

$$\neq T_c / F_{\text{vortex}} = 0$$

Kosterlitz-Thouless

$$\frac{2\pi J |\phi_0|^2}{2} = 2 \frac{e^2}{\epsilon} T_{KT}$$

$$\rightarrow T_{KT} = \frac{\pi}{2}$$

$\Rightarrow$  For  $T > T_{KT}$  vortices unbound and proliferate

$$\Rightarrow \langle e^{i\theta} \rangle = 0 \quad \text{vortex-antivortex plasma}$$

$$\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle \sim e^{-\frac{|x-y|}{\xi}} \quad \xi \sim e^{\frac{1}{T-T_{KT}}}$$

For  $T < T_K \Rightarrow$  vortices and antivortices are bound

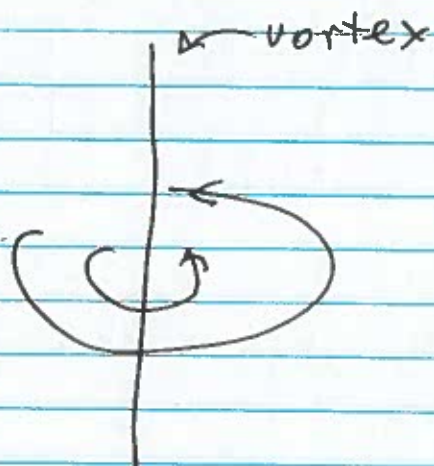
$$\Rightarrow \langle e^{i\theta} \rangle = 0 \quad \text{still}$$

$$\text{but } \langle e^{i\theta(x)} e^{-i\theta(y)} \rangle \sim \left( \frac{e}{|x-y|} \right)^{T/2\pi K}$$

$$K = \frac{J \phi_0^2}{2a^2} \quad \text{stiffness}$$

If  $D > 2$  vertices form loops

e.g.  $D=3$



$$E \sim \frac{L \phi_0^2}{2\pi} \ln\left(\frac{a}{L}\right)$$

$\Rightarrow T < T_c$  vortex loops are small

$T > T_c$  they are large

$T < T_c$

$$\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle = |\langle e^{i\theta} \rangle|^2 + e^{-R/\xi}$$

$$E \approx \frac{J \phi_0^2}{2a^2} \int d^d x (\partial \theta)^2$$

$\theta(x)$ : Goldstone boson

$T > T_c$

$$\langle e^{i\theta(x)} e^{-i\theta(y)} \rangle \approx e^{-R/\xi}$$



# Quantum Phase Transitions

Phase transitions in a quantum system as a function of a coupling constant.

Example: Ising Model in a Transverse Field

$$H = -J \sum_{\langle r, r' \rangle} \sigma_3(r) \sigma_3(r') - H_T \sum_r \sigma_1(r)$$

$\swarrow$  Pauli matrices  $\searrow$   
 $\downarrow$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Rescale  $g = \frac{J}{H_T}$

$$H \equiv -g \sum_{\langle r, r' \rangle} \sigma_3(r) \sigma_3(r') - \sum_r \sigma_1(r)$$

$\uparrow$  "pot. energy"                       $\uparrow$  "kinetic E."

① For ~~all~~  $g \ll 1 \Rightarrow$

$$H_0 = H_0 + V$$

$$H_0 = - \sum_r \sigma_1(r)$$

$$V = -g \sum_{\langle r, r' \rangle} \sigma_3(r) \sigma_3(r')$$

$$\Rightarrow |\Psi\rangle_{GS} \approx \prod_r \otimes |+\rangle_r$$

$$\sigma_1(\vec{r}) |+\rangle_{\vec{r}} = |+\rangle_{\vec{r}}$$

$$\sigma_1(\vec{r}) |-\rangle_{\vec{r}} = -|-\rangle_{\vec{r}}$$

$$\text{Clearly } \langle \Psi |_{GS} \sigma_3(\vec{r}) | \Psi \rangle_{GS} = 0$$

$$\text{Since } \sigma_3 |+\rangle = |-\rangle$$

$$\text{and } \langle + | - \rangle = 0$$

$\Rightarrow$  pert. theory has a finite radius of convergence.

$\Rightarrow$  Para magnet.

