Torus

It is more convenient to use the Landau (axial) gauge: \( A_1 = -B x_2 \), \( A_2 = 0 \),

\( \tau = x_1 + i x_2 \)

Lowest Landau level w.f.c.:

\( \Psi(x_1, x_2) = f(z) e^{-\frac{z^2}{2L_2^2}} \)

Generalized PBC's:

\[
\begin{align*}
  f(z + l_1) &= e^{i \Theta_1} f(z) \\
  f(z + il_2) &= e^{i \Theta_2 - i \pi N \phi \left( \frac{2z}{L_2} + \frac{1}{2} \right)} f(z)
\end{align*}
\]

\( \tau = \frac{i L_2}{l_1} \) is the modular parameter of the torus

\[-\frac{1}{2}(l_1 + il_2) \quad \frac{1}{2}(l_1 + il_2) \quad \frac{1}{2}(l_1 - il_2) \]

\( \frac{1}{2}(l_1 - il_2) \)

\( f(z) \) has zeros inside the rectangle

\[
\oint dz \frac{f'(z)}{f(z)} = N \phi \Rightarrow \text{total phase change around } \gamma \text{ is } 2\pi N \phi \]
If $f(z)$ is analytic inside $\gamma$ and satisfies $\text{BOF's}$, it must have $N\phi$ zeros

$$f(z) = e^{\frac{2\pi i}{L_1} \phi} \prod_{j=1}^{N\phi} \Theta_j \left( \frac{z - e^{2\pi i j}}{L_1} \right)$$

where $\Theta_j(u|\omega)$ is the first odd elliptic theta function

$$\Theta_j(u|\omega) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{i \pi (n^2 - n \frac{1}{2} \omega + \omega (2n-1) u)}$$

$k \in \mathbb{R}$, $0 \leq |k| \leq \pi \sqrt{\frac{N\phi}{L_1} \frac{L_2}{L_1}}$

$\Rightarrow f(z)$ are parametrized by $k$ and by the zeros $z_j$:

$$e^{T_1} = e^{kL_1} (-1)^{N\phi}$$

$$e^{T_2} = e^{-kL_2 + \pi i \frac{Z_0}{L_1}}$$

$$k = \frac{\Theta_1 + \pi N\phi}{L_1}, \quad Z_0 = \Theta_2 \frac{L_1}{\pi} - ik \frac{L_1 L_2}{\pi}$$

The locations of the zeros $f(z)$ are determined
s.t. \( f(z) \) form a complete set of orthogonal wave functions that are eigenstates of the angular momenta translations.

E.g. choose \( \varepsilon_{j+1} = \varepsilon_j + \frac{\hbar}{N} \) \( \Rightarrow \) the dimension of the Hilbert space is \( N^2 \).

N-particle wavefunction:

\[
\Phi_{\text{CM}}(z_1, \ldots, z_N) = \Phi_{\text{CM}}(z) \prod_{1 \leq i < k \leq N} f(z_k - z_i) \times e^{-\sum_{j=1}^{N} \frac{(x_j^2)}{2\hbar^2}}
\]

Center of mass.

\[
z = \sum_{j=1}^{N} \varepsilon_j
\]

Only the \( \Phi_{\text{CM}} \) is sensitive to the translation.

\[
f(z_{j+1}) = f(z)
f(z_{j+\frac{1}{2}}) = f(z) e^{i\frac{\pi}{L} \left( \frac{z_{j+1} - z_j}{\hbar} \right)}
\]

\[
\Phi_{\text{CM}}(z_{j+1}) = e^{i\theta_1} (-1)^{N-1} \Phi_{\text{CM}}(z)
\]

\[
\Phi_{\text{CM}}(z_{j+\frac{1}{2}}) = e^{i\theta_2} (-1)^{N-1} e^{i\frac{\pi}{L} \left( \frac{z_{j+1} - z_j}{\hbar} \right)} \Phi_{\text{CM}}(z)
\]

\[
f(z_j - z_k) = f_\nu \left( \frac{z_j - z_k}{\hbar} \right)
\]
\[ \Psi_{\text{CM}}(z) = e^{\frac{i}{\hbar} 2 \gamma_1 \frac{(z - z_0)}{L_1}} \]

\[ \begin{align*}
\text{s.t.} & \quad e^{\frac{i}{\hbar} \beta_1} = (-1)^N e^{i\theta_1} \\
& \quad e^{2\pi \frac{z_0}{L_1}} = (-1)^N e^{i\theta_2 + \beta_2} \\
\text{unique solution:} & \quad k = \frac{\pi N + \theta_1}{L_1} \\
& \quad \beta_0 = L_1 \left( \frac{\theta_2 + \frac{N}{2}}{2\pi} \right) - i h \gamma \left( \frac{N}{2} + \frac{\theta_1}{2\pi} \right) \\
\end{align*} \]

\( \Rightarrow \) The wave function of a filled LL on the torus is unique.

