There is an additional change in angular momentum by the addition \( \frac{e}{m} \) of the 2nd solenoid (but this time measured from \( w \), not \( u \)).

Also if we drag adiabatically the quasihole with charge \( \frac{e}{m} \) around the 2nd fluxord, the wave function should have a phase due to the Aharonov-Bohm effect equal to \( \frac{e}{m} \) (Kivelson & Ruckel, 1980).

How can we determine \( N(u, w) \)?

Two conditions: (1) the W.F. should be an analytic function of \( z_1, \ldots, z_N \) (lowest Landau level)

(2) It should be translation invariant (i.e. a function of differences of \( u, w, z_1, \ldots, z_N \))

The plasma analogy now is

\[
| \Phi^{\text{eff}}_m (u, w, z_1, \ldots, z_N) |^2 = e^{-V_\text{eff} (u, w; z_1, \ldots, z_N)}
\]

\[
V_\text{eff} = V(z_1, \ldots, z_N) - \frac{2}{m} \sum_{i=1}^N \ln |u - z_i| + \ln |w - z_i|
\]

\[
+ \frac{2}{m} \ln |N(u, w)|
\]
Translation invariance and analyticity are met if we choose (Halperin 1983)
\[ N(u, w) = N_0 (u-w)^{\frac{1}{m}} e^{-\frac{(|u|^2 + |w|^2)}{4\hbar^2 m}} \]

\[ \uparrow \]
branch cut!

Halperin w.f. for two quasiholes
\[ P_m^{(2)} (u, w; z_1, \ldots, z_N) = N_0 (\Theta u-w)^{\frac{1}{m}} \]
\[ \times \prod_{j=1}^{N} \frac{1}{(z_j-u)(z_j-w)} e^{-\frac{|u|^2 + |w|^2}{4\hbar^2 m}} \]
\[ \times P_m^{(2)} (z_1, \ldots, z_N) \]

\[ \Rightarrow U_{\text{eff}} = -2 \sum_{1 \leq i < k \leq N} \frac{1}{m} |z_i - z_k| \]
\[ - \frac{2}{m} \sum_{j=1}^{N} \left( \frac{1}{m} |z_j-u| + \frac{1}{m} |z_j-w| \right) \]
\[ - \frac{2}{m^2} \frac{1}{l_0^2} |u-w|^2 + \frac{1}{2m^2 \hbar^2} \sum_{j=1}^{N} |z_j|^2 \]
\[ + \frac{1}{4m^2 \hbar^2} \left( |u|^2 + |w|^2 \right) \]

\[ \Rightarrow \] N classical particles of charge 1 interact with two extra \( -\frac{1}{m} \) charged particles at \( u \) and \( w \) (and a neutralizing background)
since the Halperin WFT has a branch cut - exchanging slowly u and w causes the WFT to change by $e^{i\pi/(m, n)}$

\[ \Rightarrow \text{anyons! (fractional statistics!)} \]

Arovas, Schrieffer and Wilets (1984) used a Berry phase argument to show that

the quasiholes indeed have fractional statistics.

Jain's reinterpretation $\Rightarrow$ composite fermions

\[
\Phi_n^a(z_1, \ldots, z_N) = \prod_{j<k} \frac{1}{z_j - z_k} \exp \left\{ \sum_{j=1}^N \frac{\epsilon_j}{4\hbar^2} \right\}
\]

\[
\prod_{j<k} \frac{1}{z_j - z_k} \Phi_n^a(z_1, \ldots, z_N)
\]

\[
\uparrow \quad \text{(n-1) fluxes attached to the n-th Landau level}
\]

Jain: the Laughlin state is the same as that of a composite fermion (e^- attached to n-flux)

filling up an effective Landau level of a field $B_{eff} = B - (n-1)\Phi_0$ (partial screening)
Simple hydrodynamic picture of the FOHE (Frohlich, Tze)

I can think of the 2DEG as an irrotational fluid in a magnetic field.

