The Fractional Quantum Hall Effect (FQH)

The FQH effect is found in 2DEGs in samples of very high purity. Nowadays this is done in GaAs – AlAs heterostructures which confine the 2DEG into a bound state located some 500 Å from the sample surface (and where the donor ions reside). The e^- mobility in these samples is as high as 35×10^6 cm^2/Vs and the nominal mean free path is up to 0.5 mm! The electron densities are low and their mean separation is large compared ~~[illegible]~~ to the lattice spacing of the host crystal $\Rightarrow$ we have a system of electrons ~~[illegible]~~ on an approximately uniform background! In the regime in which the FQH is observed the magnetic fields $B \sim 10$ Tesla and the electrons fill states of <u>low</u> Landau levels.

The important length scale here is the magnetic length $l_0 = \sqrt{\frac{\hbar c}{eB}} \gg a_0$ ($l_0 \sim$ hundreds of $\mathring{A}$).

Most of the FQH states that are seen occur when the e^- fill a fraction of the single-particle states of the lowest Landau level $\Rightarrow$ the natural measure of the density is not the areal density $\rho = \frac{N}{A}$, but the filling fraction $\nu = \frac{N_e}{N_\phi}$

area

Most of the FQH states have $\nu \lesssim 1$.

~~If~~ The GaAs–AlAs samples have g factors which are large $\Rightarrow$ the electrons are fully spin polarized $\Rightarrow 2 \gtrsim \nu > 1$ typically involve states with overturned spins.

There are also samples with two 2DEGs $\Rightarrow \nu > 1$ for those samples.

Finally there are also FQH states in the N=1 Landau level. These states are <u>believed</u> to belong to a non-Abelian type of FQH state.

The FQH state is an <u>incompressible</u> state of the 2DEG in a large field B.

Incompressible means that there is a ~~gap~~ gap to all excitations. In this regime the Hall conductivity $\sigma_{xy} = \frac{\nu e^2}{h}$

where $\nu = \frac{p}{q}$ $\quad ((p,q)=1)$

Most states are at $\nu = \frac{1}{m}$ with m odd
These are the Laughlin states
There are also many states that belong to the Jain sequences

$$\nu = \frac{p}{2np \pm 1} \qquad \begin{array}{l} n = 1, 2, \ldots \\ p = 1, 2, \ldots \end{array}$$

Laughlin: $p = 1$, $m = 2n \pm 1$ (odd)

$$\hat{\rho} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix} \equiv \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$$

$$\rho_{xx} = \rho_{yy} \quad , \quad \rho_{yx} = -\rho_{xy}$$

$$\hat{\sigma} = \hat{\rho}^{-1} \Rightarrow \hat{\sigma} = \frac{1}{\rho_{xx}^2 + \rho_{xy}^2} \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$$

In the FQH, $\rho_{xx} \xrightarrow[T \to 0]{} 0 \quad (\sim e^{-W/kT})$ "gap"

and $\rho_{xy} = m \frac{h}{e^2}$ (Laughlin)

$$\rho_{xy} \gg \rho_{xx} \Rightarrow \rho_{xy} \simeq \frac{1}{\sigma_{xy}}$$

The Landau level has a degeneracy $= N_\phi$ (# flux quanta)

$\Rightarrow$ <u>any</u> interaction is ∞ strong! (KE $= 0$)

Laughlin postulated a "variational" wave function for the gnd. state

$$\Psi_m(z_1, \ldots, z_N) = \# \underbrace{\prod_{1 \le i < j \le N} (z_i - z_j)^m}_{\text{"Jastrow factor"}} e^{-\frac{1}{4\ell_0^2} \sum_{j=1}^{N} |z_j|^2}$$

$z = x + iy$

The Laughlin wave function has a large overlap with the exact w.f. for $N=8$ electrons ($\gtrsim 98\%$) including the Coulomb interactions.

To understand the physics of this state we will use the Plasma Analogy. The probability density of finding the N electrons at $z_1, \ldots, z_N$ is

$$\rho(z_1,\ldots,z_N) = |\Psi_m(z_1,\ldots,z_N)|^2 \equiv e^{-\beta U(z_1,\ldots,z_N)}$$

The norm of the state is

$$Z_{CG} = \|\Psi\|^2 = \int d^2z_1 \ldots d^2z_N \, |\Psi_m(z_1,\ldots,z_N)|^2$$
$$\equiv \int d^2z_1 \ldots d^2z_N \, e^{-\beta U(z_1,\ldots,z_N)}$$

is the same as the partition function of a gas of particles with an interaction

$$U(z_1,\ldots,z_N) = -2 \sum_{1 \le i < j \le N} \ln|z_i - z_j| + \frac{1}{2m}\sum_{i=1}^{N} |z_i|^2$$

Here I set $\beta \equiv m$ and $l_0 = 1$

This is the potential energy of a gas of N particles of charge 1 in 2D interacting with the 2D Coulomb interaction (the logarithm) and with a neutralizing background

$$V_{coul}(z_i - z_j) = -\ln|z_i - z_j|$$

$$\nabla^2\left(\frac{1}{2m}|z|^2\right) = \frac{2}{m} \Rightarrow \rho_0 = \frac{1}{2\pi m}$$

M.C. simulations find that this one-component Coulomb gas is uniform for $m \lesssim 5$

