L24 11/15/2022

There is an additional change in angular momentum caused by the addition of the 2nd solenoid.

(but this time measured from w, not u)

Also if we drag adiabatically the quasihole with charge $\frac{e}{m}$ around the 2nd fluxoid, the wave function should have a phase due to the Aharonov–Bohm effect equal to $\frac{e}{m}$ (Kivelson & Rocek, 1985)

How can we determine $N(u,w)$?

Two conditions: ① the w.f. should be an analytic function of $z_1, .., z_N$ (Lowest Landau Level)

② It should be translation invariant

(i.e. a function of differences of $u, w, z_1, .., z_N$)

The plasma analogy now is

$$|\Psi_m^{(+)}(u,w,z_1,\dots,z_N)|^2 = e^{-\beta U_{eff}(u,w,z_1,\dots,z_N)}$$

$$U_{eff} = U(z_1,\dots,z_N) - \frac{2}{m}\sum_{j=1}^{N}\left(\ln|u-z_j| + \ln|w-z_j|\right)$$

$$+ \frac{2}{m}\ln|N(u,w)|$$

Translation invariance and analyticity are met if we choose (Halperin 1983)

$$N(u,w) = N_0 \underset{\substack{\uparrow \\ \text{branch cut!}}}{(u-w)}^{1/m} e^{-(|u|^2+|w|^2)/4l_0^2 m}$$

Halperin w.f. for two quasiholes

$$\Psi_m^{(+)}(u,w;z_1,\dots z_N) = N_0 (u-w)^{1/m} \times$$

$$\times \prod_{j=1}^{N} (z_j-u)(z_j-w)\, e^{-\frac{|u|^2+|w|^2}{4l_0^2 m}}$$

$$\times \Psi_m^{(+)}(z_1,\dots,z_N)$$

$$\Rightarrow U_{eff} = -2\sum_{1\le j<k\le N} \ln|z_j - z_k|$$

$$-\frac{2}{m}\sum_{j=1}^{N}\left(\ln|z_j-u| + \ln|z_j-w|\right)$$

$$-\frac{2}{m^2}\ln|u-w| + \frac{1}{2m l_0^2}\sum_{j=1}^{N}|z_j|^2$$

$$+\frac{1}{2m^2 l_0^2}\left(|u|^2+|w|^2\right)$$

$\Leftrightarrow$ N classical particles of charge -1 interacting with two extra $-\frac{1}{m}$ charged particles

at u and w (and a neutralizing background) since the Halperin wf has a branch cut ⇒ exchanging slowly u and w causes the wf to change by $e^{\pm i\pi/m}$

⇒ anyons! (fractional statistics!)

Arovas, Schrieffer and Wilczek (1984) used a Berry phase argument to show that the quasiholes indeed have fractional statistics.

Jain's reinterpretation ⇒ composite fermions

$$\Psi_m(z_1 \dots z_N) = \prod_{j<k} (z_j - z_k)^m \, e^{-\sum_{j=1}^{N} |z_j|^2/4l_0^2}$$

$$= \underset{\underset{\text{(m-1) fluxes attached to each electron}}{\uparrow}}{\left[\prod_{j<k} (z_j - z_k)^{m-1}\right]} \; \underset{\underset{\text{full Landau level}}{\uparrow}}{\Psi_1(z_1, \dots, z_N)}$$

Jain: the Laughlin state is the same as that of a composite fermion (e^- attached to m-1 fluxes) filling up an effective Landau level of a field $B_{eff} = B - (m-1)\phi_0$ (partial screening)

Simple hydrodynamic picture of the FQHE
(Fröhlich, Zee)

I can think of the 2DEG as an incompressible fluid in a magnetic field.

The charge current $j_\mu = (j_0, \vec{j})$ must be locally conserved $\Rightarrow \partial^\mu j_\mu = 0$

$$\Rightarrow j_\mu = \underset{\substack{\nearrow \\ \text{convenience}}}{\frac{1}{2\pi}} \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda$$

$\uparrow$ one-form ≡ vector field

$a^\lambda \to a^\lambda + \partial^\lambda \Phi$ does not change j^μ

$\Rightarrow a^\lambda$ is a gauge field

What is its eff. action?