The charge current $j_\mu = (j_0, j^\perp)$ must be locally conserved $\Rightarrow \partial^\mu j_\mu = 0$

$\Rightarrow j_\mu = \frac{1}{2\pi} \epsilon_{\mu
\nu} \partial^\nu A^\perp$

Invariance

$A^\perp \Rightarrow A^\perp + \partial^\perp \Phi$ does not change $j^\mu$

$\Rightarrow A^\perp$ is a gauge field

What is its eff. action?

Incompressibility $\Rightarrow$ locality

$\Rightarrow B \not\Rightarrow$ broken time reversal (and parity)

$\Rightarrow$ it must also be gauge invariant

Coupling to the external gauge field $A^\perp$

must be $-eA^\mu j^\mu \equiv -\frac{e}{2\pi} A^\mu \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda$

$\Rightarrow$ \text{eff} = \frac{1}{4\pi} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - \frac{1}{4\pi} f_{\mu\nu}^2 - \frac{e}{2\pi} A^\mu \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda$

level 1 Chern-Simons! 
\[ g^2 \text{ has units of (length)}^{-1} \rightarrow 0 \text{ at long distances} \]

\[ \Rightarrow \frac{1}{4\pi} J^\mu \text{ is irrelevant} \]

\[ \text{Dimension 4} \]

\[ \text{Equation of motion} \]

\[ m \epsilon_{\mu \nu \lambda} \partial \nu A^\lambda = j^\mu \]

\[ \Rightarrow \quad j^\mu = -e \epsilon_{\mu \nu \lambda} \partial \nu A^\lambda \]

\[ \Rightarrow \quad m A^\mu = -e A^\mu \]

\[ \Rightarrow \quad \text{Eff} (A) = \frac{m}{4\pi} \left( \frac{e}{\mu} \right)^2 \epsilon_{\mu \nu \lambda} A^\mu \partial \nu A^\lambda \]

\[ \equiv \frac{e^2}{2\pi m} \frac{1}{2} \epsilon_{\mu \nu \lambda} A^\mu \partial \nu A^\lambda \]

\[ \Rightarrow \quad \sigma_{xy} = \frac{e^2}{m} \frac{1}{2\pi \hbar} \quad \text{FQH} \]
Chern–Simons QFT

It is a Topological QFT. It plays a central role in the theory of anyons and of FQH. We will focus on the U(1) theory (almost completely).

\[ L = k \frac{e_{\mu \nu \lambda}}{4\pi} \partial_{\mu} a_x \wedge \partial_{\nu} a_y + \int \frac{1}{2} \partial_{\mu} a^\mu \]

\( a^\mu \) is a U(1) gauge field; conserved current.

\( k \in \mathbb{Z} \) is the level of the CS theory.

Expand in components:

\[ L = a_0 \left[ j^0 - \frac{k}{2\pi} \epsilon_{ij} \partial_i a_j \right] \]

\[ \theta + k \frac{1}{2\pi} \epsilon_{ij} \partial_i a_j \]

\[ \Rightarrow \text{Gauss Law} \Rightarrow \int_{\mathbb{S}^2} \partial_i (\mathbf{j}^i) = \frac{k}{2\pi} \mathbf{b}(x) \]

\( \mathbf{b}(x) = \epsilon_{ij} \partial_i a_j \) [flux attachment]

\( \Rightarrow \) \( a_1 \) and \( a_2 \) are canonical pairs

\[ [a_i, a_j] = i \frac{2\pi}{k} \epsilon_{ij} \delta(x-y) \]
How about the Hamiltonian?

