\[ \nu_F = \frac{\alpha}{\sqrt{D t}} \]

\( M \): mass of CH group (heavy) \( ("\text{adhesively})\)

\( \Delta(x) \) is classical \( \Rightarrow M \text{-T} \)

\[ \Delta_0 = \frac{2 \Lambda \nu_F}{g} e^{-\pi \nu_F/g^2} \Lambda \approx \frac{\pi}{a_0} \]

Anti-adhesive limit: \( M \to 0 \)

\[ \lambda = \bar{\psi}_0(x) i \gamma^\mu \partial_\mu \psi_0(x) + g^2 (\bar{\psi}_0(x) \psi_0(x))^2 \]

(\( N=2 \) Gross-Neveu model)

\[ \beta_{\text{RG}}: \beta(g) = \alpha_0 \frac{\partial g}{\partial \alpha_0} = \frac{(N-1)}{\pi} g^2 + \ldots \]

(\( N=2 \)) \( \Rightarrow \) marginally relevant (asymptotically free)

\[ \langle \bar{\psi}_0 \psi_0 \rangle \neq 0 = g \langle \Delta \rangle \]

Here \( \rightarrow \) Soliton: domain wall of the dimerized state

\[ \Delta(x) \to \pm \Delta_0 \quad x \to \pm \infty \]

\( g \Delta(x) \) is a slowly varying mass
Single particle spectrum:

\[ H = -i \sigma_1 \frac{\partial}{\partial x} + m(x) \sigma_3 = \begin{pmatrix} m(x) & -i \frac{\partial}{\partial x} \\ i \frac{\partial}{\partial x} & -m(x) \end{pmatrix} \]

\[ m(x) = \rho G \Delta(x) \]

\[ \psi_0(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{-s \rho \int_0^x m(x') dx'} \end{pmatrix} \]

(there is also a non-normalizable state)

\[ \Rightarrow \text{positive } E \text{ states } \psi_p^+(x) \] and

\[ \text{negative } E \text{ states } \psi_p^-(x) + \text{ zero mode} \]

\[ \psi_p^+(x) \leftrightarrow \psi_p^-(x) \text{ under } CP \]

\[ +E \leftrightarrow -E \]

\[ E = \sqrt{p^2 + m^2} \]

Completeness:

\[ \int \frac{dp}{2\pi} \left[ \psi_p^*(x) \psi_p(x) + \psi_p^*(x) \psi_p(x) \right] + \psi_0^*(x) \psi_0(x) = \delta(x-y) \]
and

\[ \psi(x) = \alpha \psi_0 + \sum_p (b_p \ u_p(x) + d_p^+ \ \phi_p(x)) \]

\( \text{fermion} \)

\( \text{zero mode} \)

\( \{ a, a^+ \} = \{ a^+, a^+ \} = 0, \quad \{ a, a^+ \} = 1 \)

\[ Q_\pm = -\frac{e}{2} \left\{ \pm \int dx \ \psi_0^*(x) \ \psi_0(x) \right\} \]

\[ + \int dx \int dp \ \frac{1}{2m} \left[ \psi_p^*(x) \ u_p(x) - u_p^*(x) \ \psi_p(x) \right] \]

\[ = \mp \frac{e}{2} \]

\( \text{Soliton charge} = -\frac{e}{2} \]

\( \text{antisoliton} = +\frac{e}{2} \]

\[ \text{Spectral Asymmetry:} \quad \phi = \int dE \ \left( \rho_s(E) - \rho_0(E) \right) \]

\( -\alpha \rightarrow +\alpha \)

\[ \text{completeness} \Rightarrow = -\frac{1}{2} \int dE \ \left( \rho_s(E) - \rho_s(-E) \right) \]

\( -\infty \rightarrow +\infty \)

\[ = -\frac{1}{2} \text{ (zero mode!)} \]
Goldstone–Wilczek argument

\[ \mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + g \bar{\psi} \left( \phi_1 + i \gamma_5 \phi_2 \right) \psi \]

