II Torus

It is more convenient to use the Landau (axial) gauge: $A_1 = -Bx_2, \quad A_2 = 0$

$z = x_1 + i x_2$

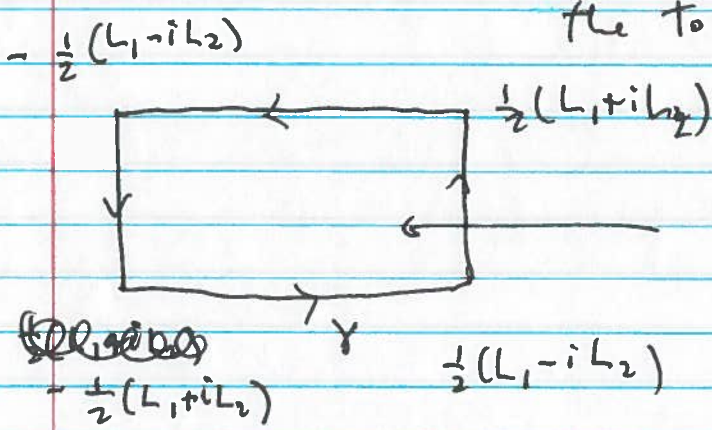
Lowest Landau level w.f.'s: $\psi(x_1, x_2) = \underbrace{f(z)}_{\text{analytic}} e^{-x_2^2/2l_0^2}$

Generalized PBC's

$$f(z+L_1) = e^{i\theta_1} f(z)$$

$$f(z+iL_2) = e^{i\theta_2 - i\pi N_\phi \left(\frac{2z}{L_1} + \tau\right)} f(z)$$

$\tau = iL_2/L_1$ is the modular parameter of the torus



$f(z)$ has N_ϕ zeros inside the rectangle

$$\oint_\gamma dz \, \frac{f'(z)}{f(z)} = N_\phi \Rightarrow \text{total phase change around } \gamma \text{ is } 2\pi N_\phi$$

If $f(z)$ is analytic inside γ and satisfies OBPC's $\Rightarrow$ it must have N_ϕ zeros

$$f(z) = e^{ikz} \prod_{j=1}^{N_\phi} \vartheta_1\left(\frac{z - \overset{\text{zeros}}{z_j}}{L_1} \Big| \tau\right)$$

where $\vartheta_1(u|\tau)$ is the first odd elliptic theta function

$$\vartheta_1(u|\tau) = i \sum_{n=-\infty}^{+\infty} (-1)^n e^{i\pi\tau\left(n-\frac{1}{2}\right)^2 + i\pi(2n-1)u}$$

$k \in \mathbb{R}$ $\qquad 0 \leq |k| \leq \pi N_\phi \frac{L_2}{L_1^2}$

$\Rightarrow$ $f(z)$ are parametrized by k and by the zeros/

$$e^{i\theta_1} = e^{ikL_1}(-1)^{N_\phi}$$

$$e^{i\theta_2} = e^{-kL_2 + i\pi\frac{z_0}{L_1}}$$

$$k = \frac{\theta_1 + \pi N_\phi}{L_1}, \quad z_0 = \theta_2 \frac{L_1}{\pi} - ik\frac{L_1 L_2}{\pi}$$

The locations of the zeros $\{z_j\}$ are determined s.t. $f(z)$ form a complete set of orthogonal wf's that are eigenstates of the magnetic translations.

e.g. choose $z_{j+1} = z_j + \frac{L_1}{N_\phi}$ $\Rightarrow$ the dimension of the H.S. is N_ϕ ✓

<u>N-particle wf:</u>

$$\Psi(z_1,\dots,z_N) = \# \, \Psi_{CM}(z) \prod_{1\le j<k\le N} f(z_j - z_k) \times e^{-\sum_{j=1}^{N} \frac{(x_2^j)^2}{2\ell_0^2}}$$

center of mass

$$z = \sum_{j=1}^{N} z_j$$

Only the Ψ_{CM} is sensitive to the translations.

$$\Rightarrow f(z+L_1) = f(z)$$

$$f(z+iL_2) = f(z)\, e^{i\pi\left(\frac{2z}{L_1}+\tau\right)}$$

$$\Psi_{CM}(z+L_1) = e^{i\theta_1}(-1)^{N-1}\Psi_{CM}(z)$$

$$\Psi_{CM}(z+iL_2) = e^{i\theta_2}(-1)^{N-1} e^{-i\pi\left(\frac{2z}{L_1}+\tau\right)}\Psi_{CM}(z)$$

$$\Rightarrow f(z_j - z_k) = \vartheta_1\left(\frac{z_j - z_k}{L_1}\Big|\tau\right)$$

$$\Psi_{CM}(z) = e^{ikz}\,\vartheta_1\!\left(\frac{z-z_0}{L_1}\middle|\tau\right)$$

s.t.

$$e^{ikL_1} = (-1)^N e^{i\theta_1}$$

$$e^{i2\pi\frac{z_0}{L_1}} = (-1)^N e^{i\theta_2 + kL_2}$$

Unique solution: $\quad k = \frac{\pi N}{L_1} + \frac{\theta_1}{L_1}$

$$z_0 = L_1\left(\frac{\theta_2}{2\pi} + \frac{N}{2}\right) - iL_2\left(\frac{N}{2} + \frac{\theta_1}{2\pi}\right)$$

$\Rightarrow$ The wave function of a filled LL on the torus is unique.