Incompressibility $\Rightarrow$ locality

$B \neq 0 \Rightarrow$ broken time reversal (and parity)

$\Rightarrow$ it must also be gauge invariant

Coupling to the external gauge field A_μ must be $-eA_\mu j^\mu \equiv -\frac{e}{2\pi} A^\mu \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda$

$$\Rightarrow \mathcal{L}_{\text{eff}} = \underbrace{\frac{m}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda}_{\text{Chern-Simons!}} - \frac{1}{4g^2} f_{\mu\nu}^2 - \frac{e}{2\pi} A^\mu \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda$$

$\uparrow$ level

g^2 has units of $(\text{length})^{-1} \longrightarrow 0$ at long distances

$$\Rightarrow \frac{1}{4g^2}\underset{\overset{\uparrow}{\text{dimension } 4}}{f_{\mu\nu}^{2}} \text{ is irrelevant}$$

Equation of motion

$$\frac{m}{2\pi}\,\varepsilon_{\mu\nu\lambda}\,\cancel{a^{\mu}}\,\partial^{\nu}a^{\lambda} = j_{\mu}$$

$$j_{\mu} = -e\,\varepsilon_{\mu\nu\lambda}\,\partial^{\nu}A^{\lambda}$$

$$\Rightarrow \quad m\,a_{\mu} = -e\,A_{\mu}$$

$$\Rightarrow \mathcal{L}_{eff}(A) = \frac{m}{4\pi}\left(\frac{e}{m}\right)^{2}\varepsilon_{\mu\nu\lambda}A^{\mu}\partial^{\nu}A^{\lambda}$$

$$\equiv \underset{\overset{\uparrow}{\sigma_{xy} = \frac{e^2}{m}\frac{1}{2\pi\underset{\overset{\shortparallel}{1}}{\hbar}}}}{\frac{e^{2}}{2\pi m}}\ \frac{1}{2}\,\varepsilon_{\mu\nu\lambda}A^{\mu}\partial^{\nu}A^{\lambda}$$

FQH!

Chern-Simons QFT

It is a Topological QFT. It plays a central role in the theory of anyons and of FQH. We will focus on the U(1) theory (almost completely)

$$\mathcal{L} = \frac{k}{4\pi}\, \varepsilon_{\mu\nu\lambda}\, a^\mu \partial^\nu a^\lambda + j_\mu a^\mu$$

$\uparrow$

a_μ is a U(1) gauge field; conserved current

k: $k \in \mathbb{Z}$ is the level of the CS theory.

Expand in components:

$$\mathcal{L} = a_0 \left[j_0 - \frac{k}{2\pi}\, \varepsilon_{ij}\, \partial_i a_j \right] + \frac{k}{2\pi}\, \varepsilon_{ij}\, a_i \partial_0 a_j$$

$\Rightarrow$ Gauss Law $\Rightarrow$ $\boxed{j_0(x) = \frac{k}{2\pi}\, b(x)}$

$b(x) = \varepsilon_{ij}\, \partial_i a_j$ $\quad\uparrow$ flux attachment!

$\Rightarrow$ a_1 and a_2 are canonical pairs

$$[a_i, a_j] = i\, \frac{2\pi}{k}\, \varepsilon_{ij}\, \delta(x-y)$$

How about the Hamiltonian?

"$\mathcal{H} = p\dot{q} - L$"

~~$\mathcal{H} = \frac{k}{2\pi} a_1 \partial_0 a_2$~~

$$\mathcal{H} = \cancel{\frac{k}{2\pi}\varepsilon_{ij}\, a_i \partial_0 a_j - \frac{k}{2\pi}\varepsilon_{ij}\, a_i \partial_0 a_j}$$

$$- a_0\left(j_0 - \frac{k}{2\pi}\varepsilon_{ij}\,\partial_i a_j\right) - \vec{j}\cdot\vec{a}$$

$\Rightarrow$ in the gauge-invariant subspace

$$j_0 = \frac{k}{2\pi}\varepsilon_{ij}\,\partial_i a_j \quad \text{(operator identity!)}$$

$\Rightarrow$ $\mathcal{H} = -\vec{j}\cdot\vec{a}$

In the absence of ~~sour~~ sources $\mathcal{H} = 0$!