If \( \phi_1, \phi_2 \) are constant

\[ H = -i \sigma \partial_x + g \phi_1 \sigma_3 + g \phi_2 \sigma_2 \quad (\text{CP is broken}) \]

\[ \phi_1 = |\phi_1| \cos \theta, \quad \phi_2 = |\phi_1| \sin \theta \]

\[ \mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + g |\phi_1| \bar{\psi} e^{i \theta} \sigma_5 \psi \]

\[ g |\phi_1| = g \sqrt{\phi_1^2 + \phi_2^2} \]

\( \phi_1 \): disroriztion, \( \phi_2 \): CDW or \( \phi^4 \).

Coupling be a gauge field \( A_\mu \rightarrow -e \bar{\gamma}_5 \gamma^\mu A_\mu \)

= induced current \((\text{Goldstone, Wilczek, 1981})\)

\[ \langle \delta_{\mu} \rangle = -\frac{i}{2\pi} \epsilon_{\mu \nu} \epsilon^{a b} \phi_a \partial_{\nu} \phi_b / |\phi_1|^2 \]

\[ |\phi|^2 = \phi_1^2 + \phi_2^2, \quad \theta(x) = \tan^{-1}(\phi_2/\phi_1) \]
Adiabatic Change

\[ \phi_1 = \text{constant and } \phi_2 = 0 \text{ at } t = 0 \]

\[ \Rightarrow \text{ slowly turn } \phi_2 \text{ on until we have a solution} \]

\[ \Rightarrow \text{ Charge } = Q = -e \int_{-\infty}^{+\infty} dx \langle \psi |\phi_2(x) \rangle = \]

\[ = \frac{e}{2\pi} \int_{-\infty}^{+\infty} dx \Theta(\Theta(x) - \Theta(-\infty)) \]

\[ = -e \frac{\Delta \Theta}{2\pi} \]

Fermion mass; \( m = \frac{\phi_1}{g} \), twist \( \phi_2 \)

\[ \phi_2(-\infty) = -\phi_0 \text{ and } \phi_2(+\infty) = +\phi_0 \]

\[ \Rightarrow Q = \frac{1}{\pi} \tan^{-1} \left( \frac{g \phi_0}{m} \right) (-e) \]

If \( m \rightarrow 0 \Rightarrow Q \rightarrow \frac{1}{2} (-e) \)

Can also be derived using bosonization.
Edge States in the A QiH State

We saw that the A QiH state can be described in terms of two Dirac fermions: 

- a "light mass" fermion and 
- a "heavy mass" fermion (the "doubler").

The A QiH state corresponds to the case where 

\[ \sigma_0 = \sigma_M \] \quad (C = 1) \]

\[ \text{sgn}(m) = \text{sgn}(M) \] and the trivial state where 

\[ \text{sgn}(m) = -\text{sgn}(MM) \].

A domain wall involves having \( m(x) \) varying slowly from \( m = \lim_{x \to \infty} m(x) \) to \( -m = \lim_{x \to \infty} m(x) \).

<table>
<thead>
<tr>
<th>( M &lt; 0 )</th>
<th>( M &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>unbroken TRS</td>
<td>broken TRS</td>
</tr>
<tr>
<td>( \sigma_{xy} = 2 )</td>
<td>( \sigma_{xy} = e^2 / 1 )</td>
</tr>
<tr>
<td>( M &gt; 0 )</td>
<td>( M &gt; 0 )</td>
</tr>
</tbody>
</table>

\[
L = \chi \left( \chi - m(x) \right) \bar{\psi} \gamma \psi + \bar{\chi} \left( \chi^2 - M \right) \chi
\]

\( \chi \) light, \( \bar{\chi} \) heavy.
The one-particle Hamiltonian of the light fermion is

$$H = -\frac{i}{\hbar} \sigma^i \partial_x - \frac{i}{\hbar} \sigma^j \partial_y + m(x) \sigma^3$$

which is Hermitian and complex.