\[ H = p \dot{q} - L \]

\[ H = \frac{k}{2\pi} \sum_{i,j} \epsilon_{ij} \partial_i \phi \cdot \partial_j \phi - \frac{k}{2\pi} \sum_{i,j} \epsilon_{ij} \partial_i \phi \cdot \partial_j \phi \]

\[ - a_0 \left( \dot{\phi} - \frac{k}{2\pi} \sum_{i,j} \epsilon_{ij} \partial_i \phi \cdot \partial_j \phi \right) + \dot{\phi} \cdot \dot{\phi} \]

\[ \Rightarrow \text{in the gauge-invariant subspace} \]

\[ \dot{\phi} = \frac{k}{2\pi} \sum_{i,j} \epsilon_{ij} \partial_i \phi \cdot \partial_j \phi \quad \text{(operator identity!)} \]

\[ \Rightarrow H = - \dot{\phi} \cdot \dot{\phi} \]

In the absence of source fields, \( H = 0 \).

Chern-Simons theory is gauge invariant on a closed manifold (if \( k \in \mathbb{Z} \)) but not on a manifold with a boundary.

\[ S(\phi_\mu + \phi_\mu) = \frac{k}{4\pi M} \int d^3x \left( \phi_\mu + \phi_\mu \right) \epsilon_{\mu
u\lambda} \partial^\nu (\phi^\lambda + \phi^\lambda) \]

\[ = \frac{k}{4\pi M} \int d^3x \sum_{\mu\nu\lambda} \epsilon_{\mu\nu\lambda} \partial^\nu \phi^\lambda + \frac{k}{2\pi M} \int \sum_{\mu\nu\lambda} \epsilon_{\mu
u\lambda} \partial^\nu \phi^\lambda \partial^\lambda \phi^\lambda \]
\[ \delta S = S(\alpha^+ + \alpha^-) - S(\alpha_\mu) = \]
\[ = \frac{k}{2\pi} \int d^3\mathbf{x} \, \epsilon_{\mu\nu\rho} \, \partial^\mu \Phi \, A^\nu \]
\[ = \frac{k}{2\pi} \int d^3\mathbf{x} \, \partial^\mu \Phi \, F^\mu_* \]
\[ F^\mu_* = \epsilon_{\mu\nu\lambda} \, A^\nu A^\lambda \]
\[ \Rightarrow \delta S = \frac{k}{2\pi} \int d^3\mathbf{x} \, \partial^\mu \Phi \, F^\mu_* \]
\[ - \frac{k}{2\pi} \int d^3\mathbf{x} \, \Phi \, \omega^\mu F^\mu_* \]
\[ \text{but } \omega^\mu F^\mu_* = 0 \quad (\text{Bianchi} \Rightarrow \text{no monopole}) \]
\[ \Rightarrow \text{Gauss: } \delta S = \frac{k}{2\pi} \int dS^\mu \Phi^\mu F^\mu_* \]
\[ \partial \mathbf{M} = \Sigma \]
If \( \Phi \) is constant on \( \partial \Sigma \Rightarrow \delta S = \frac{k}{2\pi} \Phi \times \text{flux}(\Sigma) \]
\[ \Rightarrow \text{we need boundary degrees of freedom to set gauge invariance } \Rightarrow \text{edge states} \]
The eq. of motion is \( F_{\mu \nu} = 0 \) (no source).

\[ A_{\mu} = \partial_{\mu} \psi \]

Let \( M = D \times \mathbb{R} \)

\[ \Sigma = \partial M = S_1 \times \mathbb{R} \]

\( \Rightarrow \) the action for a flat anti-symmetric field.

\[ S = \int_{S_1 \times \mathbb{R}} \frac{k}{2\pi} \partial_{\mu} \psi \partial^{\mu} \psi \]

We can break the topological invariance on the boundary with a term \( \alpha A^2 \).

\[ S = \int_{S_1 \times \mathbb{R}} \frac{k}{2\pi} \left( \partial_{\mu} \partial^{\mu} \psi - (\partial \psi)^2 \right) \]

Action of a chiral boson in 1+1 dimensions!

This is the action of the edge states!