For m large $\Rightarrow$ Wigner crystal
m small $\Rightarrow$ fluid (incompressible)

$$\langle \rho(z) \rangle = \frac{1}{Z_{CG}} \int d^2z_1 \ldots d^2z_N \; \rho(z) \; |\Psi_m(z_1, \ldots, z_N)|^2$$

$$\rho(z) = \sum_{j=1}^{N} \delta(z - z_j)$$

$$U(z_1, \ldots, z_N) \equiv U[\rho(z)]$$

$$= \int d^2z \int d^2z' \; (\rho(z) - \rho_0) \, V_{CG}(z - z') \, (\rho(z') - \rho_0)$$

$$V_{CG}(z - z') = -\ln|z - z'|$$

$\Psi_m(z_1, \ldots, z_N)$ is an eigenstate of angular momentum

$$\Psi_m(z_1, \ldots, z_N) = \sum_{\underbrace{\{k_1, \ldots, k_N\}}_{\text{pos. integers}}} C_{k_1, \ldots, k_N}\, z_1^{mk_1} \cdots z_N^{mk_N} \times \underbrace{e^{-\frac{1}{4l_0^2}\sum_{j=1}^{N}|z_j|^2}}$$

with the restrictions $\sum_{j=1}^{N} k_j = \frac{1}{2}N(N-1)$

and $C_{k_1, \ldots, k_N}$ is antisymmetric under permutations

(e^- are fermions)

Under a rotation by θ, $z_j \to z_j e^{i\theta}$

$$\Rightarrow \Psi_m(e^{i\theta} z_1, \ldots, e^{i\theta} z_N) = e^{im\frac{N}{2}(N-1)}\, \Psi_m(z_1, \ldots, z_N)$$

$\Rightarrow$ the angular momentum of Ψ_m

$L_m = \frac{1}{2} mN(N-1)$

Consider now the problem of a single particle in a magnetic field B in the Landau level state $z^n e^{-|z|^2/4l_0^2}$. Suppose we add (adiabatically) an infinitesimally thin solenoid at $z_0 = 0$ with flux Φ. The eigenstate changes to $z^{n+\alpha}\, e^{-|z|^2/4\ell_0^2}$ with $\alpha = \Phi/\Phi_0$ ($\Phi_0 = \frac{hc}{e}$ the flux quantum)

if $\Phi = \Phi_0 \Rightarrow \alpha = 1 \Rightarrow z^n e^{-|z|^2/4\ell_0^2} \to z^{n+1} e^{-|z|^2/4\ell_0^2}$

The Laughlin state reacts similarly with each $z_j^{mk_j} \to z_j^{mk_j+1}$. If we ignore the change of the coefficients $C_{k_1,\dots k_N}$

$$\Rightarrow \Psi_m(z_1,\dots,z_N) \to \left(\prod_{j=1}^{N} z_j\right) \Psi_m(z_1,\dots,z_N)$$

In general if the solenoid is inserted at z_0

$$\Rightarrow \Psi_m^{(+)}(z_0; z_1,\dots,z_N) = \prod_{j=1}^{N} (z_j - z_0)\ \Psi_m(z_1,\dots,z_N)$$

What is the physical effect?

The probab. density now is

$$|\Psi_m^{(+)}(z_0; z_1,\dots,z_N)|^2 = \prod_{j=1}^{N} |z_j - z_0|^2\ |\Psi_m(z_1,\dots,z_N)|$$

The Plasma Analogy now has a potential

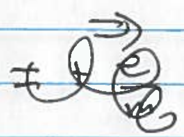
$$U(z_0; z_1, \ldots, z_N) = U(z_1, \ldots, z_N) - \frac{2}{m}\sum_{j=1}^{N} \ln|z_j - z_0|$$

This is the potential energy change of the one-comp. Coulomb gas due to the interaction with negative electric charges $\left(-\frac{1}{m}\right)$ located at z_0.

$\Rightarrow$ some charge will be expelled from near z_0.

$\Rightarrow$ energy cost ε_0

Away from $|z - z_0| \lesssim \xi$ the plasma is uniform. How much charge is expelled?



The solenoid behaves as a positive charge $+\frac{e}{m}$ (depletion of e^- charge)

The extra charge goes to the edge of the e^- droplet of radius $R \gg l_0 \Rightarrow R \to R + \delta R$ /

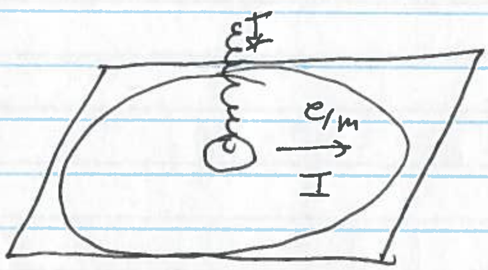
$$\left(\pi(R+\delta R)^2 - \pi R^2\right)\frac{1}{2\pi m l_0^2} = \frac{1}{m}$$

$$\frac{R}{l_0} = \sqrt{2mN} \Rightarrow \frac{\delta R}{2 l_0} = \sqrt{mN+1} - \sqrt{mN}$$

$\Rightarrow$ the highest single particle angular mom. state which enters in $\Psi_m(z_1 \cdots z_N)$ is mN

$\Rightarrow$ δR is such that the state with $mN+1$ is now occupied.

$\Rightarrow$ The ~~.~~ insertion of the solenoid is equivalent to a positively charge hole with charge $\frac{e}{m}$



On the other hand, the insertion of the flux $\phi = \phi_0$ generates a current to the edge $\Rightarrow$ ~~.~~ $\sigma_{xy} = \frac{1}{m} \frac{e^2}{h}$

$\Rightarrow$ $\Psi_m(z_1 \ldots z_N)$ exhibits the FQHE!

It also has fractionally charged quasiholes

Statistics

Suppose I now insert two ~~.~~ solenoids at locations u and w. Naively we expect

$$\Psi^{(+)}(u, w; z_1, \ldots, z_N) = N(u,w) \prod_{j=1}^{N} (z_j - u)(z_j - w) \; \Psi_m(\{z_k\})$$