See eigenstates $\psi(x, y) = e^{i p_y x} \left( \frac{u_p(x)}{v_p(x)} \right)$

$\Rightarrow$ propagating massive modes

+ modes on the wall. ("zero modes")

**Zero modes:**

$$\psi(x, y) = e^{i p_y x} \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \end{array} \right) e^{- \text{sgn}(m) \int_{-\infty}^{x} m(x') dx'}$$

$$E(p_y) = \text{sgn}(m) v_F p_y \quad \text{(just as in the IHM!)}$$

Ledge = $\psi(y, t) \left( i \partial_t - v_F \text{sgn}(m) i \partial_y \right) \psi(y, t)$

Chiral fermion $m > 0$

(anti-chiral for $m < 0$)
We saw that

\[ S_{\text{eff}} (A) = \frac{(N_+ - N_-)}{4\pi} \int d^4x \ \varepsilon_{\mu
u\lambda\sigma} A^\mu \partial^\nu A^\lambda \]

\( N_+ = \) number of fermions with topological charge \( \pm \frac{1}{e} \)

For \( x < 0 \) \( N_+ = N_- \Rightarrow S_{\text{eff}} = 0 \)

\( x > 0 \) \( N_+ = 2, N_- = 0 \)

\( \text{or} \ N_+ = N_- + 2 \)

\( \Rightarrow S_{\text{eff}} = \int_0^{2\pi} d^3x \ \varepsilon_{\mu
u\lambda} \partial^\nu A^\lambda \) \( (x < 0) \)

\( \Rightarrow S_{\text{eff}} = 0 \) \( (x > 0) \)

Under a gauge transform \( A_\mu \rightarrow A_\mu + \partial_\mu \phi \)

\[ S_{\text{eff}} (A + \partial_\mu \phi) - S_{\text{eff}} (A) = \]

\[ = - \frac{1}{\pi} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_0 \Phi (k_x = 0, k_0 \neq 0) \varepsilon_{\mu
u\lambda} k_0 A^\lambda \]

\( \Rightarrow \) gauge anomaly

\[ \langle J_\mu \rangle = \frac{\delta S_{\text{eff}}}{\delta A_\mu (x)} = \frac{1}{\pi} \varepsilon_{\mu
u\lambda} \partial^\nu A^\lambda \]
Say that $A_z$ is $\leq 1$, $B=0$ and $E_z=E$ (up to the wall).

$\Rightarrow$ current in the bulk $\rightarrow$ towards the wall along the $x_z$ axis.

On the wall the fermion zero mode contributes current on the wall!

\[ \text{current on the wall} \]

\[ \text{current} \]

\[ \text{TR I} \quad \text{A GH} \quad \text{wall} \]

Ledge ($A_z$) = $\psi(y, t) \exp(i E_z v t \sin(\omega m y + e A_y^{\text{wall}} y)) \psi^{\dagger}(x, t)$
Edge states of QSH

It is the same as two $A\Phi H$ states with ($\uparrow$ and $\downarrow$ $\sigma-$) with opposite chirality.

Same analysis $\Rightarrow$ no bulk charge current.

Edge "helical" current

$H_{\text{edge}} = R^+ p(x) (-i\nabla - 2\sigma) R_p(x) \left( R_{\uparrow}(x) (-i\nabla - 2\sigma) L_{\uparrow}(x) \right)$

$\Rightarrow$ Right movers are chiral

and left movers are anti-chiral

mass term? $R^+ p \downarrow L_{\downarrow} + \downarrow L_{\downarrow} R_p$ breaks $S_2$

In the absence of Zeeman $\Rightarrow$ the edge states are protected. Each edge carries

a conductivity of $\frac{e^2}{h} \Rightarrow \frac{2e^2}{h}$

$|\uparrow, p\rangle \leftrightarrow |\downarrow, -p\rangle$ Kramers pairs
3D $\mathbb{Z}_2$ TI Topological Insulators