(2)  $g \gg 1 \Rightarrow$  G.S. is an eigenstate of  $\sigma_3$

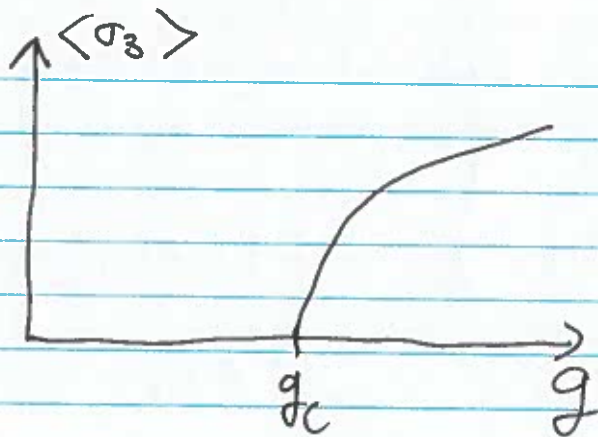
$$|\Psi\rangle_{GS} = |\uparrow \dots \uparrow\rangle \text{ or } |\downarrow \dots \downarrow\rangle$$

$$\text{and } \langle \uparrow \dots \uparrow | \sigma_3(N) | \uparrow \dots \uparrow \rangle = +1$$

$$\langle \downarrow \dots \downarrow | \sigma_3(N) | \downarrow \dots \downarrow \rangle = -1$$

If  $N \rightarrow \infty \Rightarrow$  Pert. theory for  $|\uparrow\rangle$  converges and  $|\downarrow\rangle$  converges but do not mix!

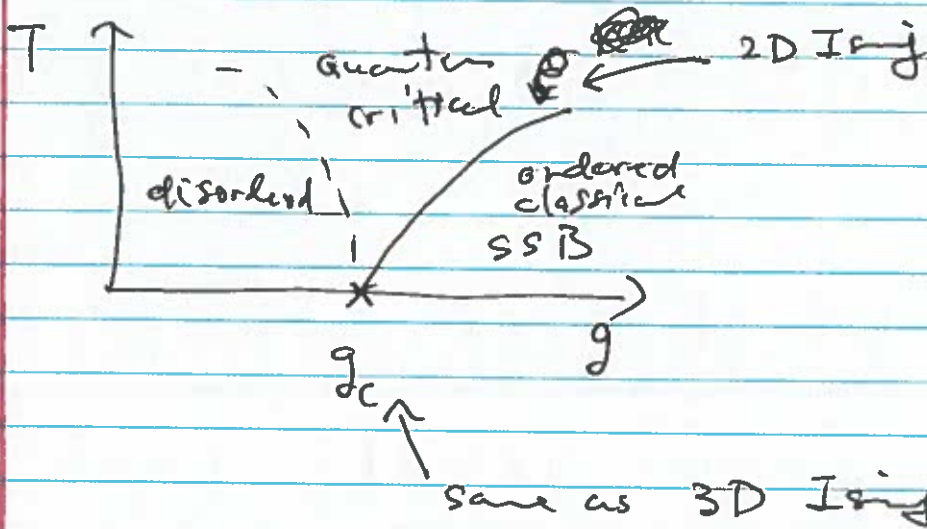




$$\langle \sigma_3 \rangle = \# |g - g_c|^\beta$$

Note : The Q. Ising Model in dimension 1d is the same as the classical model in  $D = d + 1$

Quantum Criticality



Symmetries

$R = \prod \sigma_1(r)$  ~~is~~ ~~a~~ global spin flip operator

$R^2 = I, R^{-1} = R$

$\Rightarrow \{I, R\} \cong \mathbb{Z}_2$

$[H, R] = 0$

$R^{-1} \sigma_3(r) R = -\sigma_3(r)$  (spin flip)

$R^{-1} \sigma_1(r) R = \sigma_1(r)$

If  $R|\Psi\rangle = |\bar{\Psi}\rangle$  (i.e. invariant)

$\Rightarrow \langle \bar{\Psi} | \sigma_3(r) | \Psi \rangle =$

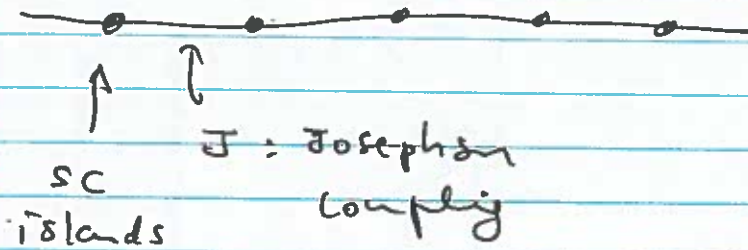
$= \langle \bar{\Psi} | R^{-1} \sigma_3(r) R | \Psi \rangle$