Note that \( \theta_1 \) and \( \theta_2 \) affect only the CM factor.

Hall conductance

\[ \langle \sigma_{xy} \rangle = \frac{e^2}{i \hbar} \int \frac{d\theta_j}{2\pi} \langle \Psi_N | \frac{\partial}{\partial \theta_j} | \Psi_P \rangle \]

\[ = \frac{e^2}{i \hbar} \int \frac{d\theta_j}{2\pi} \int_0^{L_1} dx_2 \int_0^{L_2} \frac{dx_1}{2\pi} \text{ln} \left( \frac{\gamma}{\text{cm}}(z, \theta) \right) \]

\( \Rightarrow \) if \( \Psi_{\text{cm}}(z, \theta) \) is an entire function with only one zero
\[ \oint d\theta \ln \frac{\Theta_{CM}(z, \theta)}{\theta} = 2\pi \alpha \]

\[ \Rightarrow \langle \sigma_{xy} \rangle = \frac{e^2}{h} \times 1 \]

Clem number

Quantized Hall conductance of the Hofstadter bands

Let's return to the problem of the states of charged particles on a square lattice in a magnetic field \( B \) with flux \( \Theta = 2\pi n/g \) on each plaquette. We want to first find the eigenspectra of these particles, which form bands, and then to determine the Hall conductance of each filled band. There are \( g \) bands and each band has \( L/2/g \) states (integer!).

This task can be done numerically (or by approx. methods). However, the computation of \( \sigma_{xy} \) is simplified since it is (as we will see) related to a topological invariant.
\( \Rightarrow \) we can compute the states in some limited but \( \mathbb{R}^2 \) will be exact.

The analysis here is simpler than in the continuum. The main effect of \( B \) is to generate a subsidiary structure (i.e., the magnetic unit cells) \( \Rightarrow \) we can use PBC's.

This is true since \( B \) enters through \( e^{iA_\psi(r)} \), which is invariant under \( 2\pi L_k(r) \).

The eigenvectors are

The Hofstadter states are eigenvectors of \( \hat{T}_1 \) and \( \hat{T}_2 \) (discrete magnetic translation) \( \Rightarrow \) in units of the magnetic unit cell \( H \) is periodic. \( \Rightarrow \) it is consistent to apply PBC's in real space.

However, the W.F.'s are not globally well defined in the momentum space torus, \( \mathbb{R}^2 \), the magnetic \( B \) is

\[-\pi < k_1 < \pi , \quad -\pi/2 < k_2 < \pi/2\]

We will follow the work of TKNN (Thouless, Kohmoto, den Nijs, Nightingale, PRL 1982) and of

The current operator for the lattice model

\[ \hat{J}_k \left( \hat{A} \right) = \frac{\delta H}{\delta A_k \left( \hat{A} \right)} \]  

\( H \) is arbitrary (generally interacting). We will assume that \( A_k \left( \hat{x} \right) \) enters only through the KE term.

\[ H_{KE} = \int \frac{d^3k}{(2\pi)^3} \frac{c^+ (\hat{k}) h_{KE} (\hat{k}, \hat{k}') c(\hat{k}')} {\sqrt{\eta_{\text{GB}}}} \]

\[ \text{hermitian.} \]

\( \text{MBZ: magnetic Brillouin Zone.} \)

In the presence of an electric field \( \hat{E} \), \( \hat{A} \) is shifted by \( \hat{E} t \)

\[ h_{KE} (\hat{k}, \hat{k}'; \hat{E}) = h_{KE} \left( \frac{\hat{k} + e \hat{E} t}{\hbar c}, \frac{\hat{k}' + e \hat{E} t}{\hbar c} \right) \]

\( \hat{E} \) is equivalent to a shift of the momentum of each particle by \( \frac{e}{\hbar c} \hat{E} t \) (this is also the same as a twist \( \hat{\theta} = \frac{e}{\hbar c} \hat{E} t \)).