We will now look at a 3D $\mathbb{Z}_2$ TI with a domain wall to a trivial state $(x_1, x_2, x_3)$

One-particle $H$ is

$$H = -i \mathbf{\alpha} \cdot \mathbf{\nabla} + m (x_3) \beta = H_1 + H_3$$

$$H_1 = -i \mathbf{\alpha}_1 \partial_1 - i \mathbf{\alpha}_2 \partial_2$$

$$H_3 = -i \mathbf{\alpha}_3 \partial_3 + m (x_3) \beta$$

$$m (x_3) = -m \quad (m > 0)$$

$x_3 \to \pm \infty$

(doubles are fixed)

$\psi_0^{\pm}$ is an eigenstate of $H_3 = \beta \otimes \mathbf{\alpha}_3 \quad (\gamma_3^{+} = \gamma_3^{-})$

$H_3 \psi_0^{\pm} = m \psi_0^{\pm}$

$
\Rightarrow -i \partial_3 \psi_0^{\pm} + m \psi_0^{\pm} = 0$

$$\Rightarrow \mathbf{\nabla} \psi_0^{\pm} - i (\mathbf{\alpha}_3 \cdot \nabla) \psi_0^{\pm} = 0$$

$\Rightarrow \psi_0^{\pm} = \mathbf{\nabla} (x_0, x_1, x_2) f (x_3)$
\[\pm x_2 f_\pm(x_3) = -m(x_3) \sum_{\text{all } \Psi} f_\pm(x_3)\]

\[\Rightarrow f_\pm(x_3) = f(0) e^{+ \int_0^{x_3} dx_3 m(x_3)}\]

since \(m(x_3) < 0\) for \(x_3 > 0\) and \(m(x_3) = -m\) for \(x_3 < 0\).

\[\Rightarrow\] we must choose the normalizable

solution \(f_\pm(x_3)\).

Also \(\gamma_\pm(x_0, x_1, x_2, x_3)\) must be an

up-spinor of \(\gamma\) with e.v. \(-i\).

\(\Rightarrow\) \(\gamma_++\) is a superposition of \(>0\) energy

states with \(\uparrow\) spin (und. bound) and \(<0\)

energy states with \(\downarrow\) spin, and satisfies

\[i \gamma_0 \partial_0 \psi^+ = \gamma \cdot \nabla \psi^+ = 0 \Rightarrow \text{massless }\]

(\text{Dirac in } 2+1!)

It is easier to understand these edge states

in a basis in which \(R_3\) is diagonal.
In this basis \( \gamma_0 = -1 \otimes \tau_2, \gamma_5 = 1 \otimes \tau_1 \)
and \( \gamma = i \sigma_0 \otimes \tau_3 \). The subspace of spinors
in which \( \gamma_3 = +i \) is spanned
by \((1,0,0,0)\) and \((0,0,0,1)\); the
subspace with \( \gamma_3 = -i \) is spanned by \((0,1,0,0)\)
and \((0,0,0,1)\).

In the subspace \( \gamma_3 = +i \), \( \alpha_1 = \gamma_0 \sigma_1 \) and
\( \alpha_2 = \gamma_0 \gamma_2 \) become \( \alpha_1 = \sigma_1 \) and \( \alpha_2 = -\sigma_2 \).
In the subspace with \( \gamma_3 = -i \), \( \alpha_1 = \sigma_1 \)
and \( \alpha_2 = +\sigma_2 \). In both subspaces, the
eff. \( H \) is that of a Weyl fermion in \( 2+1 \)
dimension:

\[
H_{2D} = -i \alpha_1 \partial_1 - i \alpha_2 \partial_2
\]

The states have energy \( E(\vec{p}) = \pm |\vec{p}| \).

The two subspaces are related by parity.

