L19 10/25/2022

$$\gamma_0 = -\sigma_2, \quad \gamma_1 = -i\sigma_1, \quad \gamma_2 = -i\sigma_3$$

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (2+1 \text{ dimensions})$$

~~However~~ The Dirac Hamiltonian (1 particle)

$$h = \alpha_1 p_1 + \alpha_2 p_2 + \beta m \qquad (v_F = 1)$$

$$\alpha_1 = \sigma_3, \quad \alpha_2 = -\sigma_1, \quad \beta = -\sigma_2$$

$\Rightarrow$ <u>time-reversal is broken</u> (masses with <u>same</u> sign

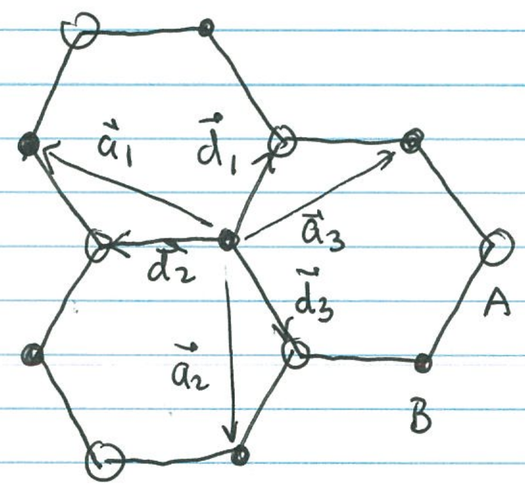
Alternatively, if we had instead site potentials $\pm\varepsilon$

$+\varepsilon$ for the ~~even~~ even-even sublattice (and odd-odd)

$-\varepsilon$ for the even-odd (and odd-even) sublattice.

we would also have two spinors but will masses of <u>opposite</u> sign. $\Rightarrow$ time reversal is equivalent ~~the~~ to an exchange of the two Dirac spinors $\Rightarrow$ T reversal is <u>not</u> broken.

II Graphene (Semenoff '84, Haldane '88, Castro Neto et al, 2009)



$$\vec d_1 = \left(\frac{1}{2\sqrt3}, \frac{1}{2}\right), \quad \vec d_2 = \left(-\frac{1}{\sqrt3}, 0\right)$$

$$\vec d_3 = \left(\frac{1}{2\sqrt3}, -\frac{1}{2}\right)$$

$$\vec a_1 = \vec d_2 - \vec d_3$$

$$\vec a_2 = \vec d_3 - \vec d_1$$

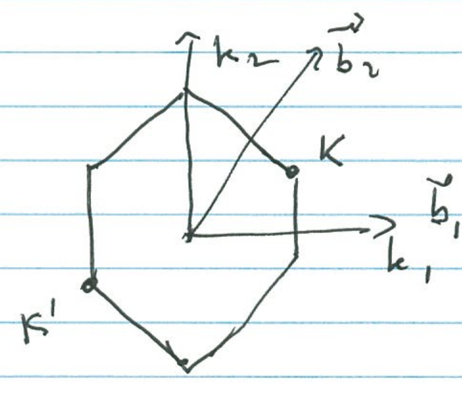
$$\vec a_3 = \vec d_1 - \vec d_2$$

On A sites $\quad \psi(\vec r_A)$

On B sites $\quad \chi(\vec r_B)$

$$H_0 = t_1 \sum_{\vec r_A, i=1,2,3} \left(\psi^\dagger(\vec r_A)\, \chi(\vec r_A + \vec d_i) + h.c.\right)$$

BZ = is a hexagon



$$\psi(\vec{r}_A) = \int_{BZ} \frac{d^2k}{(2\pi)^2} \, \psi(\vec{k}) \, e^{i\vec{k}\cdot\vec{r}_A}$$

$$\chi(\vec{r}_B) = \int_{BZ} \frac{d^2k}{(2\pi)^2} \, \chi(\vec{k}) \, e^{i\vec{k}\cdot\vec{r}_B}$$

$$H_0 = \int_{BZ} \frac{d^2k}{(2\pi)^2} \left(\psi^\dagger(\vec{k}), \chi^\dagger(\vec{k})\right) \begin{pmatrix} 0 & t_1 \sum_{i=1,2,3} e^{i\vec{k}\cdot\vec{d}_i} \\ t_1 \sum_{i=1,2,3} e^{-i\vec{k}\cdot\vec{d}_i} & 0 \end{pmatrix} \begin{pmatrix} \psi(\vec{k}) \\ \chi(\vec{k}) \end{pmatrix}$$

$$\Rightarrow \quad E_\pm(\vec{k}) = \pm t_1 \left(\left| e^{i\vec{k}\cdot\vec{d}_1} + e^{i\vec{k}\cdot\vec{d}_2} + e^{i\vec{k}\cdot\vec{d}_3} \right|^2 \right)^{1/2}$$