Note that θ_1 and θ_2 affect only the CM factor.

Hall conductance

$$\langle \sigma_{xy}\rangle = \frac{e^2}{i\hbar}\oint \frac{d\theta_j}{2\pi}\,\langle \Psi_N | \frac{\partial}{\partial\theta_j} | \Psi_N\rangle$$

$$= \frac{e^2}{i\hbar}\oint \frac{d\theta_j}{2\pi}\int_0^{L_1} dx_1 \int_0^{L_2} dx_2\, |\Psi_N|^2 \frac{\partial}{\partial\theta_j} \ln\left(\Psi_{CM}(z,\vec{\theta})\right)$$

Since $\Psi_{CM}(z,\vec{\theta})$ is an entire function with only one zero $\Rightarrow$

$$\oint d\theta_j \frac{\partial}{\partial \theta_j} \ln \Psi_{CM}(z, \vec{\theta}) = 2\pi i$$

$$\Rightarrow \langle \sigma_{xy} \rangle = \frac{e^2}{h} \times 1$$

Chern ~~number~~ number

Quantized Hall conductance of the Hofstadter bands

Let's return to the problem of the states of charged particles on a square lattice in a magnetic field B with flux $\Phi = 2\pi\, p/q$ on each plaquette. We want to first find the single particle states, which form bands, and then to determine the Hall conductance of each filled band. There are q bands and each band has $L_1 L_2 / q$ states (integer!)

This task can be done numerically (or by approx. methods). However, the computation of σ_{xy} is simplified ~~to~~ since it is (as we will see) related to a topological invariant.

$\Rightarrow$ we can compute the states in some ~~limited~~

but σ_{xy} will be exact.

The analysis here is simpler than in the continuum since the main effect of B is to generate a sublattice structure (i.e. the magnetic unit cells) $\Rightarrow$ we can use PBC's

This is true since B enters through $e^{iA_j(\vec{r})}$

which is invariant under $2\pi \ell_j(r)$.

~~The eigenstates are~~

The Hofstadter states are eigenstates of $\hat{T}_1$ and $\hat{T}_2^{\,q}$

(discrete magnetic translations) $\Rightarrow$ in units of the magnetic unit cell H is periodic. $\Rightarrow$ it is consistent to impose PBC's in real space.

However the w.f.'s are not globally well defined on the momentum space torus, ~~of~~ the magnetic BZ

$-\pi \leq k_1 \leq \pi$, $-\frac{\pi}{q} \leq k_2 \leq \frac{\pi}{q}$

We will follow the work of TKNN (Thouless, Kohmoto, den Nijs, Nightingale, PRL 1982) and of

Kohmoto (Ann. Phys. 1985).

The current operator for the lattice model

is $\hat{J}_k(\vec{r}) = \dfrac{\delta H}{\delta A_k(\vec{r})}$ link $(\vec{r}, \vec{r}+\hat{e}_k)$

H is <u>arbitrary</u> (generally interacting). We will assume that $A_k(\vec{r})$ enters only through the KE term.

$$H_{KE} = \int_{MBZ} \frac{d^2k}{(2\pi)^2} \int \frac{d^2k'}{(2\pi)^2}\, c^\dagger(\vec{k})\, h_{KE}(\vec{k},\vec{k}')\, c(\vec{k}')$$

hermitian.

MBZ: magnetic BZ

In the presence of an electric field $\Rightarrow \vec{A}$ is shifted by $\vec{E}t$

$$\Rightarrow h_{KE}(\vec{k},\vec{k}';\vec{E}) \equiv h_{KE}\left(\vec{k} + \frac{e}{\hbar c}\vec{E}t,\ \vec{k}' + \frac{e}{\hbar c}\vec{E}t\right)$$

$\Rightarrow \vec{E}$ is equivalent to a shift of the momentum of each particle by $\frac{e}{\hbar c}\vec{E}t$ (this is also the same as a twist $\vec{\theta} = \frac{e}{\hbar c}\vec{E}t$)

