alpha(\vec{\theta})$

But $\Psi_\alpha(\vec{\theta})$ is defined up to a ~~phase~~ phase

$\Rightarrow$ we have a ray of states.

$\Rightarrow$ for each $\vec{\theta} \in T \longrightarrow$ ray or bundle of states associated with $\Psi_\alpha(\vec{\theta})$

$S_1 \times S_1$

The 2-torus $T \simeq S_1 \times S_1$ is partitioned into a set of regions $T_I, T_{II}, \ldots$ on which Ψ_α^I, $\Psi_\alpha^{II}, \ldots,$ are smoothly defined.

These state vectors differ by gauge transf. which are smooth functions $f(\vec{\theta})$ on the overlap between adjacent regions. ~~The~~ The transition function $f(\vec{\theta})$ is a smooth map from the closed curve ~~area~~ $\gamma \subset T_I \cap T_{II} \rightarrow$ group of $U(1)$ phases $e^{i f(\vec{\theta})}$

is $\gamma \cong U(1) \Rightarrow$ we have a map $U(1) \rightarrow U(1)$

These maps can be classified ~~by~~ into homotopy classes

* Each class is defined by a winding # C_1

This map is known as the principal $U(1)$ bundle over the 2-torus T. The 1-form field

$A_k(\vec{\theta})$ defines a connection. (Berry connection)

$dA \equiv A_k \, d\theta_k$ is the 1-form.

curvature 2-form $F = dA$ (Berry curvature)

* Q: Is the Hall conductance always an integer?

NO. We made the assumption that Φ_α is unique and single-valued. If Φ_α is multivalued (i.e. if there is more than one state on the Torus) $\Rightarrow$ traversing once over $S_1 \times S_1$ will map one gnd. state to another gnd. state

We will see that this is what happens in the FQH case.

Quantized Hall conductance of a ~~B~~ non-interacting system.

Consider a system of spinless fermions in the lowest Landau level. Assume that the # of fluxes $N_\phi = N_e$ (# of electrons)

$\Rightarrow$ the LLL is full.

(I) Disk geometry

Use the circular gauge and the w.f.'s are

$$\psi_m(z) = \# \; z^m \, e^{-|z|^2/4l_0^2}$$

$$m = 0, 1, \ldots, N_\phi - 1 \quad ; \quad z = x + iy$$

The many-body w.f. of N electrons $(N = N_\phi)$ is a Slater determinant

$$\Psi_N(z_1, \ldots, z_N) = \# \begin{vmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_N \\ \vdots & & \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} e^{-\frac{1}{4l_0^2}\sum_{i=1}^{N} |z_i|^2}$$

Vandermonde determinant

$$\Psi_N(z_1, \ldots, z_N) = \# \prod_{1 \leq j < k \leq N} (z_j - z_k) \, e^{-\frac{1}{4l_0^2}\sum_{i=1}^{N} |z_i|^2}$$

which is antisymmetric