Zeros at the K and K' points

$$\vec{q}_K \cdot \vec{d}_2 = -\frac{2\pi}{3}, \quad \vec{q}_{K'} = -\vec{q}_K$$

$$\vec{q}_K \cdot \vec{d}_1 = \frac{2\pi}{3}, \quad \vec{q}_K \cdot \vec{d}_2 = -\frac{2\pi}{3}, \quad \vec{q}_K \cdot \vec{d}_3 = 0$$

Near K and K' (semi-metal)

$$\Psi_1(\vec{q}) = \begin{pmatrix} e^{-i\pi/6} \, \psi_K(\vec{q}) \\ e^{i\pi/6} \, \chi_K(\vec{q}) \end{pmatrix}$$

$$\Psi_2(\vec{q}) = \begin{pmatrix} e^{-i\pi/6} \, \psi_{K'}(\vec{q}) \\ e^{i\pi/6} \, \chi_{K'}(\vec{q}) \end{pmatrix}$$

where $\psi_K(\vec{q}) = \psi(\vec{q}_K + \vec{q})$ etc.

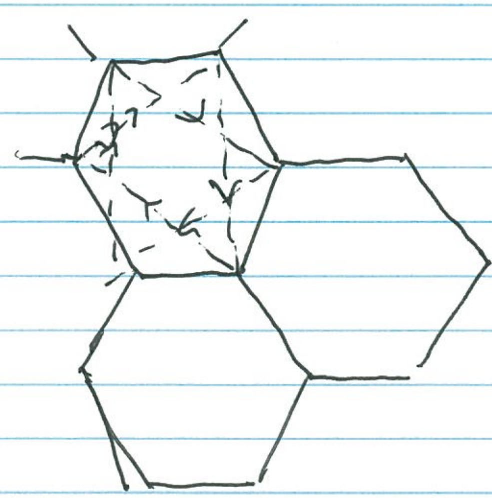
$$\Rightarrow H_0 = \int \frac{d^2 q}{(2\pi)^2} \sum_{a=1,2} \psi_a^\dagger(\vec{q})\, v_F\, (\sigma_1 q_1 + \sigma_2 q_2)\, \psi_a(\vec{q})$$

$$\equiv \int d^2 x \sum_{a=1,2} \psi_a^\dagger(x)\, \overset{v_F}{\downarrow}\, (i\sigma_1 \partial_1 + i\sigma_2 \partial_2)\, \psi_a(\vec{x})$$

$v_F = \frac{\sqrt{3}}{2} t_1$

Haldane's Model (add $t_2 e^{\pm i\phi}$) (staggered flux)

site potential $\pm \varepsilon$ (A and B)



Now $H_0 = \int_{BZ} \frac{d^2 k}{(2\pi)^2} (\psi^\dagger(k), \chi^\dagger(k)) \underbrace{\{\ \ \}}_{\mathcal{H}(\vec{k})} \begin{pmatrix} \psi(k) \\ \chi(k) \end{pmatrix}$

$\mathcal{H}(\vec{k}) = h_0(\vec{k})\, \mathbb{1} + \vec{h}(\vec{k}) \cdot \vec{\sigma}$

$$\Rightarrow E_{\pm}(\vec{k}) = h_0(\vec{k}) \pm \|\vec{h}(\vec{k})\|$$

$$\|\vec{h}(\vec{k})\| = (h(\vec{k}) \cdot h(\vec{k}))^{1/2}$$

$$h_0(k) = 2t_2 \cos\phi \sum_{i=1}^{3} \cos(\vec{k} \cdot \vec{a}_i)$$

$$h_1(\vec{k}) = t_1 \sum_{i=1}^{3} \cos(\vec{k} \cdot \vec{d}_i)$$

$$h_2(\vec{k}) = t_1 \sum_{i=1}^{3} \sin(\vec{k} \cdot \vec{d}_i)$$

$$h_3(\vec{k}) = \varepsilon + 2t \sin\phi \sum_{i=1}^{3} \sin(\vec{k} \cdot \vec{a}_i)$$