The Kubo formula for the Hall conductivity can be written as

\[ (\tau_{xy})_k = -i \hbar \frac{1}{2} \frac{\delta}{\delta A_k} \frac{\delta}{\delta A_y} \]
For a non-interacting system this expression is a sum over occupied states \( |n\rangle \) with \( E_n < E_F \)

\[
\hat{Q}_{xy}(\alpha) = \frac{e^2}{h} \sum_{\nu \in \text{occupied}} \frac{\partial^2}{\partial \epsilon_{k_\nu} \partial \epsilon_{k_\nu}} \langle n | \hat{\sigma} | m \rangle
\]

The one-particle states \( |n\rangle \) are labelled by a band index \( r \) (\( r = 1, \ldots, B - 1 \)) and by \( \vec{k} \in \text{the unit MBZ} \).

\( \Psi_r(\vec{k}) \) are eigenstates / \( \Psi_{r+1}(\vec{k}) = \Psi_r(\vec{k}) \)

Let us define a formal perturbation in powers of a parameter \( \lambda \) (eq. 1)

\[
\text{Hopping Eqs.}
\]

\[
\text{hopping}
\]

\[
\lambda \hat{T} \left[ e^{i \vec{k}_1 \cdot \vec{r}_{k+1}} \Psi_{r+1}(\vec{k}_1, \vec{k}) + e^{-i \vec{k}_1 \cdot \vec{r}_{k-1}} \Psi_{r-1}(\vec{k}_1, \vec{k}) \right]
\]

\[
- \lambda \hat{T} \cos(\vec{k}_2 + 2\pi \vec{r}_r) \Psi_r(\vec{k}_1, \vec{k}_2) = E(\vec{k}_1, \vec{k}_2) \Psi_r(\vec{k}_1, \vec{k}_2)
\]

\( \Rightarrow \) \( F + \hat{G} \) linearly independent solutions \( \{ \Psi_r(\vec{k}) \} \)

(\( F = 1, \ldots, B \)) and each solution has e.v. \( E_j(\vec{k}) \)

(Landau-Hofstadter bands)
Suppose that $E_p$ there are exactly $p$ filled Landau-Hofstadter bands. This is the state $|0\rangle$

due the Hall conductance $C_{xy}$ is a sum over the contributions of the filled bands,

$$
C_{xy} = \frac{e^2}{\hbar} \sum_{n=1}^{\infty} \frac{1}{2\pi} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \sum_{p=1}^{\infty} E_j \partial_j \psi_p^*_n (k) \partial_j \psi_p^*_n (k)
$$

define the Berry connection for each band

\begin{align*}
A_j^B (k) &= \sum_{p=1}^{\infty} \frac{1}{2\pi} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \sum_{p=1}^{\infty} E_j \partial_j \psi_p^*_n (k) \partial_j \psi_p^*_n (k)
\end{align*}

Berry curvature

$$
E_j \partial_j A_{\theta n}^B (k) = \sum_{p=1}^{\infty} E_j \partial_j \psi_p^*_n (k) \partial_j \psi_p^*_n (k)
$$

$$
C_{xy} = \frac{e^2}{\hbar} \int \frac{d^2k}{(2\pi)^2} \ E_j \partial_j A_{\theta n}^B (k)
$$

which is the Chern number (the winding number of the wave function as he traces over the MBZ)

Chern number of the
The Chern number of the $n$-th occupied band: $I_n$

$$I_n = \frac{1}{\pi} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{\varepsilon_{\mathbf{k}+\mathbf{q}}^2} \delta_{\mathbf{p}, \mathbf{k}^0} \partial_{\varepsilon_{\mathbf{k}}^0} \frac{\partial \varepsilon_{\mathbf{k}}^0}{\partial k_1} \partial_{\varepsilon_{\mathbf{k}+\mathbf{q}}^0} \frac{\partial \varepsilon_{\mathbf{k}+\mathbf{q}}^0}{\partial k_2}$$

is a topological invariant of the $n$-th band.

$\Rightarrow I_n \neq 0 \Leftrightarrow$ the states are not globally well defined over the MBZ.

How to compute the Chern numbers

Consider the Landau-Harper eqn in the limit $\lambda \to \infty$ (quasi 1D limit). We will not here do an expansion in powers of $\lambda$ (to the lowest $\neq 0$ order).

At $\lambda = 0$, $E^{(0)}_p(k^0) = \delta_{p,n}$

with $E^{(0)}_n(k^0) = 2t \cos(k_x + 2\pi \frac{n}{\xi})$

which are generally non-degenerate.

There are band crossings at $(k_x, 0)$ and $(k_x, \frac{\pi}{\xi})$

e.g. the $n=1$ and $n=2$ bands cross at $k_x = \frac{\pi}{\xi}$,

the $n=2$ band crosses the $n=3$ band at $(k_x, 0)$, etc.

$\Rightarrow$ for $n$ even, the $n$th and the $n-1$th band cross at