L2 8/25/2022

$\exists\ T_c$ / $F_{vortex} = 0$ Kosterlitz–Thouless

$$\frac{\cancel{2\pi}}{2}\, J \frac{|\phi_0|^2}{\cancel{2\pi}} = 2\, T_{KT}$$

$$\rightarrow \quad T_{KT} = \frac{\pi}{2}$$

$\Rightarrow$ For $T > T_{KT}$ vortices unbound and proliferate

$\Rightarrow \langle e^{i\theta} \rangle = 0$ vortex-antivortex plasma

$$\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle \approx e^{-\frac{|x-y|}{\xi}} \qquad \xi \sim e^{\#/(T-T_{KT})}$$

For $T < T_K \Rightarrow$ vortices and antivortices are bound

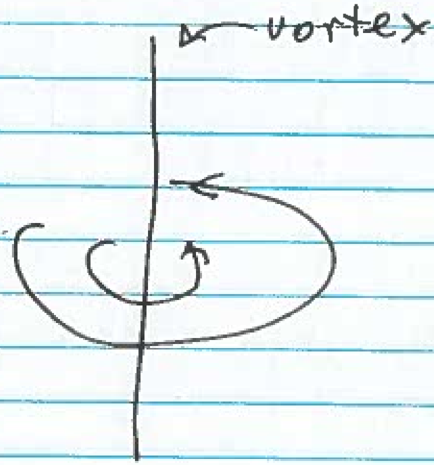
$\Rightarrow \langle e^{i\theta} \rangle = 0$ still

but $\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle \approx \left(\frac{a}{|x-y|} \right)^{T/2\pi K}$

$K = \frac{J\phi_0^2}{2a^2}$ stiffness

If $D > 2$ vortices form loops

e.g $D=3$



$$E \sim \frac{L\phi_0^2}{2\pi} \ln\left(\frac{a}{L}\right)$$

$\Rightarrow$ $T < T_c$ vortex loops are small

$T > T_c$ they are large

$T < T_c$

$$\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle = |\langle e^{i\theta} \rangle|^2 + e^{\#/R}$$

$$E \approx \frac{J\phi_0^2}{2a^2} \int d^d x \, (\partial\theta)^2$$

$\theta(x)$: Goldstone boson

$T > T_c$

$$\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle \cong e^{-R/\xi}$$

Quantum Phase Transitions

Phase transitions in a quantum system as a function of a coupling constant.

Example: Ising Model in a Transverse Field

Pauli matrices

$$H = -J \sum_{\langle r,r' \rangle} \sigma_3(\vec{r})\, \sigma_3(\vec{r}\,') - H_T \sum_{\vec{r}} \sigma_1(\vec{r})$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Rescale $g = \dfrac{J}{H_T}$

$$H = -g \underbrace{\sum_{\langle r,r' \rangle} \sigma_3(r)\, \sigma_3(r')}_{\text{"pot. energy"}} - \underbrace{\sum_{r} \sigma_1(r)}_{\text{"kinetic E."}}$$

(1) For ~~case~~ $g \ll 1 \Rightarrow$

$$H = H_0 + V$$

$$H_0 = -\sum_{r} \sigma_1(r)$$

$$V = -g \sum_{\langle r,r' \rangle} \sigma_3(r)\, \sigma_3(r')$$

$$\Rightarrow |\Psi\rangle_{GS} \cong \prod_r \otimes |+\rangle_r$$

$$\sigma_1(\vec{r})|+\rangle_{\vec{r}} = |+\rangle_r$$

$$\sigma_1(\vec{r})|-\rangle_{\vec{r}} = -|-\rangle_{\vec{r}}$$

Clearly $\langle \Psi | \sigma_3(\vec{r}) | \Psi \rangle_{GS} = 0$
GS

Since $\sigma_3 |+\rangle = |-\rangle$

and $\langle + | - \rangle = 0$

$\Rightarrow$ pert. theory has a finite radius of convergence.

$\Rightarrow$ Paramagnet.

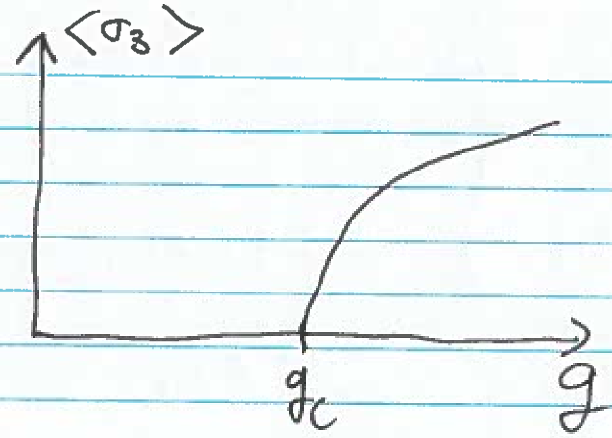
(2) $g \gg 1 \Rightarrow$ G.S. is an eigenstate of σ_3