\( \therefore \gamma_3 = +i \) has positive chirality
-1 negative.
there are states bound to the wall with wave functions

$$\psi_0^+(x_0, x_1, x_2, x_3) = \eta_+^* \left( x_0, x_1, x_2 \right) e^{-i \int_0^{x_3} \left[ m(x_3) \right] \, dx_3}$$

where $$x_3 \eta_+ = i \eta_+$$ and obeys the eqn.

$$\left( i \partial_\perp \right) \eta_+ = 0 \Rightarrow 2D \text{ massless Weyl fermion}.$$ 

At the opposite surface it also has fermionic two-component massless fermions with opposite chirality.

**External Magnetic Field**

Now

$$H_1 = \left[ -i \epsilon \left( \nabla - i \frac{e A}{\hbar c} \right) \right] \perp - g B \Sigma_3$$

$$H_3 = - \frac{i}{2} \Sigma_3 \Sigma_3 + m(x_3) \beta$$

$$\Sigma_3 = \text{diag} \left( \frac{1}{2}, \frac{1}{2} \right)$$

$$\Sigma_3$$ plays the role of a mass term in the edge Weyl fermion, $$m = -g B \Sigma_3$$.
\[ \left[ \Sigma_3, r_3 \right] = 0 \]

In the subspace of the normalizable zero modes of \( H_3 \) (which we called \( Y_0^+ \)) the eff. Hamiltonian in 2+1 dimensions is:

\[ H_{2D} = \alpha \cdot (\dot{r} + eA) - g B r_3 \]

\[ \alpha_1 = 0, \quad \alpha_2 = \frac{e}{c}, \quad (\sin \varphi, i \gamma_5 \gamma_0 = \pm \gamma_0^+) \]

\( r_\perp \) are tangent to the wall.

\( B \perp \) wall opens a gap

\( B \parallel \) wall \( \Rightarrow \) shift \( r \rightarrow r' \) (twist of \( \text{BC's} \))

Relativistc Landau Levels

\[ E_{n,\sigma} = \pm (\ell \pi + 1) B - \sigma B + m^2 \]

\[ E_0 = m \quad n = 0, \quad \sigma = \pm 1 \]

\( N_\phi \)-fold degeneratet

\( n = 0, \sigma = \uparrow \) empty if \( m > 0 \)

\( \text{full if } m < 0 \)

\( \Rightarrow \) charge and spin accumulation at the wall.

How much?

\( H_3 \) is the same as the \( H \) for a 1D soliton.

\( \Rightarrow \) induced charge

\[ Q = \frac{1}{2} \text{sgn}(m) N_\phi \frac{e^2}{4\pi} \frac{L}{B} \sqrt{m} \]
Where does this charge come from?

⇒ This extra charge is compensated by an equal and opposite charge at the opposite surface.

where the fermions have \( \gamma_3 = -i \) (anti-wall)

⇒ Charge polarization ⇔ Topological magneto-electric effect.

suppose we have an \( \vec{E} \) field \( \parallel \) wall

⇒ We expect the charge \((\text{its.com})\) to move with \( v = \frac{c |\vec{E}|}{|\vec{B}|} \)

\[ J_i = \sigma_{xy} E_{ij} E_j \]

\[ \sigma_{xy} = \frac{1}{4\pi} \frac{e^2}{\hbar} \text{ sqm/(m)} \]

\[ = \left( \frac{1}{2} \right) \frac{e^2}{\hbar} \text{ sqm/(m)} \]

\[ \frac{e}{\hbar} \text{ note} \]

\[ \frac{e}{\hbar} \text{ note} \]

⇒ No net current \( \Rightarrow \) spin current

\( \text{charge} \)

\( \text{parity anomaly on each surface that is cancelled as a whole} \)

⇒ anomaly in flow
Equivalently, in these systems there is an anomalous Hall effect at the surface with
\[ \sigma_{xy} = \frac{1}{2} \frac{e^2}{h} (N_+ - N_-) \]
where \( N_\pm \) is the # of Weyl fermions with \( \pm \) chirality at the surface.