The Kubo formula for the Hall conductivity can be written as $(\sigma_{xy})_\alpha = -i\hbar L_1 L_2 \dfrac{\delta}{\delta A_j} \langle \alpha | \dfrac{\delta}{\delta A_j} | \alpha \rangle$

For a non-interacting system this expression is a sum over occupied states $|n\rangle$ with $E_n < E_F$

$$(\sigma_{xy})_\alpha = \frac{e^2}{\hbar} \sum_{\{n\}} \varepsilon_{jl} \frac{\partial}{\partial k_j} \langle n| \frac{\partial}{\partial k_l} |n\rangle$$

occupied

The one-particle states $|n\rangle$ are labelled by ~~the~~ band index r ($r = 1, \dots, q-1$) and by $\vec{k}$ in the ~~reduced~~ MBZ

$\psi_r(\vec{k})$ are eigenstates / $\psi_{r+q}(\vec{k}) = \psi_r(\vec{k})$

Let us define a formal pert. expansion in powers of a parameter λ ($\lambda \to 1$)

Harper Eqn.

hopping amplitude

$$-\lambda t \left[e^{ik_1} \psi_{r+1}(k_1, k_2) + e^{-ik_1} \psi_{r-1}(k_1, k_2) \right]$$

$$-2t \cos\left(k_2 + 2\pi \frac{p}{q} r\right) \psi_r(k_1, k_2) = E(k_1, k_2)\, \psi_r(k_1, k_2)$$

$\Rightarrow \exists$ q linearly indep. solutions $\{\psi_r^{(j)}(\vec{k})\}$ ($j = 1, \dots, q$) and each solution has e.v. $E_j(\vec{k})$

(Landau-Hofstadter bands)

Suppose that E_F / there are exactly r filled Landau-Hofstadter bands. ⇒ this is the state $|\alpha\rangle$

⇒ the Hall conductance $(\sigma_{xy})_\alpha$ is a sum over the contrib.'s of the r filled bands.

$$(\sigma_{xy})_\alpha = \frac{e^2}{i\hbar} \sum_{n=1}^{r} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi/q}^{\pi/q} \frac{dk_2}{2\pi} \sum_{p=1}^{q} \varepsilon_{j\ell}\, \partial_j \psi_p^{(n)*}(\vec{k})\, \partial_\ell \psi_p^{(n)}(\vec{k})$$

⇒ define the Berry connection for each band on the MBZ

$$A_j^{(n)}(\vec{k}) = \sum_{p=1}^{q} \psi_p^{(n)*}(\vec{k}) (-i) \partial_{k_j} \psi_p^{(n)}(\vec{k})$$

⇒ Berry curvature

$$\varepsilon_{j\ell}\, \partial_{k_j} A_\ell^{(n)}(\vec{k}) = \sum_{p=1}^{q} \varepsilon_{j\ell}\, \partial_j \psi_p^{(n)*}(\vec{k})\, \partial_\ell \psi_p^{(n)}(\vec{k})$$

$$\Rightarrow (\sigma_{xy})_\alpha = \frac{e^2}{\hbar} \int_{MBZ} \frac{d^2k}{(2\pi)^2}\, \varepsilon_{j\ell} \frac{\partial}{\partial k_j} A_\ell^{(n)}(\vec{k})$$

which is the Chern number (the winding # of the w.f. as $\vec{k}$ traces over the MBZ)

~~Chern number of the~~

The Chern # of the n-th occupied band: I_n

$$I_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_1 \int_{-\pi/q}^{\pi/q} dk_2 \; \varepsilon_{j\ell} \sum_{p=1}^{q} \frac{\partial}{\partial k_j} \psi_p^{(n)*}(\vec{k}) \frac{\partial}{\partial k_\ell} \psi_p^{(n)}(\vec{k})$$

is a topological invariant of the n-th band.

$\Rightarrow$ $I_n \neq 0 \Leftrightarrow$ the states are not globally well defined over the MBZ.

How to compute the Chern numbers

Consider the Landau-Harper eqn in the limit $\lambda \to 0$ (quasi 1D limit!). We will do an expansion in powers of λ (to the lowest $\neq 0$ order).

At $\lambda = 0$ $\quad \psi_p^{(n)}(\vec{k}) = \delta_{pn}$

with e.v.'s $\quad E_n^{(0)}(\vec{k}) = -2t \cos\left(k_2 + 2\pi \frac{p}{q} n\right)$

which are generally non-degenerate.

There are band crossings at $(k_1, 0)$ and $(k_1, \frac{\pi}{q})$

e.g. the $n=1$ and $n=2$ bands cross at $k_2 = \pi/q$,

the $n=2$ band crosses the $n=3$ band at $(k_1, 0)$, etc.

$\Rightarrow$ for n even the nth and the n-1 th band cross at