$|\Psi\rangle_{Gs} = |\uparrow \ldots \uparrow\rangle$ or $|\downarrow \ldots \downarrow\rangle$

and $\langle \uparrow \ldots \uparrow | \sigma_3(r) | \uparrow \ldots \uparrow \rangle = +1$

$\langle \downarrow \ldots \downarrow | \sigma_3(r) | \downarrow \ldots \downarrow \rangle = -1$

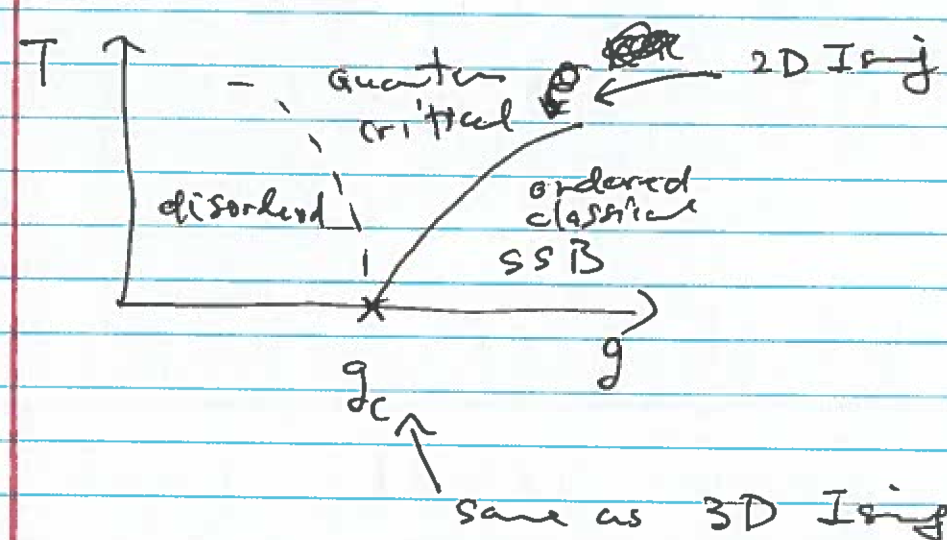
If $N \to \infty \Rightarrow$ Pert. theory for $|\uparrow\rangle$ converges and $|\downarrow\rangle$ converges but do not mix!



$$\langle \sigma_3 \rangle = \# \, |g - g_c|^{\beta}$$

Note: The Q. Ising Model in dimension d is the same as the classical model in $D = d+1$

Quantum Criticality



Symmetries

$R = \prod_r \sigma_1(r)$ global spin flip operator

$R^2 = I$, $R^{-1} = R$

$\Rightarrow \{I, R\} \equiv \mathbb{Z}_2$

$[H, R] = 0$

$R^{-1} \sigma_3(r) R = -\sigma_3(r)$ (spin flip)

$R^{-1} \sigma_1(r) R = \sigma_1(r)$

If $R|\Psi\rangle = |\Psi\rangle$ (i.e. invariant)

$\Rightarrow \langle\Psi| \sigma_3(r) |\Psi\rangle =$

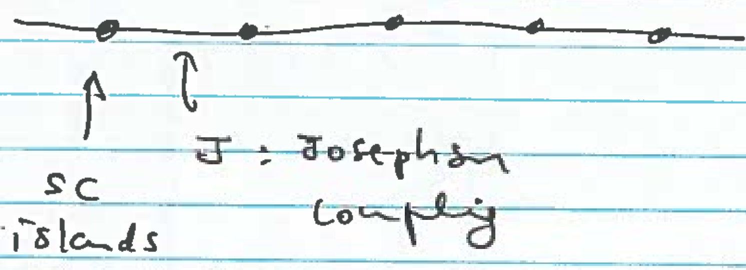
$= \langle\Psi| R^{-1} \sigma_3(r) R |\Psi\rangle$

$= -\langle\Psi| \sigma_3(r) |\Psi\rangle \Rightarrow$

$\Rightarrow \langle\Psi| \sigma_3(r) |\Psi\rangle = 0$

$\Rightarrow \langle\Psi| \sigma_3(r) |\Psi\rangle \neq$ iff $R|\Psi\rangle \neq |\Psi\rangle$

Josephson Junction array



$$H = \sum_r \frac{n^2(r)}{2C} - J \sum_{\langle r, r' \rangle} \cos\left(\theta(r) - \theta(r')\right)$$

C ← capacitance

$n(r)$ = electric charge

$n \in \mathbb{Z}$ same as angular momentum L

$$[n(r), \theta(r')] = i\,\delta_{r,r'}$$

$$e^{\pm i\theta}\,|n\rangle = |n \pm 1\rangle$$

$$g = JC$$

$$H = \sum_r \frac{L^2(r)}{2} - g \sum_{\langle r, r' \rangle} \cos\left(\theta(r) - \theta(r')\right)$$