$= - \langle \bar{\Psi} | \sigma_3(r) | \Psi \rangle \Rightarrow$

$\Rightarrow \langle \bar{\Psi} | \sigma_3(r) | \Psi \rangle = 0$

$\Rightarrow \langle \bar{\Psi} | \sigma_3(r) | \Psi \rangle \neq 0$  iff  $R|\Psi\rangle \neq |\Psi\rangle$

# Josephson Junction array



$$H = \sum_r \frac{n(r)^2}{2C} - J \sum_{\langle r, r' \rangle} \cos(\theta(r) - \theta(r'))$$

$n(r)$  = electric charge  
 capacitance

$n \in \mathbb{Z}$  same as angular momenta  $L$

~~$\langle n | \theta | n \rangle = 0$~~

$$[n(r), \theta(r')] = i \delta_{r, r'}$$

$$e^{\pm i c \theta} |n\rangle = |n \pm 1\rangle$$

$$g = J C$$

$$H = \sum_r \frac{L^2(r)}{2} - g \sum_{\langle r, r' \rangle} \cos(\theta(r) - \theta(r'))$$

(1)  $g \ll 1$

$$|\psi\rangle_{GS} \sim \prod_r |0\rangle_r$$

$$L(r) |0\rangle = 0$$



$$\langle \Psi | e^{i\Theta(r)} | \Psi \rangle_{OS} = 0$$

Symmetry transf.

$$R(\alpha) = e^{i \sum_r \alpha L(r)}$$

$$R(\alpha) | \Theta \rangle = | \Theta + \alpha \rangle$$

$$R^{-1}(\alpha) = R(-\alpha) \quad \alpha \in (0, 2\pi)$$

$$R(\alpha) R(\beta) = R(\alpha + \beta) \quad U(1)$$

$$R^{-1}(\alpha) e^{i\Theta(r)} R(\alpha) = e^{i(\Theta(r) + \alpha)}$$

$$\Rightarrow \text{if } |\Psi\rangle / R(\alpha) |\Psi\rangle = |\Psi\rangle$$

$$\Rightarrow \langle \Psi | e^{i\Theta(r)} | \Psi \rangle = 0$$

$$\Rightarrow \text{for } g < g_c \quad R(\alpha) |\Psi\rangle = |\Psi\rangle$$

$$\text{for } g > g_c \quad R(\alpha) |\Psi\rangle \neq |\Psi\rangle$$

and the symmetry is broken spontaneously



$g \gg g_0 \Rightarrow$  expand the cosine in powers

$$\cos(\theta(r) - \theta(r')) \approx 1 - \frac{1}{2} (\theta(r) - \theta(r'))^2 + \dots$$

$$H \approx \sum_r \frac{L(r)^2}{2} + \frac{g}{2} \sum_r (\theta(r+1) - \theta(r))^2 + \dots$$

$$H \approx \frac{1}{2a} \sum_r a L(r)^2 + \frac{ga}{2} \sum_r a \left( \frac{\theta(r+1) - \theta(r)}{a} \right)^2 + \dots$$

$$[\theta(r), \frac{L(r')}{a}] = i \frac{\delta_{r,r'}}{a}$$

$$\tilde{\Pi}(r) \equiv \frac{1}{a} L(r)$$

$$\tilde{H} \equiv \frac{H}{a} = \frac{1}{2} \sum_r a \left( \frac{L(r)}{a} \right)^2 + \frac{g}{2} \sum_r a \left( \frac{\theta(r+1) - \theta(r)}{a} \right)^2 + \dots$$

$a \rightarrow 0$

$$[\theta(x), \tilde{\Pi}(x')] = i \delta(x-x')$$

$$\frac{\tilde{H}}{\sqrt{g}} = \frac{1}{2} \int dx \left( \frac{1}{\sqrt{g}} \tilde{\Pi}^2 + \sqrt{g} (\partial_x \theta)^2 \right)$$

Free field with excitations  $\omega$

$$\omega(k) = \sqrt{g} |k| \text{ Goldstone boson}$$