At low energies (i.e. close to K and K') $\Rightarrow$

$\Rightarrow$ ~~$H_0 = \int d^2k$~~

bispinor

$$H_0 = \int d^2x \sum_{a=1,2} \psi_a^{\dagger}(x) \left(i\alpha_1 \partial_1 + i\alpha_2 \partial_2 + m_a \beta\right) \psi_a(x)$$

$$\alpha_1 = \sigma_2, \quad \alpha_2 = \sigma_2, \quad \beta = \sigma_3$$

$$\gamma_0 = \beta = \sigma_3, \quad \gamma_1 = \beta\alpha_1 = i\sigma_2, \quad \gamma_2 = \beta\alpha_2 = -i\sigma_1$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

<u>Action</u> $$S = \int d^2x \sum_{a=1,2} \overline{\psi}_a(x) \left(i\gamma^\mu \partial_\mu - m_a\right) \psi_a(x)$$

Time Reversal: Θ anti-linear $\mathcal{T}$

$$\Theta\,\psi(x,y,t) = -\gamma_1\,\psi(-x,-y,t) = -i\sigma_2\,\psi(-x,-y,t)$$

Parity

$$\mathcal{P}\,\psi(x,y,t) = i\gamma_2\,\psi(x,-y,t) = \sigma_1\,\psi(x,-y,t)$$

$$\Theta\,\mathcal{H}(\vec{p};m)\,\Theta^{-1} = -i\sigma_2\,h^*(-\vec{p},m) = p_1\sigma_1 + p_2\sigma_2 - m\sigma_3$$
$$= \mathcal{H}(\vec{p},-m)$$

$$\mathcal{P}\,\mathcal{H}(\vec{p},m)\,\mathcal{P}^{-1} = \sigma_1(p_1\sigma_1 + p_2\sigma_2 + m\sigma_3) = p_1\sigma_1 + p_2\sigma_2 - m\sigma_3$$

$\mathcal{P}$ is equivalent to $\mathcal{T}$

two masses:

$$m_1 = \frac{3}{2}\frac{t_2}{t_1}\sin\phi - \frac{2}{\sqrt{3}}\frac{\varepsilon}{t_1}$$

$$m_2 = \frac{3}{2}\frac{t_2}{t_1}\sin\phi + \frac{2}{\sqrt{3}}\frac{\varepsilon}{t_1}$$

If $\varepsilon = 0 \Rightarrow m_1 = m_2$ (and same sign)

If $\phi = 0 \Rightarrow m_1 = -m_2$ (general)

In general $m_1 \neq m_2$ (sign included)

Quantization of the Anomalous QH effect

The following arguments apply to both models.

Consider applying a weak electromagnetic field $A_\mu(x)$. We will compute the response in two different (but consistent) ways.

(I) We will work at low energies and use Feynman diagrams (only 1!) to compute the response

(II) We will work out a two-band topological invariant (using the full BZ).

In both models we have two bands (and two species of Dirac fermions)

(I) $\mathcal{L} = \bar{\Psi}_a i \not{\partial} \Psi_a - m_a \bar{\Psi}_a \Psi_a - e A_\mu \bar{\Psi}_a \gamma^\mu \Psi_a$

$a = 1, 2$; m_1 and m_2 are generally $\neq$ and can have equal or opposite signs.

$$Z[A] = \int \mathcal{D}\bar{\Psi}_a \mathcal{D}\Psi_a \exp\left(iS(\Psi_a, \bar{\Psi}_a, A)\right)$$

$$S = \int d^3x \, \mathcal{L}$$

$$Z[A] \simeq \exp\left(\frac{i}{2}\int d^3x \int d^3y\; A_\mu(x)\, \Pi^{\mu\nu}(x,y)\, A_\nu(y) + \dots\right)$$

Here it is important that A_μ is weak since this is a perturbative calculation

$J_\mu(x) = \bar{\Psi}_a \gamma_\mu \Psi_a$ (sum over a)

↑ total local current (gauge-invariant)

spinor indices

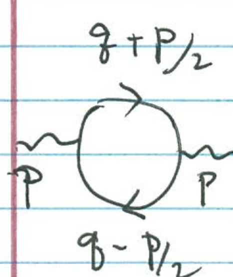
$$S_{eff}(A) = \frac{i}{2}\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sum_{a=1}^{2} \mathrm{tr}\left[S_a\left(\frac{p}{2}+q\right) \gamma^\mu S_a\left(-\frac{p}{2}, q\right) \gamma^\nu \right]$$

$A_\mu(p)\, A_\nu(-p)$

see sec.(10.4)

$S_a(p) \equiv$ Dirac Propagator

$$= \frac{1}{p_\mu \gamma^\mu - m_a}$$ (iε procedure implied)



← diagram

Both species of fermions contribute but in a subtly different way!