Chern-Simons theory is gauge invariant on a closed manifold (if $k \in \mathbb{Z}$) but not on a manifold with a boundary.

$$S(a_\mu + \partial_\mu \Phi) = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x\, (a_\mu + \partial_\mu \Phi)\, \varepsilon_{\mu\nu\lambda}\, \partial^\nu (a^\lambda + \partial^\lambda \Phi)$$

$$= \frac{k}{4\pi}\int_{\mathcal{M}} d^3x\, \varepsilon_{\mu\nu\lambda}\, a^\mu \partial^\nu a^\lambda + \frac{k}{2\pi}\int_{\mathcal{M}} \varepsilon_{\mu\nu\lambda}\, \partial^\mu \Phi\, \partial^\nu a^\lambda$$

⇒ Change

$$\delta S = S(a_\mu + \partial_\mu \Phi) - S(a_\mu) =$$

$$= \frac{k}{2\pi}\int_{\mathcal{M}} d^3x \; \varepsilon_{\mu\nu\lambda}\, \partial^\mu \Phi \, \partial^\nu a^\lambda$$

$$\equiv \frac{k}{2\pi}\int_{\mathcal{M}} d^3x \; \partial^\mu \Phi \, F^*_\mu$$

$$F^*_\mu \equiv \varepsilon_{\mu\nu\lambda}\, \partial^\nu a^\lambda$$

$$\Rightarrow \delta S = \frac{k}{2\pi}\int_{\mathcal{M}} d^2x \; \partial^\mu \left[\Phi \, F^*_\mu \right]$$

$$- \frac{k}{2\pi}\int_{\mathcal{M}} d^3x \; \Phi \, \partial^\mu F^*_\mu$$

but $\partial^\mu F^*_\mu = 0$ (Bianchi ⇔ no monopoles!)

⇒ Gauss ⇒ $\delta S = \frac{k}{2\pi}\int dS^\mu \, \Phi \, F^*_\mu$

$$\partial\mathcal{M} \equiv \Sigma$$

If Φ is constant on $\partial\mathcal{M}$ ⇒ $\delta S = \frac{k}{2\pi} \Phi \times \text{flux}(\Sigma)$

⇒ we need boundary degrees of freedom to get gauge invariance ⇒ edge states

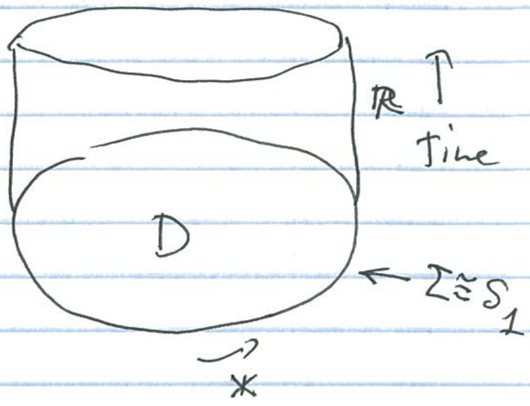
The eq. of motion is $F_{\mu\nu}=0$ (no sources)

$\Rightarrow$ a_μ must be a pure gauge

$a_\mu = \partial_\mu \varphi$

Let $M = \underset{\underset{\text{disk}}{\uparrow}}{D} \times \underset{\underset{\text{time}}{\uparrow}}{\mathbb{R}}$

$M \equiv$



$\Rightarrow$ $\Sigma = \partial M \equiv S_1 \times \mathbb{R}$

$\Rightarrow$ the action for a flat config.

$$S = \int_{S_1\times\mathbb{R}} d^2x \, \frac{k}{2\pi}\, \partial_0\varphi\, \partial_1\varphi$$

We can break the topological invariance on the boundary with a term $\propto a_*^2 \equiv (\partial_*\varphi)^2$

$$S = \int_{S_1\times\mathbb{R}} d^2x \, \frac{k}{2\pi}\left(\underset{\underset{\text{commutation relation}}{\uparrow}}{\partial_0\varphi\, \partial_1\varphi} - \underset{\underset{\text{energy}}{\uparrow}}{(\partial_x\varphi)^2}\right)$$

Action of a chiral boson in 1+1 dimensions!

This is the action of the edge states!