**Bulk Effective Action**

Let \( \mathbb{R} \times \mathbb{R} \) be the bulk of the system (\( \mathbb{R} \) being time) and \( \partial \mathbb{R}^+ \times \mathbb{R} \) the surface (top)
(\( \partial \mathbb{R}^- \times \mathbb{R} \) for the bottom surface)

\[ S_{\text{eff}}(A) = \int_{\mathbb{R} \times \mathbb{R}} d^3 x \, \frac{1}{4\pi} \text{sgn}(m) \varepsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda \]

\[ - \int_{\partial \mathbb{R}^+ \times \mathbb{R}} d^3 x \, \frac{1}{4\pi} \text{sgn}(m) \varepsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda \]

\[ = \int_{\mathbb{R} \times \mathbb{R}} d^4 x \, \frac{\Theta}{8\pi} \varepsilon^{\mu \nu \lambda \sigma} \partial_\mu A_\nu \partial_\lambda A_\sigma \]

**total derivative!**

\[ \Theta = \text{Re} \Theta \text{sgn}(m) \pi \]
Pontryagin Index
\[ Q = \int d^4x \frac{1}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \]

\( S_4 \) manifold without boundaries \( \approx S^1 \)-sphere

\( Q \) classifies the maps \( S_4 \to S_4 \)

\( \Theta \): theta angle (axion field)

Time-Reversal Invariance \( \Rightarrow \) \( \Theta = 0, \pi \) (mod \( 2\pi \))

In pple the effective action can also have a bulk Maxwell term

\[ S_{\text{eff}} = \int d^4x \left( -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\Theta}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \]

\[ = \int d^4x \left[ -\frac{1}{2g^2} \left( \nabla^2 \tilde{E} - \tilde{B}^2 \right) + \frac{\Theta}{8\pi^2} \tilde{E} \cdot \tilde{B} \right] \]

Speed of light inside

The insulator
The Callan-Harvey Effect

These results can also be derived from a theory of Dirac fermions in 3+1 dimensions.

(Callan and Harvey, 1985) (generalization of Goldstone and Wilets)

\[ L = \bar{\psi} i \gamma^\mu \gamma^4 + g \phi_1 \bar{\psi} \gamma^4 + g \phi_2 \bar{\psi} \gamma^4 \]

\((\phi_1, \phi_2)\) is a 2-component scalar field

\[ L = \bar{\psi} i \gamma^4 + g |\phi_1|^2 e^{i\phi_2} \gamma^4 \]

\(\phi_1 + i \phi_2 = 1 \phi_1 e^{i\phi_2} \) axion

Induced current:

\[ \langle J_\mu \rangle = -i \frac{\epsilon_{\mu \nu \lambda \rho}}{16 \pi^2} \langle \phi^* \gamma^\nu \phi - \phi^* \gamma^\nu \phi \rangle F_{\lambda \rho} \]

(axial anomaly)

\[ \Rightarrow \langle J_\mu \rangle = \frac{\epsilon_{\mu \nu \lambda \rho}}{8 \pi^2} \epsilon_{\nu \lambda \rho} \theta F_{\lambda \rho} \]

Domain wall: \( g \phi_1 = m \) and \( \phi_2 \to \frac{1}{2} \phi_0 \) \( x_3 \to \pm \infty \)

Chiral angle exchange \( \Delta \theta = \theta(x_3 \to +\infty) - \theta(x_3 \to -\infty) \)

\[ = -2 \tan^{-1} \left( \frac{2 \phi_0}{m} \right) \]
\[ Q = e \int d^3 x \langle \mathbf{j}_0 \rangle = \frac{e^2}{\hbar c} \frac{\Delta \theta}{2\pi} \frac{B_3 L^2}{2\pi} \]

\[ \text{TIR} \Rightarrow m \to 0 \Rightarrow \Delta \theta \to -\pi \]

\[ \Rightarrow \lim_{m \to 0} Q = -\frac{e^2 B_3 L^2}{\hbar c} \text{ accumulated charge} \]

\[ \text{on the wall} \]