$$\Rightarrow \Pi_{\mu\nu}(p) = \sum_{a=1,2} \int \frac{d^3q}{(2\pi)^3}\, i\, \mathrm{tr}\left[S_a\left(q+\frac{p}{2}\right)\gamma^\mu S_a\left(q-\frac{p}{2}\right)\gamma^\nu \right]$$

$$\equiv (p^2 g^{\mu\nu} - p^\mu p^\nu)\, \Pi_0(p^2) \qquad (p^2 = p_0^2 - \vec{p}^{\,2})$$

$$- i\, \varepsilon^{\mu\nu\lambda} p_\lambda\, \Pi_A(p^2)$$

Check: $p^\mu \Pi_{\mu\nu}(p) = 0 \Rightarrow$ conserved!
(gauge-invariance)

$$\Pi_0(p^2) = -\frac{|m_1|}{4\pi p^2} + \frac{1}{8\pi\sqrt{p^2}}\left(\frac{4m_1^2}{p^2}+1\right)\sinh^{-1}\left(\frac{1}{\sqrt{\frac{4m_1^2}{p^2}-1}}\right)$$

$$+ \; m_2 \leftrightarrow m_1$$

$$\Pi_A(p^2) = -\frac{m_1}{2\pi\sqrt{p^2}}\sinh^{-1}\left(\frac{1}{\sqrt{\frac{4m_1^2}{p^2}-1}}\right) + m_1 \leftrightarrow m_2$$

If $|p^2| \ll |m_a|^2 \Rightarrow$ we can expand in powers of p^2

$$\Rightarrow \mathcal{L}^{(2)}_{\text{eff}}(A_\mu) = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\sigma_{xy}}{4}\varepsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} + \cdots$$

(this is also a gradient expansion!) Chern-Simons!

$$\frac{1}{g^2} = \frac{1}{\pi}\left(\frac{1}{|m_1|} + \frac{1}{|m_2|}\right); \quad \sigma_{xy} = \frac{1}{4\pi}\left(\mathrm{sgn}(m_1) + \mathrm{sgn}(m_2)\right)$$

(1) If $\text{sgn}(m_1) = -\text{sgn}(m_2)$

$\Rightarrow \sigma_{xy} = 0$! (Insulator)

(2) If $\text{sgn}(m_1) = +\text{sgn}(m_2)$

$\Rightarrow \sigma_{xy} = \frac{1}{2\pi}$ (in units of $\frac{e^2}{h}$)

(Anomalous QH effect)

This result is consistent with the general concept that for free lattice fermions $\sigma_{xy} = \frac{1}{2\pi} C_1$ (where C_1 is the first Chern #)

It may seem peculiar that the result depends on the sign of the mass. Alternatively we could have used two different choices of the definition of the γ-matrices (say by exchanging γ_1 and γ_2). This amounts to a change of parity $x_1 \to -x_1$, $x_2 \to x_2$ for each fermion. In other words, when

We make a choice of the def. of the Dirac γ-matrices we are choosing a definition of a ~~frame~~ frame (right or left handed). ~~If we do~~

Notice that each Dirac fermion contributes to σ_{xy} with $\frac{1}{2}\,\frac{1}{2\pi}\,\frac{e^2}{h}$

$\Rightarrow$ σ_{xy} is "$\frac{1}{2}$ quantized". Since $\sigma_{xy} = \frac{C_1}{2\pi}$

$\Rightarrow$ we must have an even # of Dirac fermions.

This is a particular case of a general theorem (due to Holger Nielsen and Masao Ninomiya ~1981) which proves that any lattice model of Dirac fermions must have an even # of species ("fermion doubling"). This is a consequence of the assumption that the fermion KE is local and periodic.

In QFT the result that $\sigma_{xy} = \frac{1}{2} \frac{\text{sgn}(m)}{2\pi} \frac{e^2}{h}$ per fermionic species is known as the <u>parity anomaly</u>. Analogs of this anomaly occur in <u>odd</u> space-time dimensions (0+1, 2+1, 4+1, ...)

II <u>Topological Invariant</u>

This is a generalization of TKNN to a two-band model. The formulation that I give here follows the work of X. L. Qi, Y. S. Wu and S. C. Zhang, 2006, ~~and~~ ~~Xg QP, the~~

As we saw, in general σ_{xy} is defined as

$$\sigma_{xy} = \lim_{\omega \to 0} \frac{i}{\omega} \lim_{\vec{Q} \to 0} \Pi_{xy}(\omega, \vec{Q})$$ (notice the order of limits!)

For a free fermion system the current correlator is