(1) $g \ll 1$

$$|\Psi\rangle_{GS} \simeq \prod_r |0\rangle_r \qquad L(r)\,|0\rangle = 0$$

$${}_{GS}\langle\Psi| e^{i\Theta(r)} |\Psi\rangle_{GS} = 0$$

Symmetry transf.

$$R(\alpha) = e^{i\sum_r \alpha L(r)}$$

$$R(\alpha)|\theta\rangle = |\theta+\alpha\rangle$$

$$R^{-1}(\alpha) = R(-\alpha) \qquad \alpha\in(0,2\pi)$$

$$R(\alpha)\,R(\beta) = R(\alpha+\beta) \qquad U(1)$$

$$R^{-1}(\alpha)\, e^{i\Theta(r)}\, R(\alpha) = e^{i(\Theta(r)+\alpha)}$$

$\Rightarrow$ if $|\Psi\rangle$ / $R(\alpha)|\Psi\rangle = |\Psi\rangle$

$$\Rightarrow \langle\Psi| e^{i\Theta(r)} |\Psi\rangle = 0$$

$\Rightarrow$ for $g < g_c$ $R(\alpha)|\Psi\rangle = |\Psi\rangle$

for $g > g_c$ $R(\alpha)|\Psi\rangle \neq |\Psi\rangle$

and the symmetry is broken. spontaneously

$g \gg g_c \Rightarrow$ expand the cosine in powers

$$\cos(\theta(r)-\theta(r')) \cong 1 - \frac{1}{2}(\theta(r)-\theta(r'))^2 + \cdots$$

$$H \cong \sum_r \frac{L^2(r)}{2} + \frac{g}{2}\sum_r (\theta(r+1)-\theta(r))^2 + \cdots$$

$$H \cong \frac{1}{2a}\sum_r a\, L^2(r) + \frac{ga}{2}\sum_r a \left(\frac{\theta(r+1)-\theta(r)}{a}\right)^2 + \cdots$$

$$\left[\theta(r), \frac{L(r')}{a}\right] = i\,\frac{\delta_{r,r'}}{a}$$

$$\Pi(r) \equiv \frac{1}{a} L(r)$$

$$\tilde{H} \equiv \frac{H}{a} = \frac{1}{2}\sum_r a\left(\frac{L(r)}{a}\right)^2 + \frac{g}{2}\sum_r a\left(\frac{\theta(r+1)-\theta(r)}{a}\right)^2 + \cdots$$

$a \to 0$

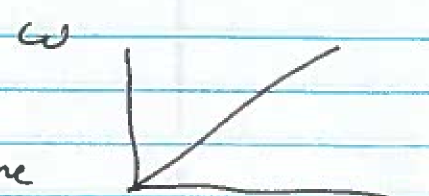
$$[\theta(x), \Pi(x')] = i\,\delta(x-x')$$

$$\frac{\tilde{H}}{\sqrt{g}} = \frac{1}{2}\int dx \left(\frac{1}{\sqrt{g}}\Pi^2(x) + \sqrt{g}\,(\partial_x \theta(x))^2\right)$$

Free field with excitations

$$\omega(k) = \sqrt{g}\,|k| \quad \text{Goldstone}$$

boson



The picture is the same for $d>1$

* Goldstone bosons $(\omega \sim |k|)$
* vortices with energy $\sim \ln \frac{L}{a}$ $(d=2)$

Correlators

$d=1$

$$\left\langle e^{\frac{i}{\sqrt{g}}\theta(x)} e^{-\frac{i}{\sqrt{g}}\theta(x')} \right\rangle_{GS} = e^{-\frac{1}{2g}\langle (\theta(x)-\theta(x'))^2 \rangle}$$

(equal-time)

$$\langle T\theta(x,t)\ \theta(x',t')\rangle_{GS} = \int \frac{d\omega\, dk}{(2\pi)^2} \frac{e^{i(k(x-x')-\omega(t-t'))}}{\omega^2 - k^2 + i\epsilon}$$

equal-time

$$\langle \theta(x,0)\ \theta(x',0)\rangle_{GS} = \int \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{|k|} \sim \frac{1}{2\pi} \ln \frac{|x-x'|}{a}$$

$$\Rightarrow \left\langle e^{i\frac{\theta(x)}{\sqrt{g}}} e^{-i\frac{\theta(x')}{\sqrt{g}}} \right\rangle \sim \frac{1}{|x-x'|^{1/2\pi g}}$$

(no long range order)

$d \geq 2$ $\quad \langle \quad \rangle \sim \text{const} + \frac{1}{|x-x'|^{\#}}$